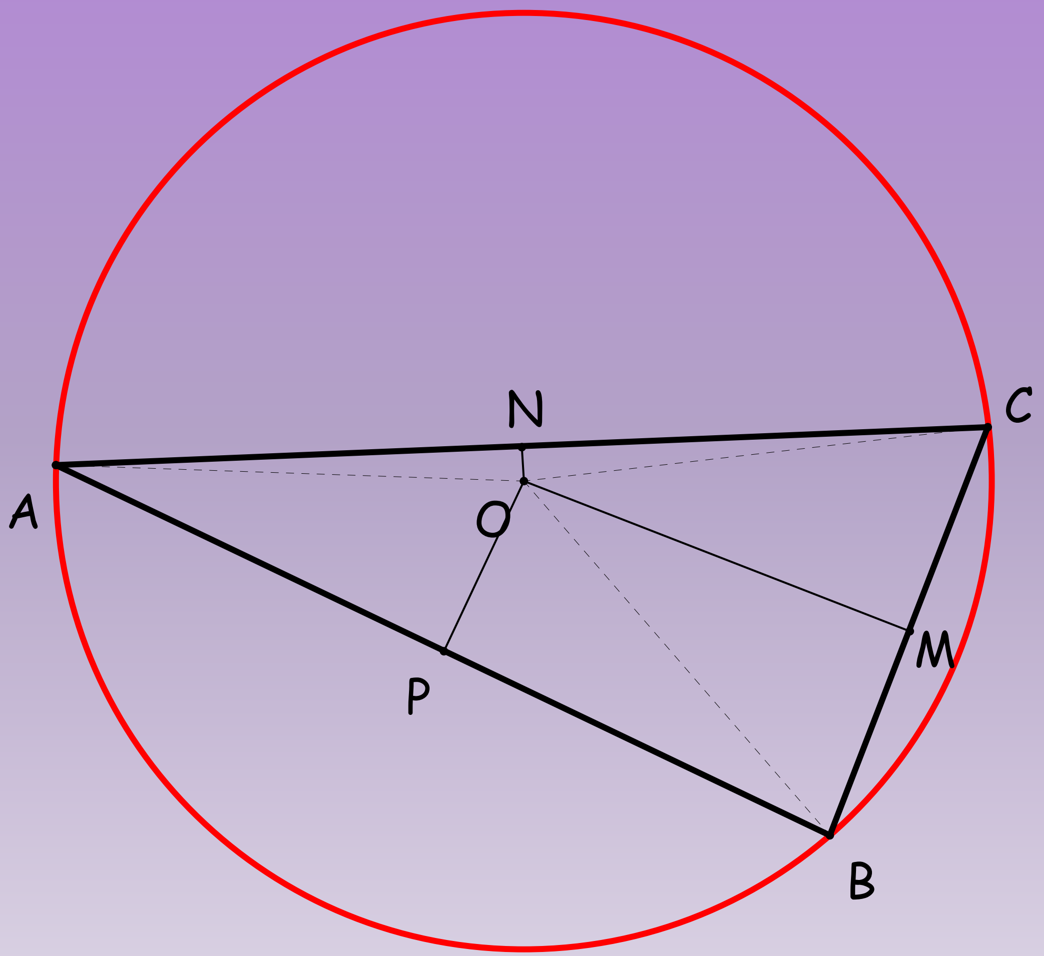
MATH 6118

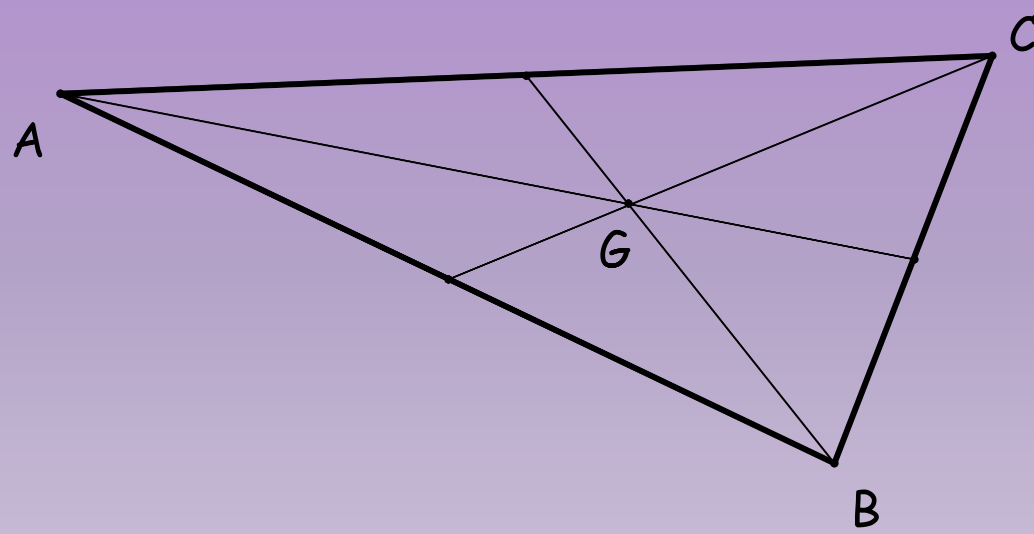
Collinearity

There are three kinds of mathematicians – those who can count and those who can't.

