Chapter 5

Collinearity of Triangle Points

In the last section we looked at when lines were concurrent. In this section, we will go the "opposite way" and look when points are collinear.

5.1 The Euler Line

What happens if the circumcenter, O, coincides with the centroid, G? That would mean that the medians of the triangle are the perpendicular bisectors as well. This will force the triangle to be an equilateral triangle. What happens if the triangle is not equilateral? Is there any relationship between the circumcenter and the centroid? They will be distinct.

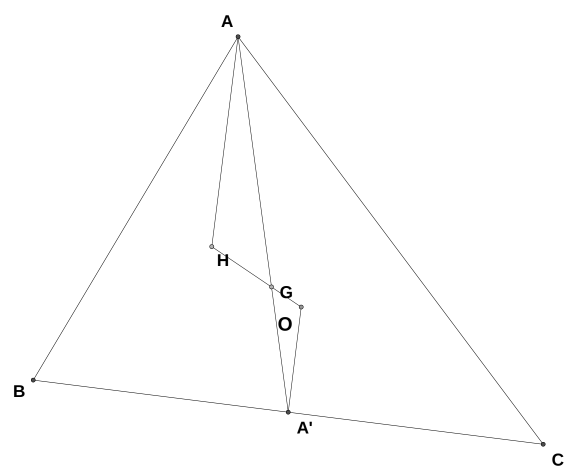

Figure 5.1: Euler line

Theorem 5.1 (The Euler Line) *The circumcenter O, the centroid G, and the orthocenter H are collinear. Furthermore, G lies between O and H and*

$$\frac{|OG|}{|GH|} = \frac{1}{2}.$$

This line is called the *Euler line*. It was not discovered in any ancient writings and apparently, Leonhard Euler (1707–1783) was the first to discover this result.

5.2 Pedal Triangles and the Simson Line

The Euler line is not unique in the study of triangles. There are other interesting points and lines associated to any triangle.

A *cyclic quadrilateral* is a quadrilateral that can be inscribed in a circle. The following is left for homework.

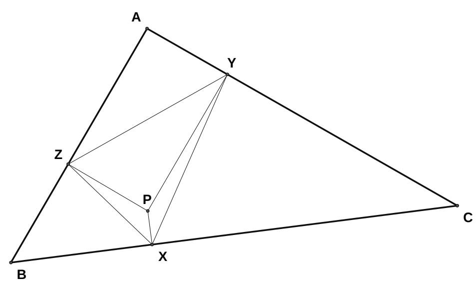

Figure 5.2: Pedal Triangle

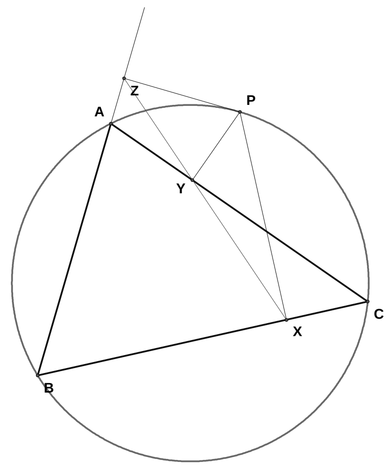

Figure 5.3: Simson Line

Theorem 5.2 *A convex quadrilateral $ABCD$ is a cyclic quadrilateral if and only if $\angle ABC + \angle CDA = 180°$.*

Let $\triangle ABC$ be an arbitrary triangle and let P be a point either inside or outside the triangle. Let X be the foot of the perpendicular to the extended side BC and through P. Define points Y and Z on the extended sides AC and AB respectively, similarly. The triangle $\triangle XYZ$ is called the *pedal triangle* with respect to the point P and the triangle $\triangle ABC$.(**cf.** Figure 5.2

Lemma 5.1 *Let P be a point inside $\triangle ABC$, and let $\triangle XYZ$ be the pedal triangle with respect to P. Then $\angle APB = \angle ACB + \angle XZY$.*

