Chapter 4

Concurrency of Lines in a Triangle

Three of the results we mentioned in the last section were about the centroid, the incenter, the circumcenter and the orthocenter. Each of these is the point of concurrency of the medians, the angle bisectors, the perpendicular bisectors, and the altitudes, respectively. There are many ways that these results are proven, but rarely do we have an opportunity to see how these might be pulled together into a more unified approach. If you are doing these in your high school classroom you might use ***Geometer's Sketchpad*** to show the students that these are correct, but often no proof is given. Some of the proofs are cumbersome, but we want to look at a different approach — one using a theorem of Giovanni Ceva.
Notation: We will use $K(\triangle ABC)$ to denote the area of $\triangle ABC$.

4.1 Ceva's Theorem

Theorem 4.1 *The three line containing the vertices A, B, and C of $\triangle ABC$ and intersecting opposite sides at points L, M, and N, respectively, are concurrent if and only if*

$$\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = 1.$$

This is clearly an "algebraic" approach to a geometrical problem. We should look at different configurations of triangles to see if there are different cases we will need to consider in proving this. There are two cases in which the lines drawn from the vertices may intersect the sides and be concurrent. These appear in Figure 4.1.

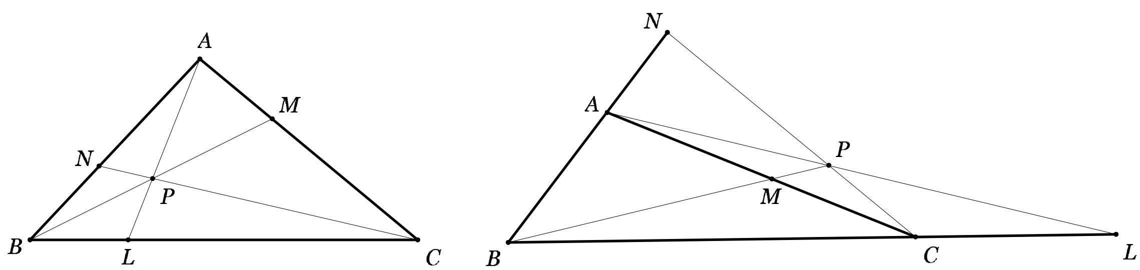


Figure 4.1:

We will argue from the figure on the left. The same argument will work for the figure on the right, but it should be checked.[1]

Proof: This is an *if and only if* statement so we have two things to prove. First we will show that if they are concurrent then the product is 1.

Assume that AL, BM, and CN intersect in a point, P. Because $\triangle ABL$ and $\triangle ACL$ have the same altitude

$$\frac{K(\triangle ABL)}{K(\triangle ACL)} = \frac{BL}{LC}.$$

Similarly,

$$\frac{K(\triangle PBL)}{K(\triangle PCL)} = \frac{BL}{LC},$$

so

$$\frac{K(\triangle ABL)}{K(\triangle ACL)} = \frac{K(\triangle PBL)}{K(\triangle PCL)}.$$

Now, a simple property of proportions

$$\frac{a}{b} = \frac{c}{d} = \frac{a-c}{b-d}$$

gives us that

$$\frac{BL}{LC} = \frac{K(\triangle ABL) - K(\triangle PBL)}{K(\triangle ACL) - K(\triangle PCL)} = \frac{K(\triangle ABP)}{K(\triangle ACP)}.$$

If we repeat this process only using BM instead of AL, we get that

$$\frac{CM}{MA} = \frac{K(\triangle BMC)}{K(\triangle BMA)} = \frac{K(\triangle PMC)}{K(\triangle PMA)},$$

and then

$$\frac{CM}{MA} = \frac{K(\triangle BMC) - K(\triangle PMC)}{K(\triangle BMA) - K(\triangle PMA)} = \frac{K(\triangle BCP)}{K(\triangle BAP)}.$$

Now use CN instead of AL and we get

$$\frac{AN}{NB} = \frac{K(\triangle ACN)}{K(\triangle BCN)} = \frac{K(\triangle APN)}{K(\triangle BPN)},$$

giving

$$\frac{AN}{NB} = \frac{K(\triangle ACN) - K(\triangle APN)}{K(\triangle BCN) - K(\triangle BPN)} = \frac{K(\triangle ACP)}{K(\triangle BCP)}.$$