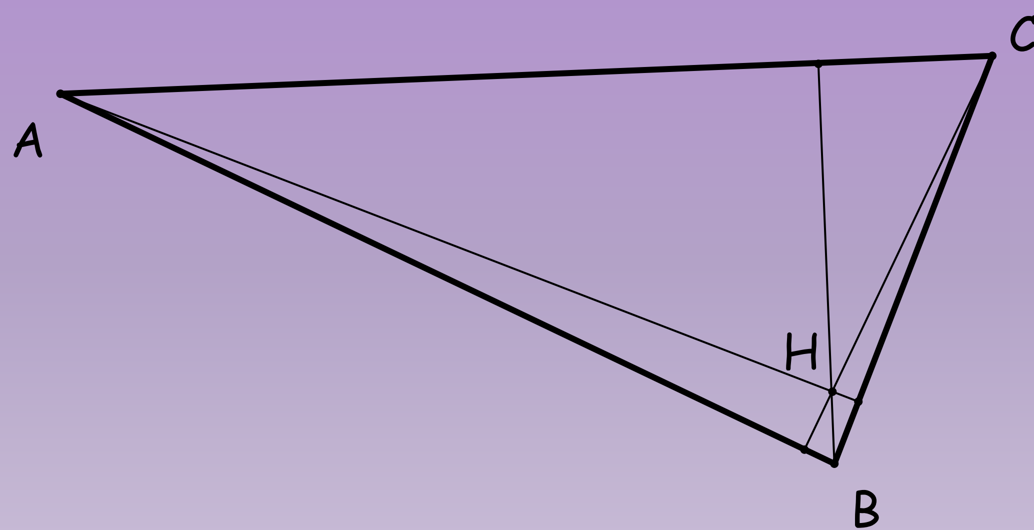
Circumcenter



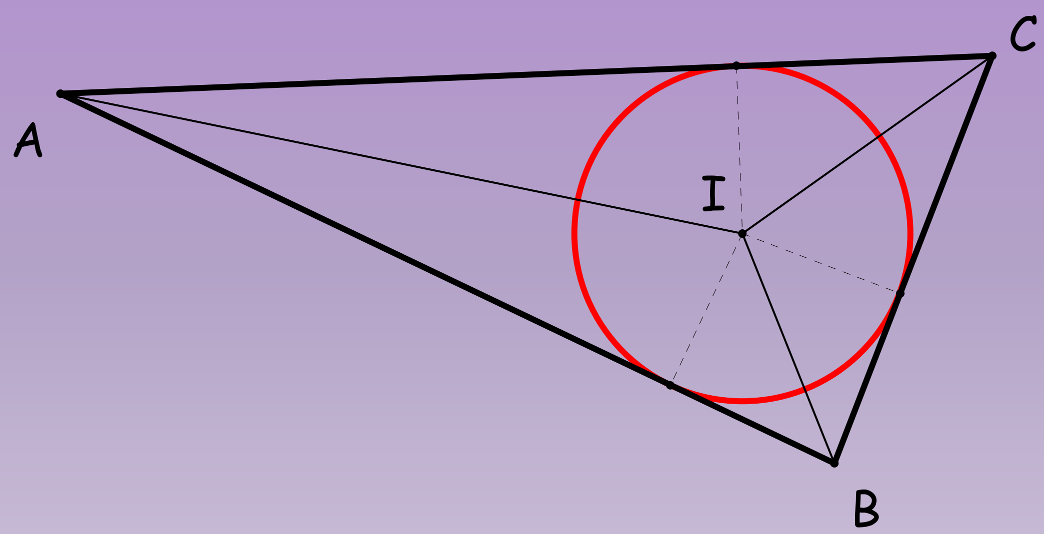
Centroid



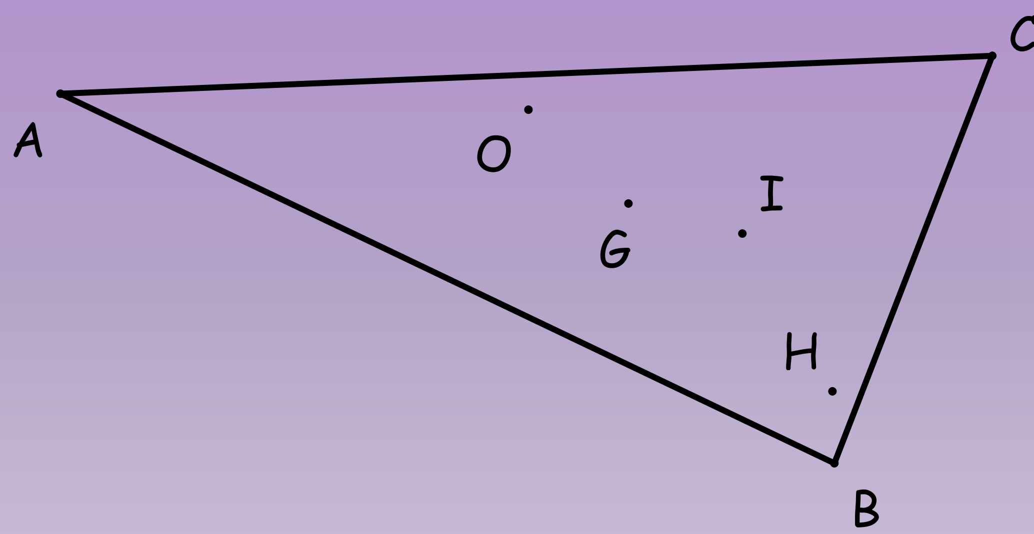
Orthocenter



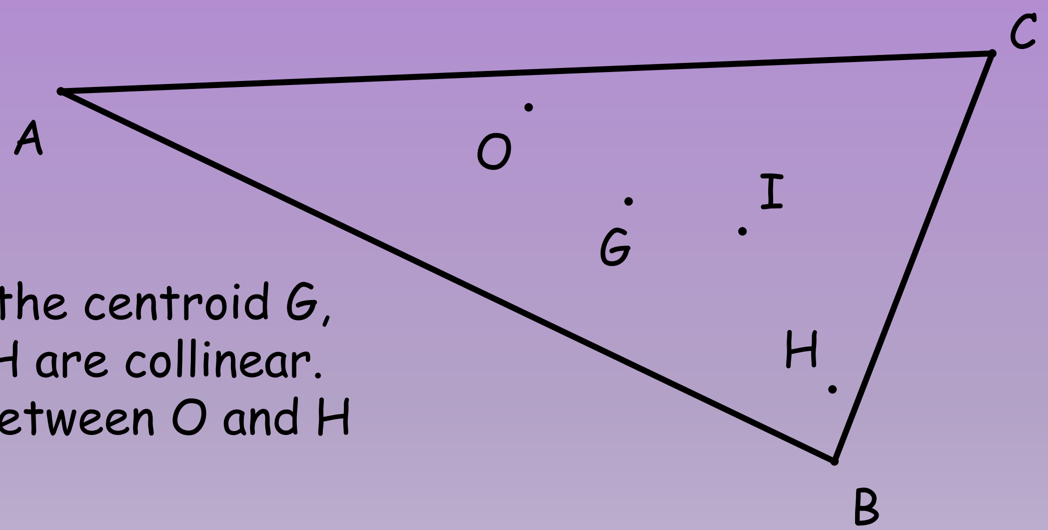
Incenter



The 4 Centers so far



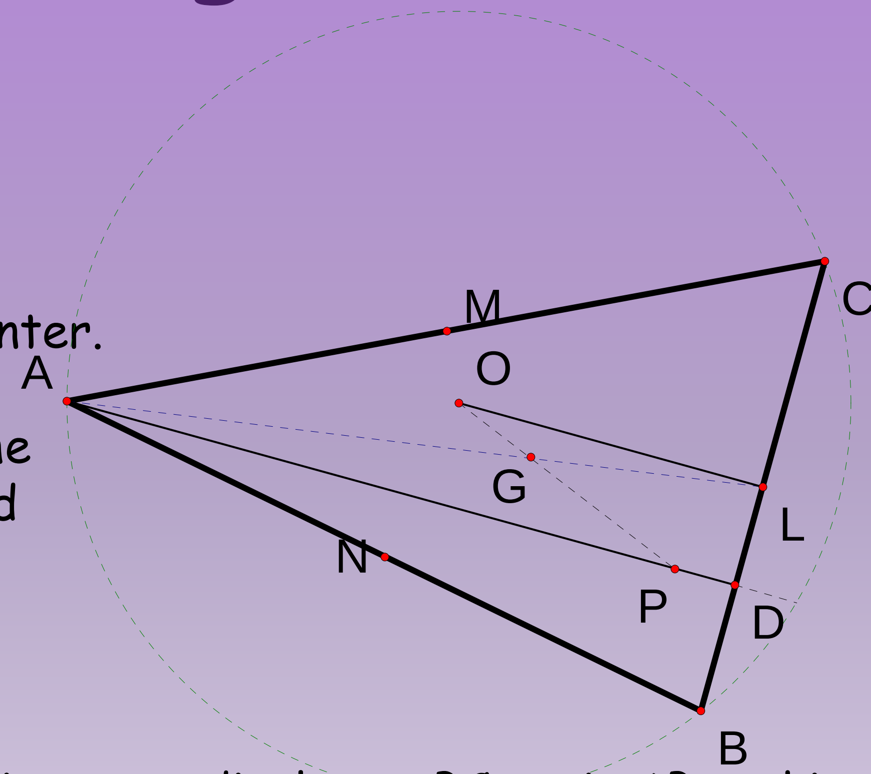
The Euler Segment



The circumcenter O, the centroid G, and the orthocenter H are collinear. Furthermore, G lies between O and H and