PROOF: Let CP intersect AB at C'. Then write

$$\angle APB = \angle APC' + \angle C'PB.$$

Since $\angle ABC'$ is an exterior angle of $\triangle APC$, we have that $\angle APC' = \angle PAC + \angle ACP$. Now, $\angle PZA = \angle AYP = 90°$, so they sum to $180°$ and $AYPZ$ is a cyclic quadrilateral. Thus,

$$\angle PAC = \angle PAY = \angle PZY,$$

which implies $\angle APC' = \angle PZY + \angle ACP$. Similarly, $\angle C'PB = \angle XZP + \angle PCB$. Thus,

$$\begin{aligned} \angle APB &= \angle APC' + \angle C'PB \\ &= (\angle PZY + \angle XZP) + (\angle ACP + \angle PCB) \\ &= \angle XZY + \angle ACB, \end{aligned}$$

as desired. ∎

Theorem 5.3 (The Simson Line) *Let Γ be the circumcircle for $\triangle ABC$. Let P be a point on Γ, and let $\triangle XYZ$ be the pedal triangle with respect to P. Then $\triangle XYZ$ is a degenerate triangle,* i.e. *the points X, Y, Z are collinear. This line is called the Simson line.*

PROOF: Without loss of generality, we may assume P lies on the arc AC. Then $\angle APB = \angle ACB$, since they subtend the same arc. Hence, by Lemma 5.1 $\angle XZY = 0$. That is $\triangle XYZ$ is degenerate. Thus, X,Y, and Z are collinear. ∎

5.3 Triangle Centers and Relative Lines

Recall that an excircle of a triangle $\triangle ABC$ is a circle outside the triangle that is tangent to all three of the lines that extend the sides of the triangle. We have three such circles, each tangent to a side and the extensions of the other two sides.

Lemma 5.2 *The lines connecting the point of tangency of each excircle of $\triangle ABC$ to the opposite vertex will intersect in a point, called the Nagel point, N. (**cf.** Figure 5.4)*

One more point of interest is the center of the incircle for $\triangle ABC$'s medial triangle. This circle is called the Spieker circle and its center is called the Spieker point, S.

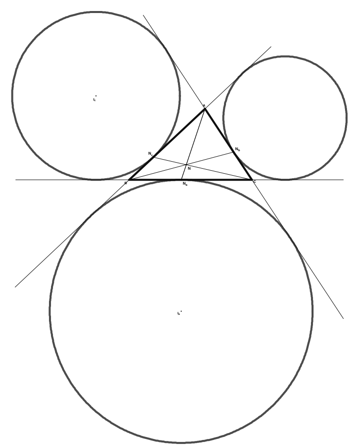

Figure 5.4: Nagel Point

Lemma 5.3 *The Nagel segment is a line segment from the incenter, I, to the Nagel point, N, which contains the Spieker point, S, and the centroid, G.(**cf.** Figure 5.5)*

We display the following results about the Nagel segment and the Spieker circle without proof.

Lemma 5.4 *For $\triangle ABC$,*

1. *The Spieker circle is the incircle of $\triangle ABC$'s medial triangle.*

2. *The Spieker circle has radius one-half of $\triangle ABC$'s incircle.*

3. *The Spieker circle is the incircle of the triangle whose vertices are the midpoints of the segments that join $\triangle ABC$'s vertices with its Nagel point.*

4. *The Spieker circle is tangent to the sides of $\triangle ABC$'s medial triangle where that triangle's sides are cut by the lines that join $\triangle ABC$'s vertices with its Nagel point.*

Note the similarity to the nine-point circle. In addition, we have the following.

Lemma 5.5 *The Spieker point is the midpoint of the Nagel segment. The centroid is one-third of the way from the incenter to the Nagel point.*

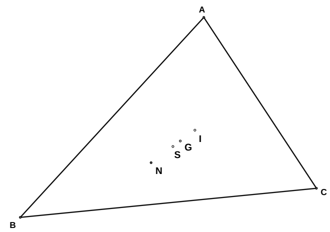

Figure 5.5: Nagel Segment

These theorems of concurrence we have considered to now are related to the concurrence of three lines. Lines are not the only items of interest in geometry. Miquel's Theorem considers the concurrence of sets of three circles associated with a triangle.