Now, the result follows:

$$\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = \frac{K(\triangle ACP)}{K(\triangle BCP)} \cdot \frac{K(\triangle ABP)}{K(\triangle ACP)} \cdot \frac{K(\triangle BCP)}{K(\triangle BAP)} = 1.$$

Now, we need to prove that if this product is 1, then they are concurrent. To do that we will assume that $\overleftrightarrow{BM}$ and $\overleftrightarrow{AL}$ intersect at a point P. Construct the line $\overleftrightarrow{PC}$ and let its intersection with $\overleftrightarrow{AB}$ be N'. Then $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN'}$ are concurrent. Thus,

$$\frac{AN'}{N'B} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = 1.$$

[1] Of course, it has been checked millions of times — or so we would think. What if everyone just assumed that "somebody else" will check it, so I don't have to? Where would that put us?

Our hypothesis was that

$$\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = 1.$$

Therefore,

$$\frac{AN'}{N'B} = \frac{AN}{NB},$$

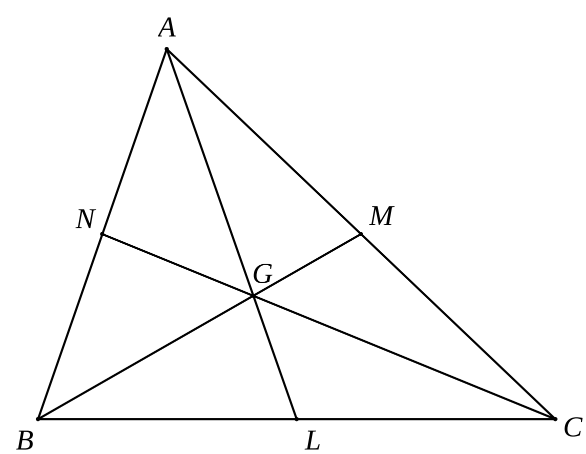
so N and N' have to coincide, proving concurrency. ∎

4.2 Medians and Centroid

In $\triangle ABC$ let A', B', and C' be the midpoints of the sides BC, AC, and AB respectively. The line segments AA', BB', and CC' are called the *medians* of $\triangle ABC$.

Theorem 4.2 *The three medians of a triangle $\triangle ABC$ intersect at a common point G.*

The common point of intersection is called the *centroid* of the triangle $\triangle ABC$.

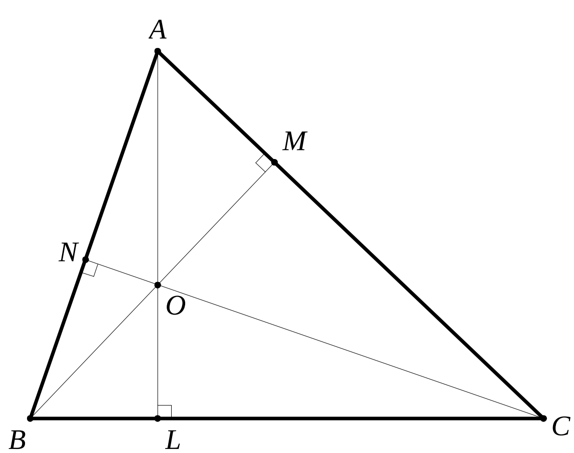


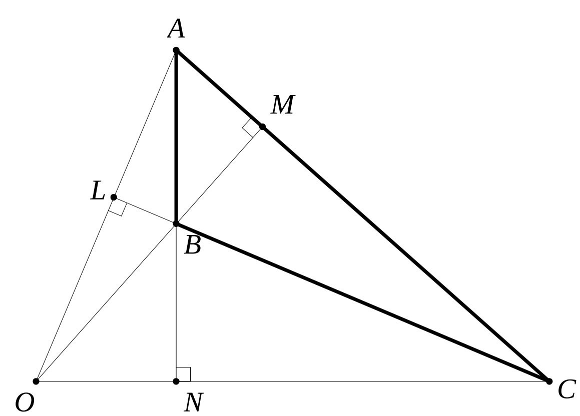
PROOF: We know that AL, BM, and CN are the medians, so $AN = NB$, $BL = LC$, and $CM = MA$. Therefore,

$$(AN)(BL)(CM) = (NB)(LC)(MA)$$
$$\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = 1.$$

Therefore, by Ceva's Theorem they are concurrent. ∎

Theorem 4.3 *In an arbitrary triangle, the three altitudes intersect in a common point, called the* ***orthocenter****.*