$$\frac{GH}{OG} = 2$$

The Euler Segment

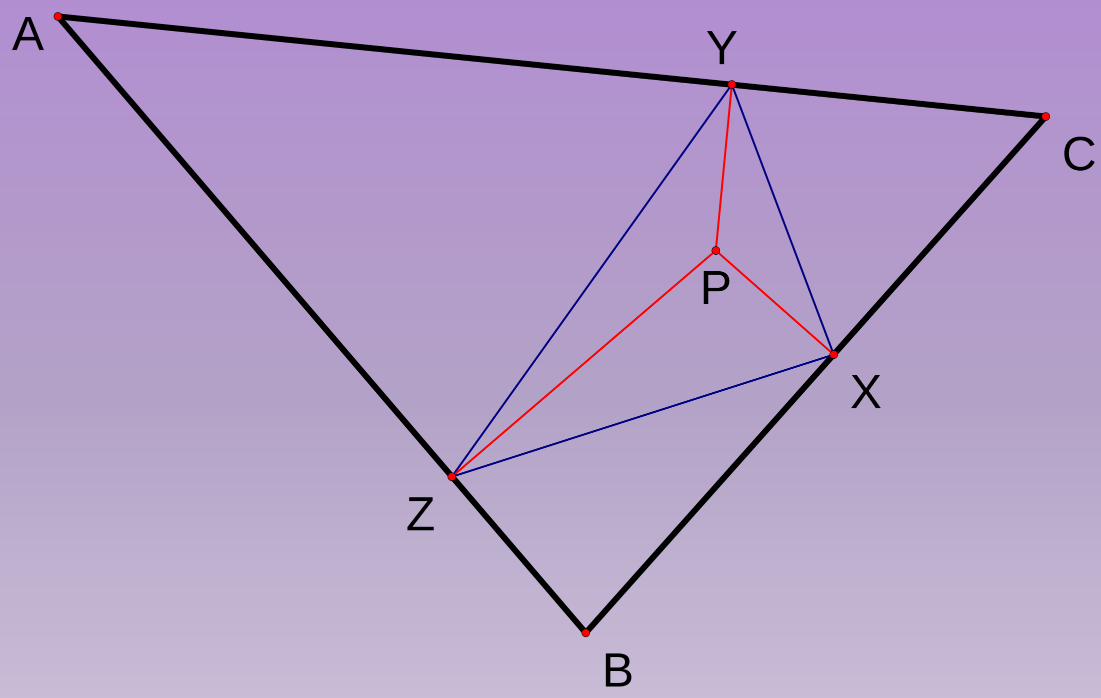


Proof 1: (Symmetric Triangles)
Extend OG twice its length to a point P, that is GP = 2OG. We need to show that P is the orthocenter.

Draw the median, AL, where L is the midpoint of BC. Then, GP = 2OG and AG = 2GL and by vertical angles we have that $\angle AGH \cong \angle LGO$

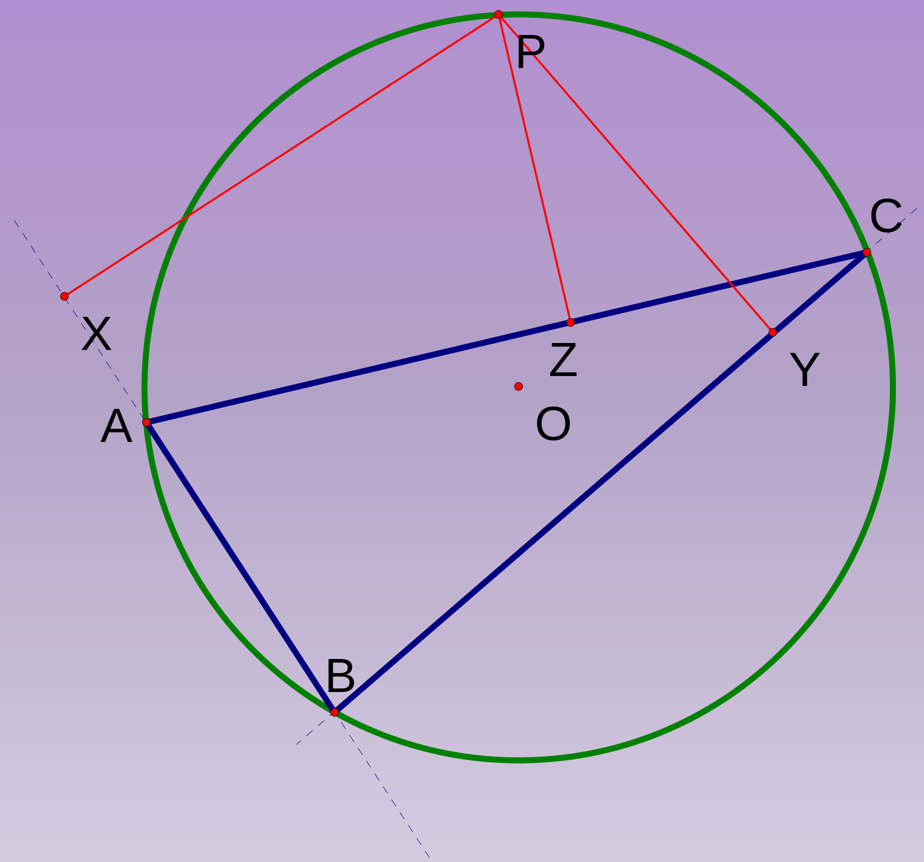
Then $\triangle AHG \sim \triangle LOG$
and OL is parallel to AP. Since OL is perpendicular to BC, so it AP, making P lie on the altitude from A. Repeating this for each of the other vertices gives us our result. By construction GP = 2OG.

The Pedal Triangle



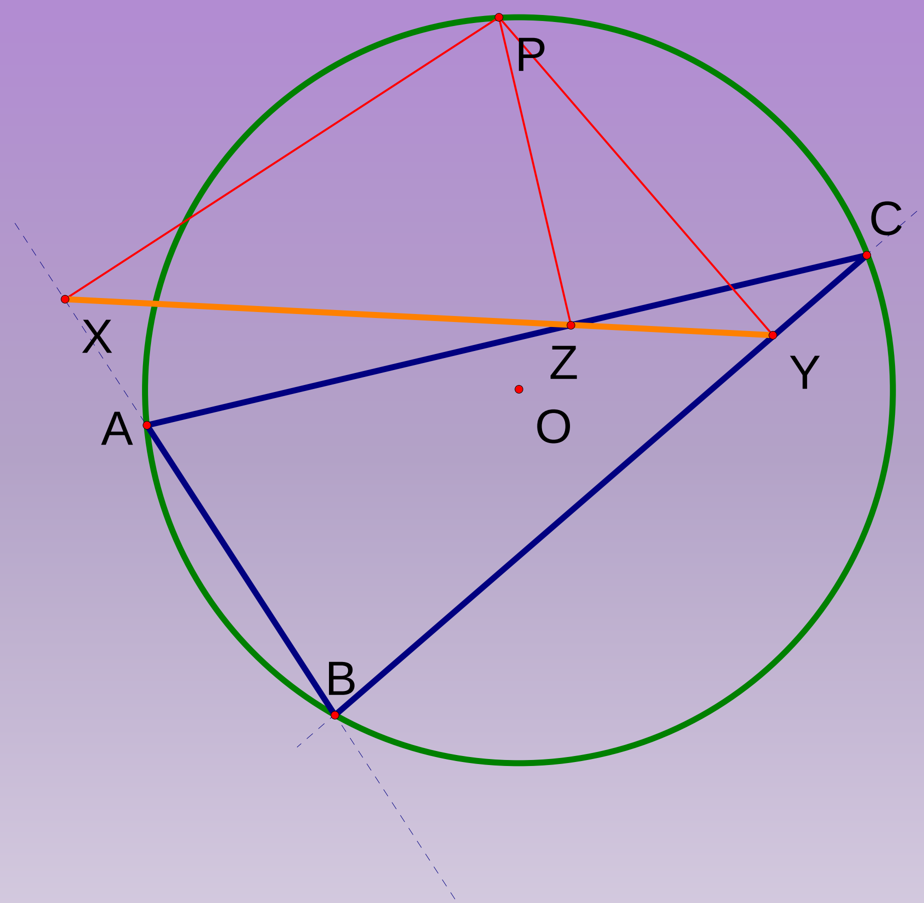
Let P be any point not on the triangle and drop perpendiculars
P to the (extended) sides. The three points form the vertices
The pedal triangle associated with P.

The Pedal Triangle from the Circumcircle



Let P be on the circumcircle. What does its pedal triangle look like?

The Simson Line



X, Y, and Z seem collinear? Are they, and are they always?