Theorem 5.4 (Miquel's Theorem) *If three points are chosen, one on each side of a triangle, then the three circles determined by a vertex and the two points on the adjacent sides meet at a point called the Miquel point.*

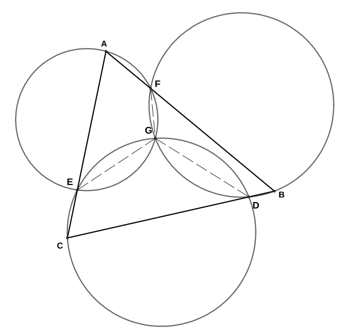

Figure 5.6: Miquel Point

Proof: Let $\triangle ABC$ be our triangle and let D,E, and F be arbitrary points on the sides of the triangle. Construct the circles determined by pairs of these points and a vertex. Consider two of the circles, C_1 and C_2, with centers I and J. They must intersect at D, so they must intersect at a second point, call it G. In circle C_2, we have that the angles $\angle EGD$ and $\angle ECD$ are supplementary. In circle C_1 $\angle FGD$ and $\angle ABD$ are supplementary.

Then,

$$\begin{aligned}
\angle EGD^\circ + \angle DGF^\circ + \angle EGF^\circ &= 360^\circ \\
(180^\circ - \angle C^\circ) + (180^\circ - \angle B^\circ) + \angle EGF^\circ &= 360^\circ \\
\angle EGF^\circ &= \angle C^\circ + \angle B^\circ \\
&= 180^\circ - \angle A^\circ
\end{aligned}$$

so that $\angle EGF$ and $\angle EAF$ are supplementary, and hence E, A, F, and G form a cyclic quadrilateral. Thus, all three circles are concurrent. Note that you must modify this proof, slightly, if the Miquel point is outside of the triangle.

5.4 Morley's Theorem

Theorem 5.5 (Morley's Theorem) *The adjacent trisectors of the angles of a triangle are concurrent by pairs at the vertices of an equilateral triangle.*

The following proof is due to John Conway.

Proof: Let the angles A,B,and C measure 3α, 3β, and 3γ respectively. Let x^+ mean $x + 60^\circ$. Now, we have that $\alpha + \beta + \gamma = 60^\circ$, since $3\alpha + 3\beta + 3\gamma = 180^\circ$. Then there certainly exist seven abstract triangles having the angles:

1	2	3	4	5	6	7
$\alpha^{++}, \beta, \gamma$	$\alpha, \beta^{++}, \gamma$	$\alpha, \beta, \gamma^{++}$	$\alpha, \beta^+, \gamma^+$	$\alpha^+, \beta, \gamma^+$	$\alpha^+, \beta^+, \gamma$	$0^+, 0^+, 0^+$

since in every case the triple of angles adds to 180 degrees. Now these triangles are only determined up to scale, i.e., up to similarity. Determine the scale by saying that certain lines are all to have the same length.

Triangle number 7, with angles 0^+, 0^+, 0^+, is clearly equilateral, so we can take all its edges to have some fixed length L. Then arrange the edges joining B^+ to C^+ in triangle 4,

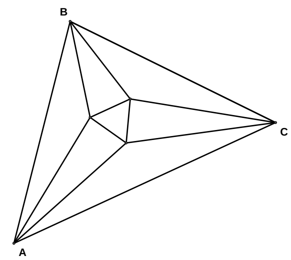

Figure 5.7: Morley's Theorem

C^+ to A^+ in triangle 5, and A^+ to B^+ in triangle 6 also to have length L. We will scale the other triangles appropriately.

Then it's easy to see that these all fit together to make up a triangle whose angles are $3A$, $3B$, $3C$, and which is therefore similar to the original one, so proving Morley's theorem. To see this, you just have to check that any two sides that come together have the same length, and that the angles around any internal vertex add to 360 degrees. The latter is easy, and the former is proved using congruences such as that that takes the vertices A, C^+, B^+ of triangle number 4 to the points A, B^{++}, Y of triangle number 2. ∎

5.5 General Collinearity and Duality

We used Ceva's Theorem to tell us when three lines are concurrent. It would be useful to have a similar theorem to tell us when three points lie on a single line, or a theorem about *collinearity*. In some sense this belongs to the principle of duality that runs through mathematics. In geometry given a statement concerning points and lines in a plane, when the word *point* is replaced by the word *line* and the word *line* is replaced by the word *point*, then the resulting statement is called the *dual* of the original statement. Dual statements are interesting when they are true. Thus, the principle of duality makes collinearity the dual concept to concurrency.