PROOF: The argument will be for the triangle on the left (acute). It is left to you to check that the same argument works for the triangle on the right (obtuse).

$$\triangle ANC \sim \triangle AMB \rightarrow \frac{AN}{MA} = \frac{AC}{AB} \tag{4.1}$$

$$\triangle BLA \sim \triangle BNC \rightarrow \frac{BL}{NB} = \frac{AB}{BC} \tag{4.2}$$

$$\triangle CMB \sim \triangle CLA \rightarrow \frac{CM}{LC} = \frac{BC}{AC} \tag{4.3}$$

Multiplying these three quantities together we get:

$$\frac{AN}{MA} \cdot \frac{BL}{NB} \cdot \frac{CM}{LC} = \frac{AC}{AB} \cdot \frac{AB}{BC} \cdot \frac{BC}{AC} = 1.$$

Thus, the altitudes are concurrent by Ceva's Theorem. ∎

Definition 4.1 *A **cevian** is a line segment which joins a vertex of a triangle with a point on the opposite side (or its extension).*

Using Ceva's Theorem we can prove the following results.

Theorem 4.4 *The bisector of any interior angle of a nonisosceles triangle and the bisectors of the two exterior angles at the other vertices are concurrent.*

Theorem 4.5 *In triangle $\triangle ABC$ let $P \in AB$ and $Q \in AC$ so that $PQ \parallel BC$. Then PC and QB intersect at a point on the median AM.*

Theorem 4.6 *In triangle $\triangle ABC$ where CD is the altitude to AB and P is any point on CD, AP intersects CB at a point Q and BP intersects CA at a point R. Then $\angle RDC \cong \angle QDC$.*

4.3 Incircles and Law of Cosines

Theorem 4.7 *The angle bisectors of a triangle intersect at a common point I called the incenter, which is the center of the unique circle inscribed in the triangle (called the incircle).*

PROOF: Consider the angle $\angle ABC$ and let D be a point on the angle bisector. Let E and E' be the points on BA and BC, respectively, so that $\angle BED$ and $\angle BE'D$ are right angles. Thus, $\triangle BED \cong \triangle BE'D$ by AAS, since they share BD. Thus, $|DE| = |DE'|$ and the circle centered at D with radius $|DE|$ is tangent to both BA and BC.

Since AL is the angle bisector of $\angle A$, we have that

$$\frac{AB}{AC} = \frac{BL}{LC}.$$

Similarly we have that

$$\frac{BC}{BA} = \frac{CM}{AM} \text{ and } \frac{CA}{CB} = \frac{AN}{BN}.$$

Therefore
$$\frac{AN}{MA}\cdot\frac{BL}{NB}\cdot\frac{CM}{LC}=\frac{AN}{NB}\cdot\frac{BL}{LC}\cdot\frac{CM}{MA}=\frac{CA}{CB}\cdot\frac{AB}{AC}\cdot\frac{BC}{BA}=1.$$

Therefore, the angle bisectors are concurrent. The first paragraph shows that this point of concurrency is equidistant from each of the three sides, and we are done. ∎

Let the *inradius* r be the radius of the incircle. Let $s=\frac{1}{2}(a+b+c)$ be the *semiperimeter* of $\triangle ABC$.

Theorem 4.8 *If r is the inradius of $\triangle ABC$, and s is the semiperimeter of $\triangle ABC$. Then*

$$\text{area}(\triangle ABC)=|\triangle ABC|=rs.$$

PROOF: Left for the reader. ∎

Theorem 4.9 (Law of Cosines) *For any triangle $\triangle ABC$, we have*

$$c^2=a^2+b^2-2ab\cos(C).$$

PROOF: Let D be the altitude dropped from A to BC. Then by the Pythagorean Theorem

$$c^2=|AD|^2+|DB|^2.$$

Now,

$$\begin{aligned}|AD| &= b\sin(C)\\ |DB| &= |a-b\cos(C)|\end{aligned}$$

Thus,

$$\begin{aligned}c^2 &= b^2\sin^2(C)+a^2-2ab\cos(C)+b^2\cos^2(C)\\ c^2 &= a^2+b^2-2ab\cos(C)\end{aligned}$$