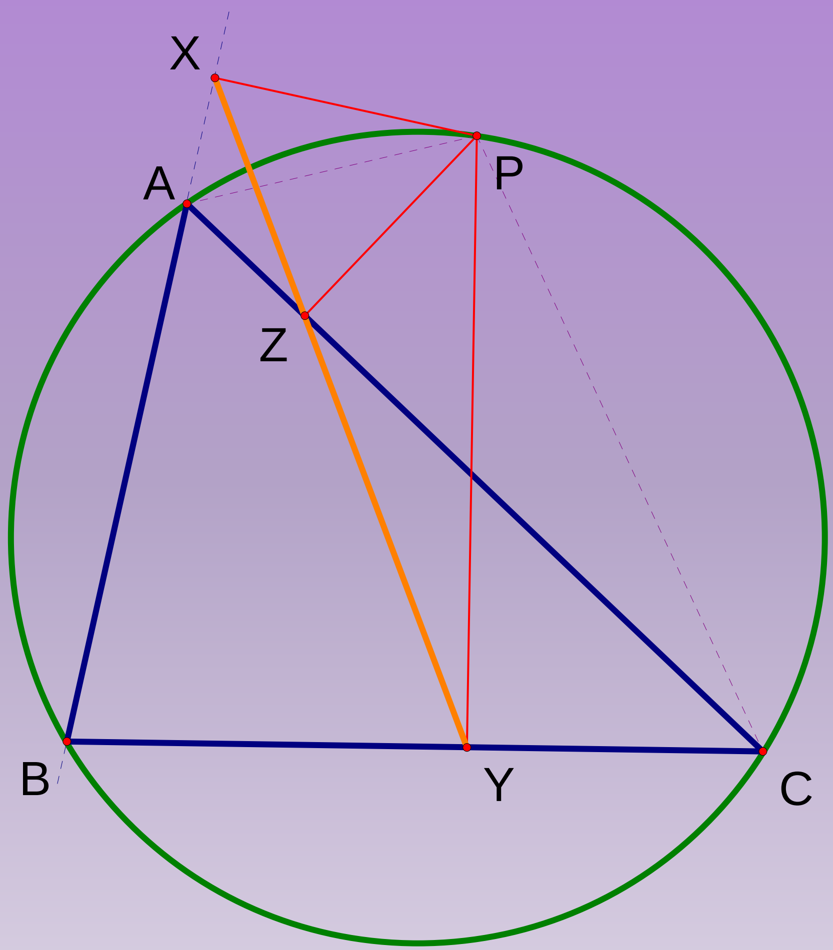
The Simson Line

Theorem: The feet of the perpendiculars from a point to the sides of a triangle are collinear if and only if the point lies on the circumcircle.

The Simson Line

Proof: First, assume that P is on the circumcircle.

WLOG we can assume that P is on arc *AC* that does not contain B and P is at least as far from *C* as it is from *A*. If necessary you can relabel the points to make this so.

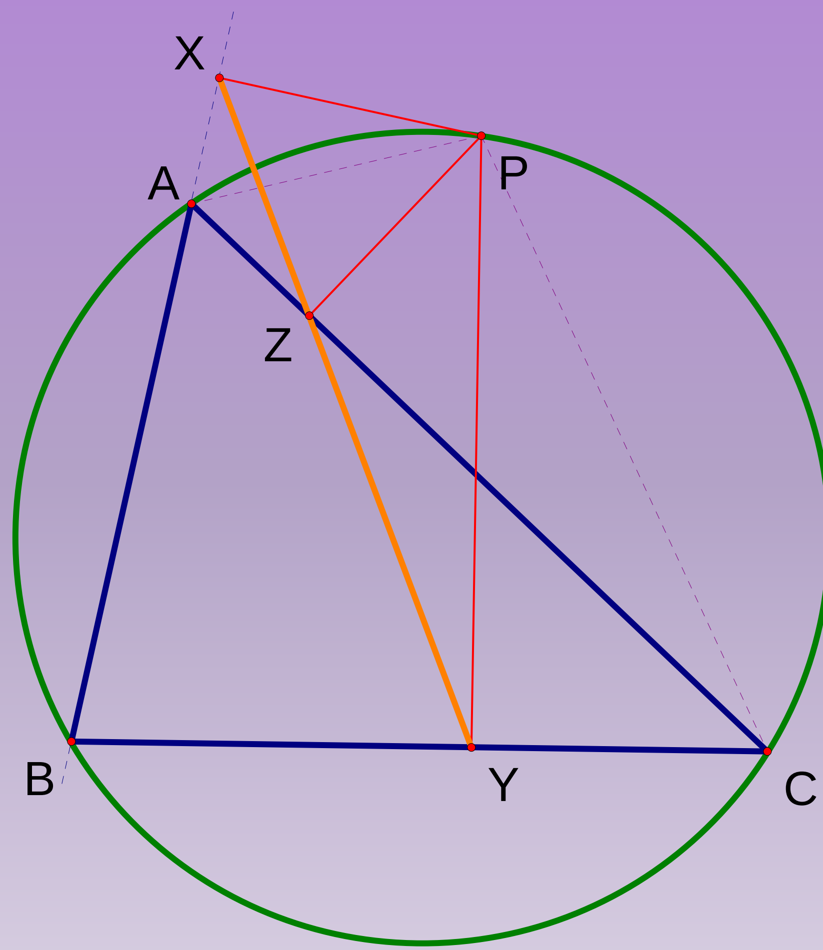


The Simson Line

P lies on the circumcircle of triangle △YBX because

∡PYB = 90 = ∡PXB. This makes □PXBY a cyclic quadrilateral by 3.2.5 2(b) since opposite angles add up to 180.

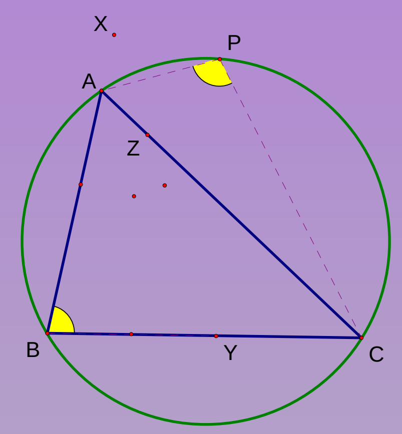
(Likewise P lies on the circumcircle of △YZC and △AZX.)

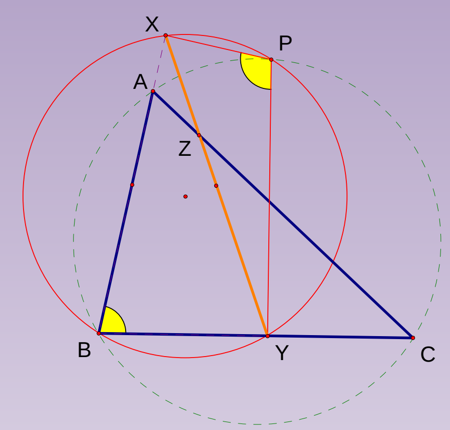


The Simson Line

$\measuredangle APC = 180 - \measuredangle B$

$= \measuredangle XPY$





The Simson Line

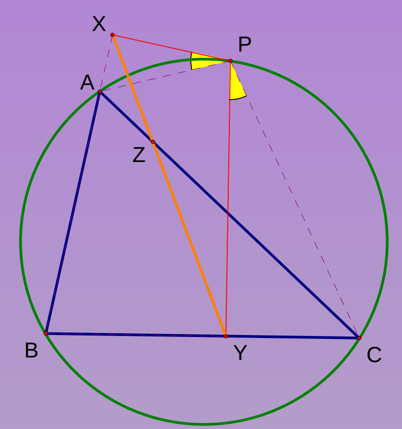
Now, subtract $\measuredangle APY$ and we get that $\measuredangle YPC = \measuredangle XPA$. Now, Y, C, P and Z are concyclic

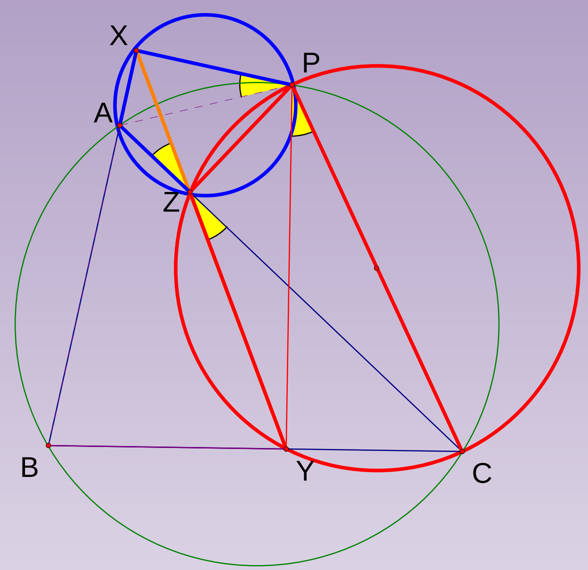
$$\measuredangle YPC = \measuredangle YZC.$$