Look at a couple of examples.

Statement	**Dual Statement**
1. Two distinct points determine a unique line.	1. Two distinct lines determine a unique point.
2. Any point is coincident with an infinite number of lines.	2. Any line is coincident with an infinity number of points.
3. Only one triangle is determined by three noncollinear points.	3. Only one trilateral is determined by three nonconcurrent lines.

5.5.1 Menelaus' Theorem

Much of Greek geometry was lost prior to the Renaissance and many of the theorems discovered by the early Italian mathematicians were already known to the Greeks. In fact, it was the rediscovery of the theorem of Menelaus of Alexandria that led Ceva to his theorem.

Theorem 5.6 (Menelaus' Theorem) *The three points P, Q, and R one the sides $\overleftrightarrow{AC}$, $\overleftrightarrow{AB}$, and $\overleftrightarrow{BC}$, respectively, of $\triangle ABC$ are collinear if and only if*

$$\frac{AQ}{QB} \cdot \frac{BR}{RC} \cdot \frac{CP}{PA} = -1.$$

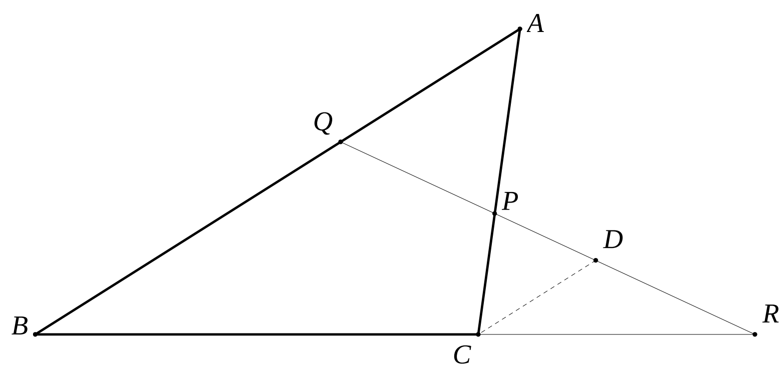

Figure 5.8: Menelaus' Theorem

Proof: Again, as in Ceva's Theorem, we have a biconditional statement so we have to prove this theorem both ways.

Assume that P, Q, and R are collinear on line ℓ. Construct a line through C parallel to AB that intersects ℓ at the point D. Then clearly $\triangle DCR \sim \triangle QBR$ which says that

$$\frac{DC}{QB} = \frac{RC}{BR} \text{ or } DC = \frac{(QB)(RC)}{BR}.$$

Likewise, $\triangle PDC \sim \triangle PQA$ so

$$\frac{DC}{AQ} = \frac{CP}{PA} \text{ or } DC = \frac{(AQ)(CP)}{PA}.$$

Thus,

$$\frac{(QB)(RC)}{BR} = \frac{(AQ)(CP)}{PA} \text{ or } (QB)(RC)(PA) = (AQ)(CP)(BR).$$

Therefore, dividing we get

$$\frac{AQ}{QB} \cdot \frac{BR}{RC} \cdot \frac{CP}{PA} = 1.$$

Now, taking direction into account we see that $\dfrac{BR}{RC}$ is a negative ratio, whereas $\dfrac{AQ}{QB}$ and $\dfrac{CP}{PA}$ are positive ratios. Therefore,

$$\frac{AQ}{QB} \cdot \frac{BR}{RC} \cdot \frac{CP}{PA} = -1.$$

Now, assume that P, Q, and R lie on $\overleftrightarrow{AC}$, $\overleftrightarrow{AB}$, and $\overleftrightarrow{BC}$, respectively, and that

$$\frac{AQ}{QB} \cdot \frac{BR}{RC} \cdot \frac{CP}{PA} = -1.$$

Let the line containing P and Q intersect BC at R'. Then we have just shown that

$$\frac{AQ}{QB} \cdot \frac{BR'}{R'C} \cdot \frac{CP}{PA} = -1.$$

Since we are given that

$$\frac{AQ}{QB} \cdot \frac{BR}{RC} \cdot \frac{CP}{PA} = -1$$

we must have that $\dfrac{BR}{RC} = \dfrac{BR'}{R'C}$, which indicates that R and R' coincide, proving collinearity. ∎