as we needed. ∎

Theorem 4.10 (Heron's Formula) *For any triangle $\triangle ABC$*

$$|\triangle ABC|=\sqrt{s(s-a)(s-b)(s-c)}.$$

PROOF: Note that
$$|\triangle ABC|=\frac{1}{2}ab\sin(C).$$

By the Law of Cosines,
$$\cos(C)=\frac{a^2+b^2-c^2}{2ab}$$

Thus, applying some algebra

$$\begin{aligned}
|\triangle ABC| &= \frac{1}{2}ab\sqrt{1-\cos^2(C)} \\
&= \frac{1}{2}ab\frac{\sqrt{4a^2b^2-(a^2+b^2-c^2)^2}}{2ab} \\
&= \frac{1}{4}\sqrt{(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)} \\
&= \frac{1}{4}\sqrt{((a+b)^2-c^2)(c^2-(a-b)^2)} \\
&= \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(c-a+b)(c+a-b)} \\
&= \sqrt{\frac{a+b+c}{2}\frac{a+b-c}{2}\frac{-a+b+c}{2}\frac{a-b+c}{2}} \\
&= \sqrt{s(s-a)(s-b)(s-c)}
\end{aligned}$$

Heron's formula is named for Heron of Alexandria, who lived sometime between 100 BC and 300 AD. Scholars state that the formula dates back to at least Archimedes (ca. 250 BC). ∎

4.4 The Circumcenter and its Spawn

We have seen the *centroid*—center of mass — and the *incenter*. There is yet another center of a triangle. We remember that given any three points there is a unique circle passing through them. How do you find that circle?

Take the perpendicular bisectors of the sides of a triangle formed by the three points. These bisectors meet in a common point, called the *circumcenter*. The radius of the circumcircle is called the circumradius.

Theorem 4.11 *Given a triangle $\triangle ABC$, the perpendicular bisectors of the sides are concurrent. The point is the center of a circle which passes through the vertices of the triangle. The point is called the circumcenter of the triangle.*

PROOF: We must have that two of the perpendicular bisectors intersect. Let p_1 and p_2 denote the perpendicular bisectors of AB and AC respectively. If p_1 is parallel to p_2, then since AC is perpendicular to p_2, AC is perpendicular to p_1. Since AB is perpendicular to p_1, then AB must be parallel to AC or they coincide. Thus, we would not have a triangle.[2] Thus, two perpendicular bisectors intersect in a point O. Let M denote the midpoint of AB. Then $\triangle AOM \cong \triangle BOM$, since the angle at M is a right angle, $AM \cong BM$, and $OM \cong OM$. Hence, $AO \cong BO$. Using AC we can also show that $AO \cong CO$. Thus, the triangles $\triangle BON$ and $\triangle CON$ are congruent, where N is the midpoint of BC. Hence, ON is perpendicular to BC and we are done. ∎

PROOF: II: Can we prove this one by Ceva's Theorem? First note that we do not have any cevians at this point! The perpendicular bisectors do not go through the vertices opposite the sides. How could we use it? Clearly, it is a theorem about concurrency, so it would seem to be a good candidate for Ceva's Theorem.

[2] This actually uses a result that is equivalent to Euclid's fifth postulate.

What we will do is to introduce a second triangle made by connecting the three points L, M, and N. Now $\triangle LMN$ is called the *medial triangle* of $\triangle ABC$ since L, M, and N are the midpoints of the sides.

Since L is the midpoint of BC and N is the midpoint of AB, we have that $\triangle BNL \sim \triangle BAC$. Thus, $\angle BNL \cong \angle BAC$ and that makes NL parallel to AC. Since ME is perpendicular to AC, it is perpendicular to NL, making it an altitude of $\triangle LMN$. Likewise, we can show that each of the perpendicular bisectors of the sides of $\triangle ABC$ is an altitude of $\triangle LMN$. Since the altitudes are concurrent, the perpendicular bisectors are concurrent. ∎

Let R denote the radius of the circumcircle.