Therefore,

$$\measuredangle YZC = \measuredangle XZA$$

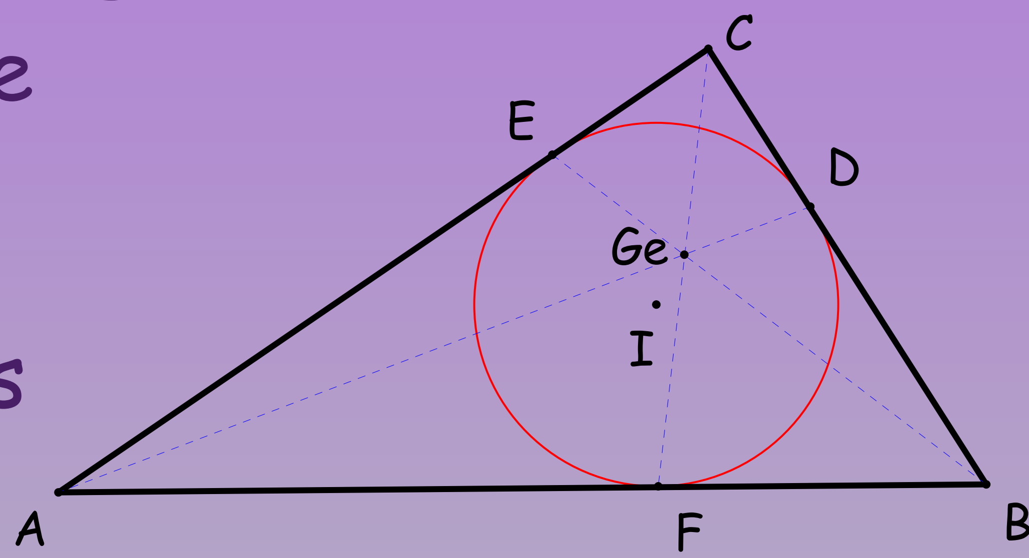
making the points collinear.





The Gergonne Point

Let D, E, F be the points where the inscribed circle touches the sides of the triangle ABC. Then the lines AD, BE and CF intersect at one point.



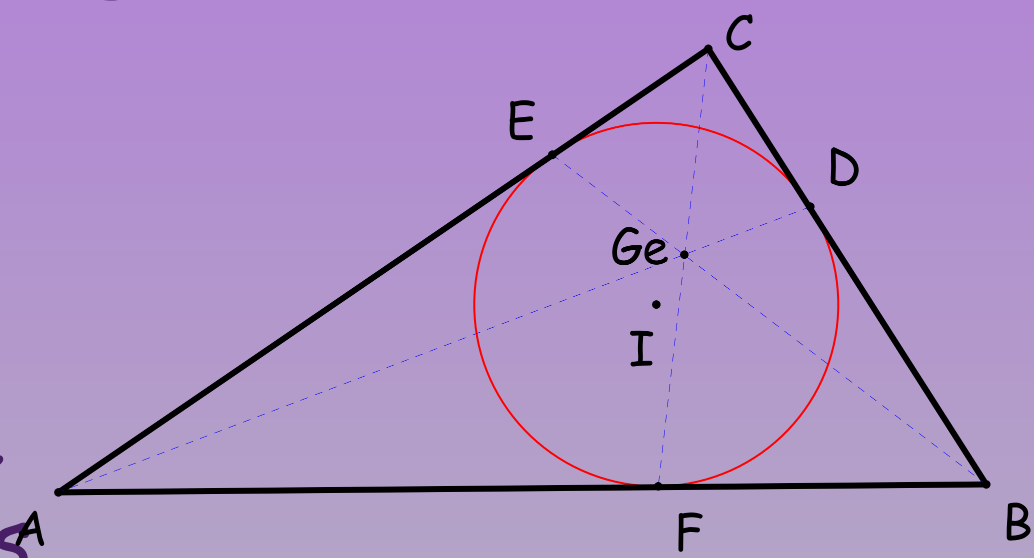
The Gergonne Point

AF = AE

BF = BD

CD = CE

because they are external tangents to a circle.

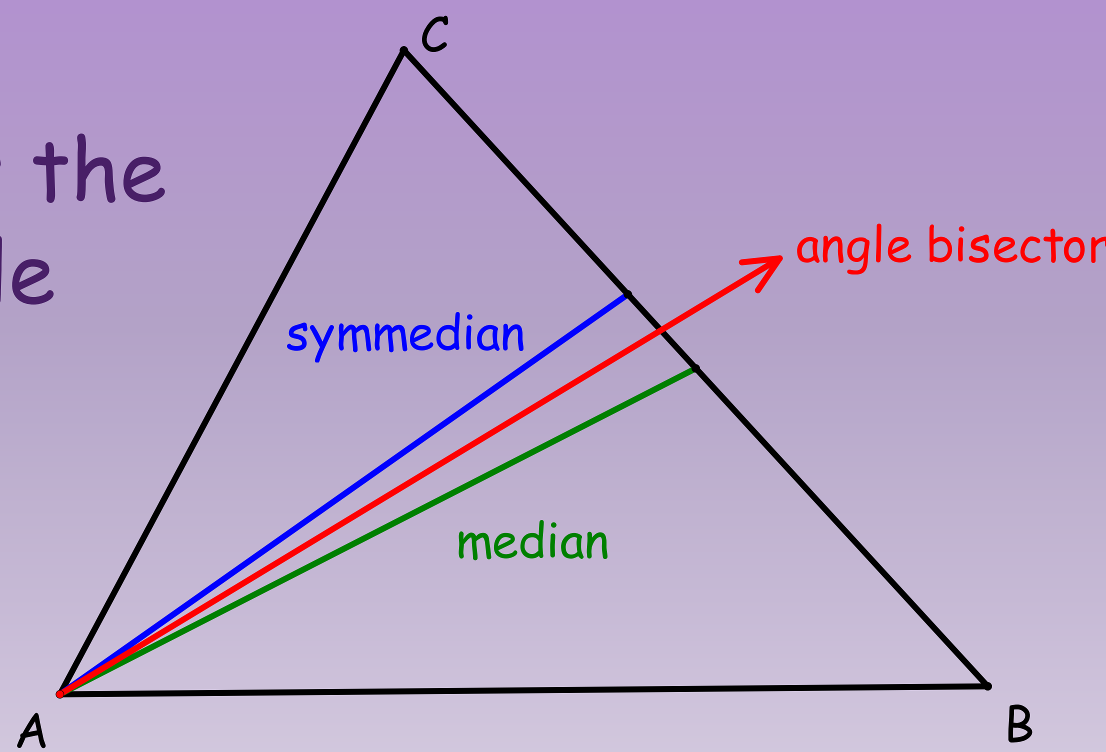


So $\frac{AF}{FB} \cdot \frac{BD}{CD} \cdot \frac{CE}{AE} = \frac{AF}{AE} \cdot \frac{BD}{BF} \cdot \frac{CE}{CD} = 1$

By Ceva's Theorem they are concurrent.

The Lemoine Point

The symmedians of a triangle are the reflections of medians across the associated angle bisectors.



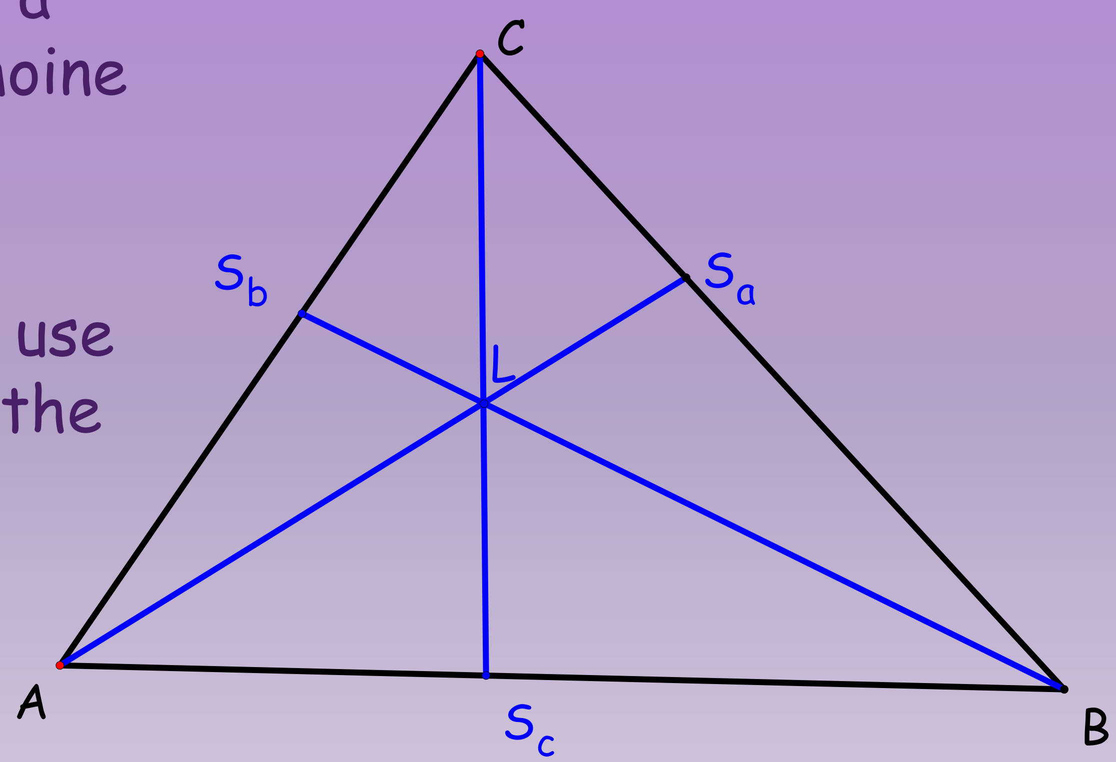
The Lemoine Point

The symmedians As_a, BS_b, and CS_c intersect in a point called the Lemoine point.

Proof: We will make use of two ways to find the area of a triangle:

$$K = \frac{1}{2} ab \sin C$$

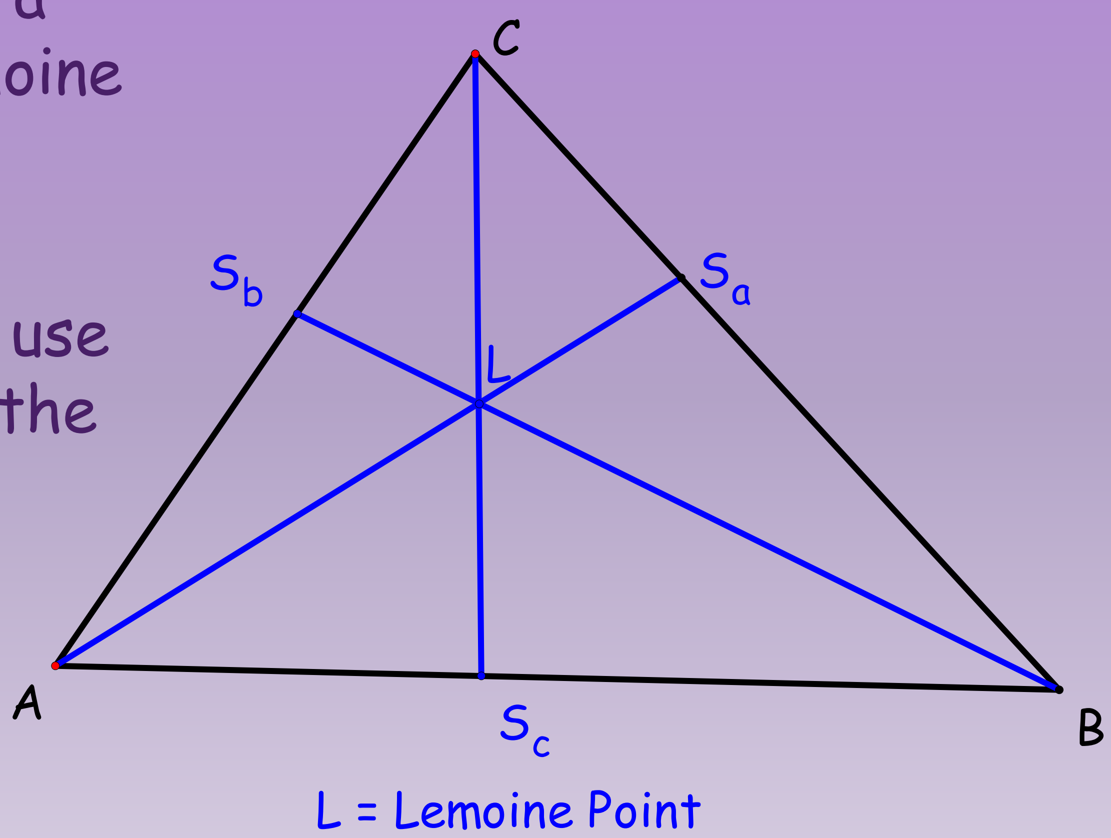
$$K = \frac{1}{2} ch_c$$



L = Lemoine Point

The Lemoine Point

The symmedians As_a, BS_b, and CS_c intersect in a point called the Lemoine point.



Proof: We will make use of two ways to find the area of a triangle:

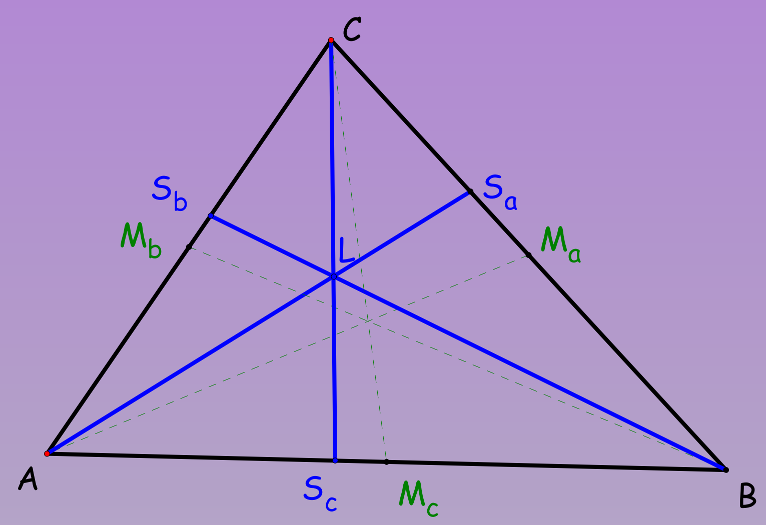
$$K = \frac{1}{2} ab \sin C$$

$$K = \frac{1}{2} ch_c$$

The Lemoine Point

$$\frac{K(\triangle BAS_a)}{K(\triangle AM_aC)} = \frac{BS_a}{CM_a} = \frac{AB \cdot AS_a}{AM_a \cdot AC}$$

$$\frac{K(\triangle AS_aC)}{K(\triangle AM_aB)} = \frac{CS_a}{BM_a} = \frac{AC \cdot AS_a}{AM_a \cdot AB}$$



Divide the second by the first

$$\frac{BS_a}{CM_a} \cdot \frac{BM_a}{CS_a} = \frac{AB^2}{AC^2}$$

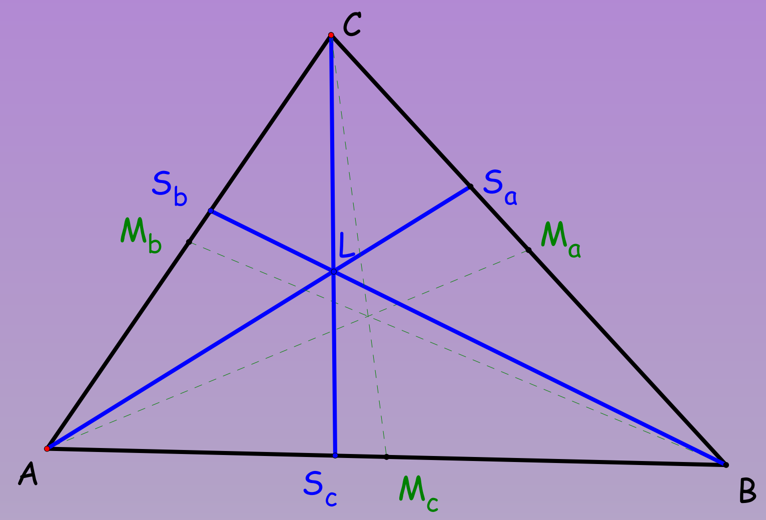
Or, since $Bm_a = CM_a$

$$\frac{BS_a}{CS_a} = \frac{AB^2}{AC^2}$$

The Lemoine Point

Similarly,

$$\frac{CS_b}{AS_b} = \frac{BC^2}{AB^2} \text{ and } \frac{AS_c}{BS_c} = \frac{AC^2}{BC^2}$$