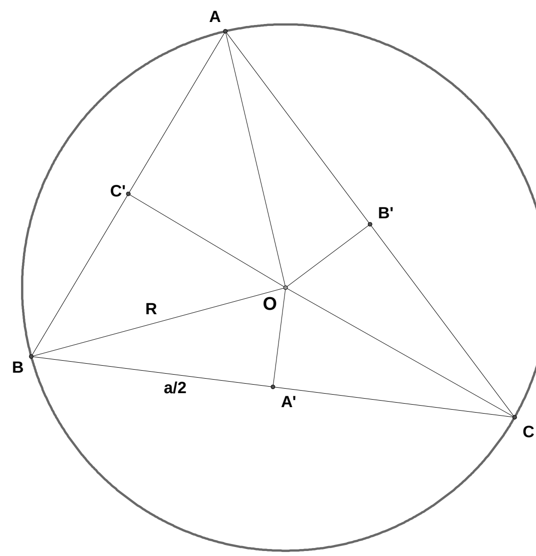


Figure 4.2: Circumcenter

Theorem 4.12 (Extended Law of Sines) *In triangle $\triangle ABC$*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Proof: In $\triangle ABC$, let ON be the perpendicular bisector of BC. Then $\triangle BOC$ is isosceles, $\angle BON \cong \angle CON$ and $BN = CN = a/2$. By the Star Trek Lemma $\angle BOC = 2A$. Thus, $\angle BON = \angle A$. Thus,

$$R \sin A = \frac{a}{2}$$

and

$$2R = \frac{a}{\sin A}.$$

Similarly,

$$2R = \frac{b}{\sin B} = \frac{c}{\sin C},$$

as we needed. ∎

Most of us remember the Law of Sines, but few of us ever ask "What is the common ratio given in the Law of Sines?" Now you know, that common ratio is twice the radius of the circumcircle.

4.5 The Gergonne Point

Theorem 4.13 *The lines containing a vertex of a triangle and the point of tangency of the opposite side with the inscribed circle are concurrent. This point of concurrency is called the **Gergonne point** of the triangle.*

Proof: Let the incircle γ be tangent to the sides AB, AC, and BC at the points N, M, and L, respectively. Then, in our proof of the incenter, we showed that $AN = AM$, $BN = BL$, and $CM = CL$. Therefore,

$$\frac{AN}{MA} \cdot \frac{BL}{NB} \cdot \frac{CM}{LC} = 1.$$

Thus, by Ceva's Theorem, these segments are concurrent. ∎

4.6 More Triangle Centers

The few centers we have seen only begin to scratch the surface of what is known about the different triangle centers and central lines of triangles. I will mention only a few more here. The best location to find information about triangle centers is the *Triangle Centers* website.

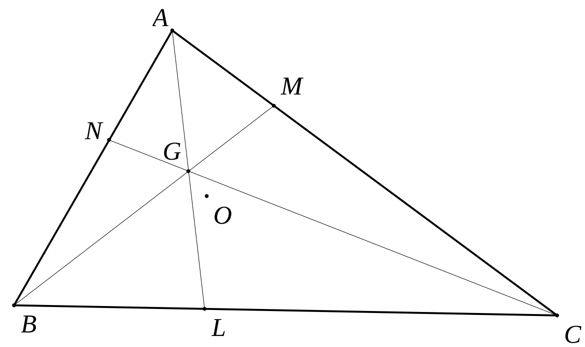


Figure 4.3: Gergonne point

Let $\triangle ABC$ be an arbitrary triangle. We want to consider the equilateral triangle constructed on each side of the triangle $\triangle ABC$. That is $\triangle A'BC$ is the equilateral triangle on side BC, $\triangle AB'C$ is the equilateral triangle on side AC, and $\triangle ABC'$ is the equilateral triangle on side AB.

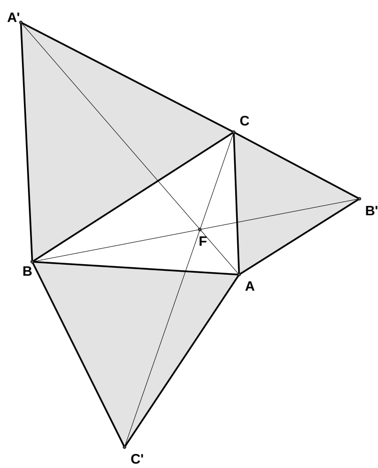


Figure 4.4: Fermat point

The lines AA', BB', and CC' meet in the *Fermat point*. This is said to be the first triangle center discovered after ancient Greek times. It arose from a problem posed by the great French mathematician, Pierre Fermat. The problem requests the solver to find the point P in the triangle for which the sum $PA+PB+PC$ is minimal. Torricelli proved that the Fermat point is the solution if each angle of the triangle $\triangle ABC$ is less than $120°$. The Fermat point is also known as the *first isogonic center*. This is because the angles $\angle BFC$, $\angle CFA$ and $\angle AFB$ are all equal.