Multiply these together and

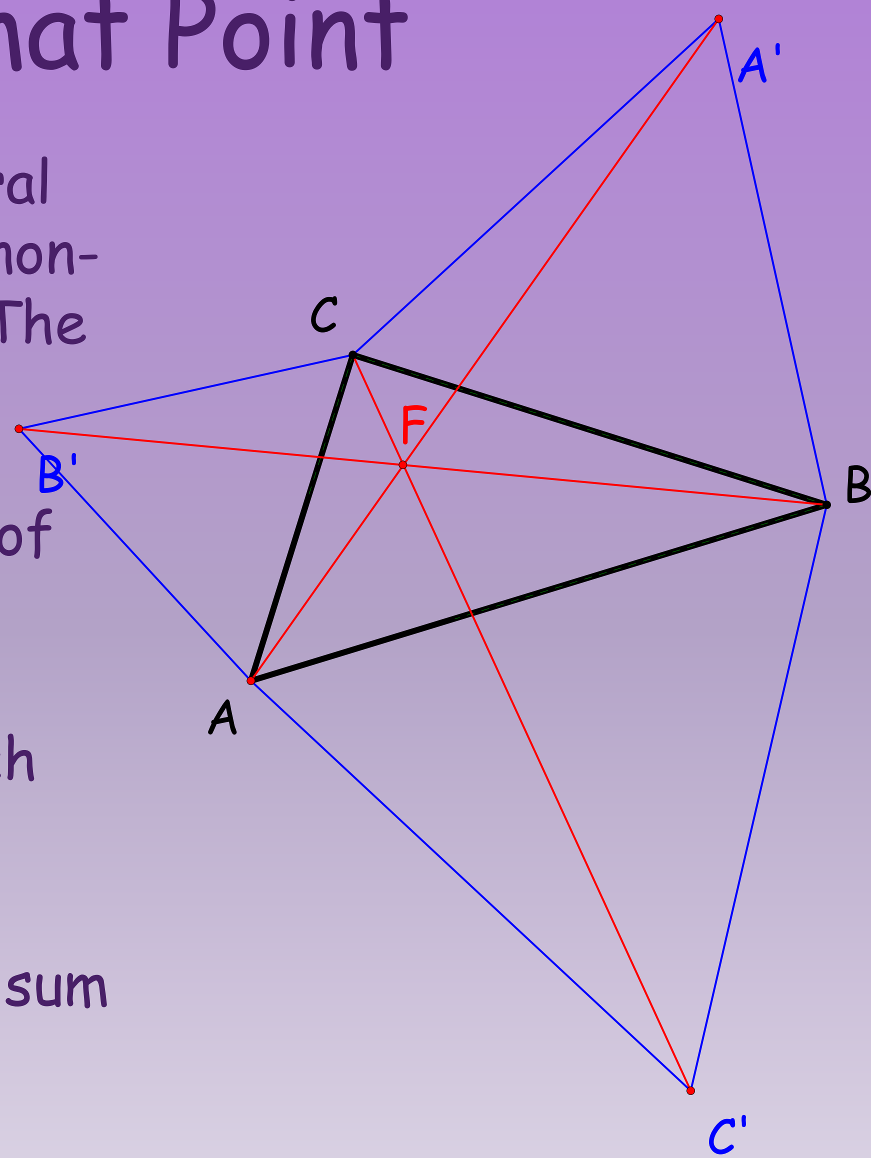
Ceva's Theorem gives us that they are concurrent

$$\frac{BS_a}{CS_a}\frac{AS_c}{BS_c}\frac{CS_b}{AS_b} = \frac{AB^2}{AC^2}\frac{AC^2}{BC^2}\frac{BC^2}{AB^2} = 1$$

The Fermat Point

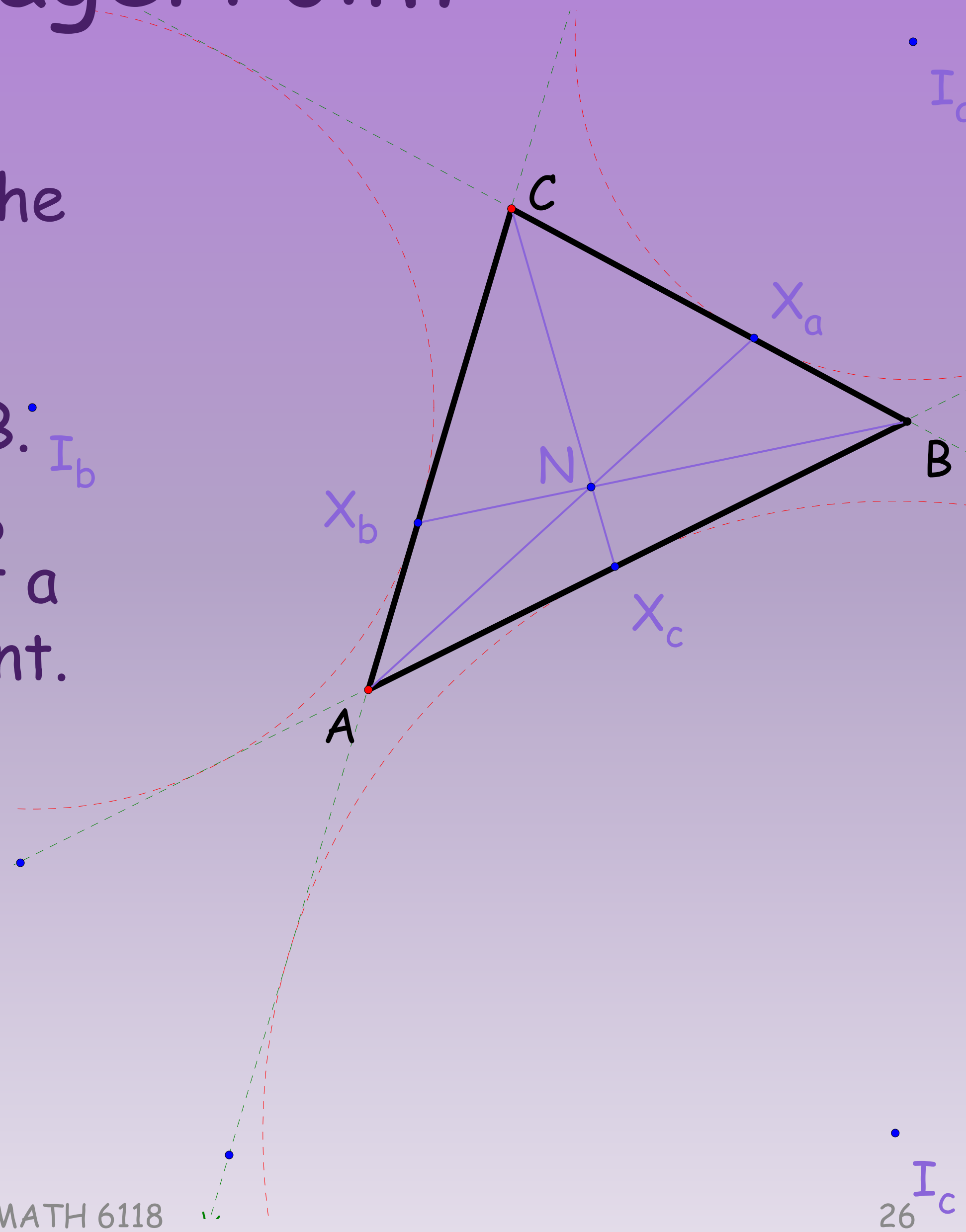
Given △ABC construct equilateral triangles on each side. Call the non-triangle vertices A′, B′, and C′. The lines AA′, BB′, and CC′ are concurrent. This point is the Fermat point and has a number of nice properties.

1. The 3 angles between F and each of the vertices are each 120, so it is the equiangular point of the triangle.
2. The Fermat point minimizes sum of the distances to the vertices.



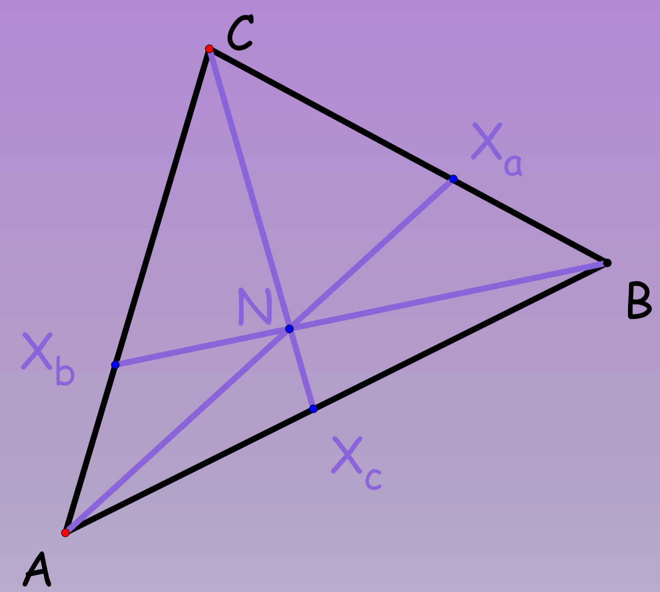
The Nagel Point

Let X_a be the point of tangency of side BC and the excircle with center I_a. Similarly define points X_b and X_c on sides AC and AB. Then three lines AX_a, BX_b and CX_c are concurrent at a point called the Nagel point.



The Nagel Point

X_a has the unique property of being the point on the perimeter that is exactly half way around the triangle from *A*.



$$AB + BX_a = AC + CX_a$$

If p denotes the semiperimeter, then

$$BX_a = p - AB = p - c \text{ and } CX_a = p - AC = p - b$$

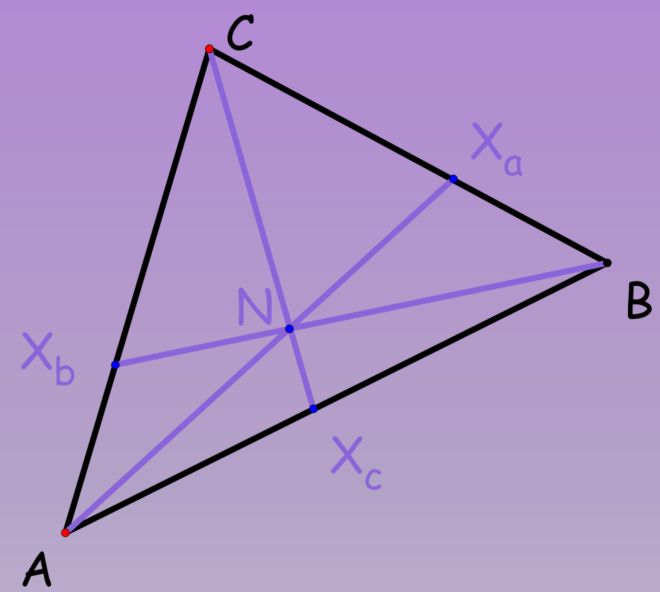
$$\frac{BX_a}{CX_a} = \frac{p - c}{p - b}$$

The Nagel Point

Doing this for the other two points gives:

$$\frac{CX_b}{AX_b} = \frac{p-a}{p-c}$$

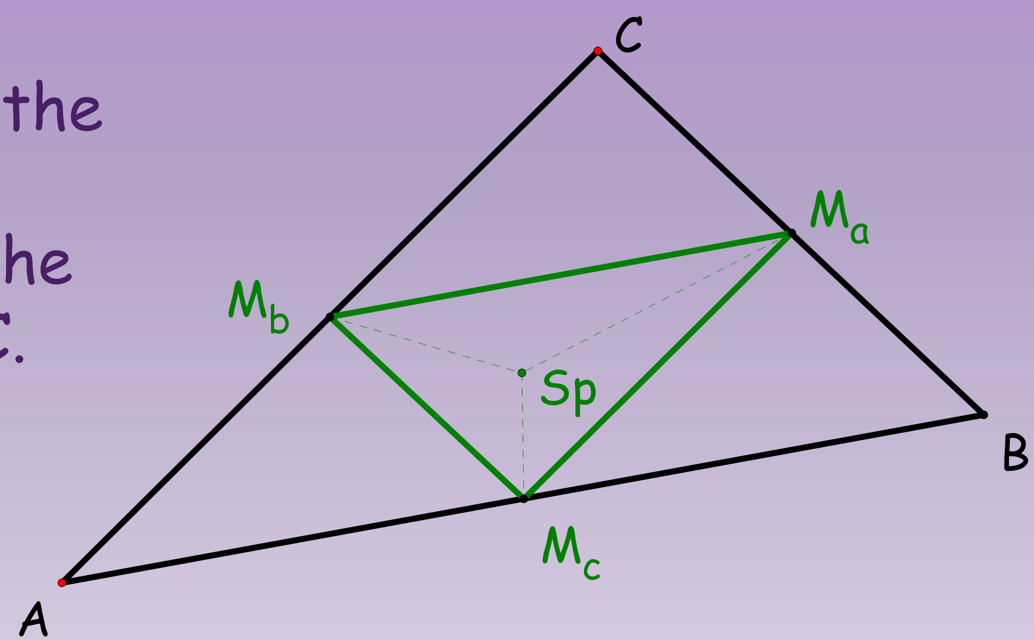
$$\frac{AX_c}{BX_c} = \frac{p-b}{p-a}$$



Applying Ceva's Theorem gives us the result.

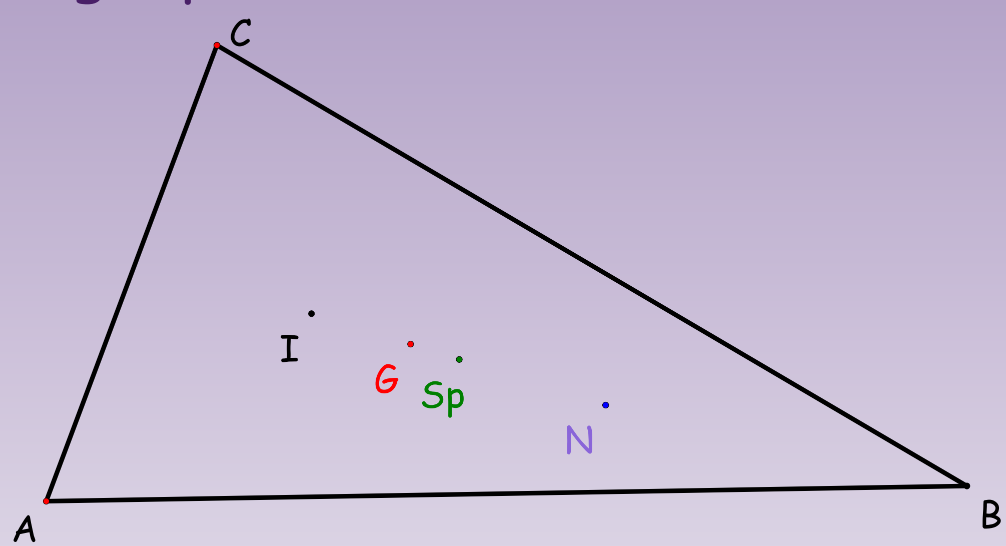
The Spieker Point

Let M_a, M_b, M_c denote the midpoints of sides BC, AC, and AB, respectively. The triangle $\triangle M_aM_bM_c$ is called the medial triangle to $\triangle ABC$. Let Sp denote the incenter of the medial triangle. Sp is called the Spieker point of $\triangle ABC$.



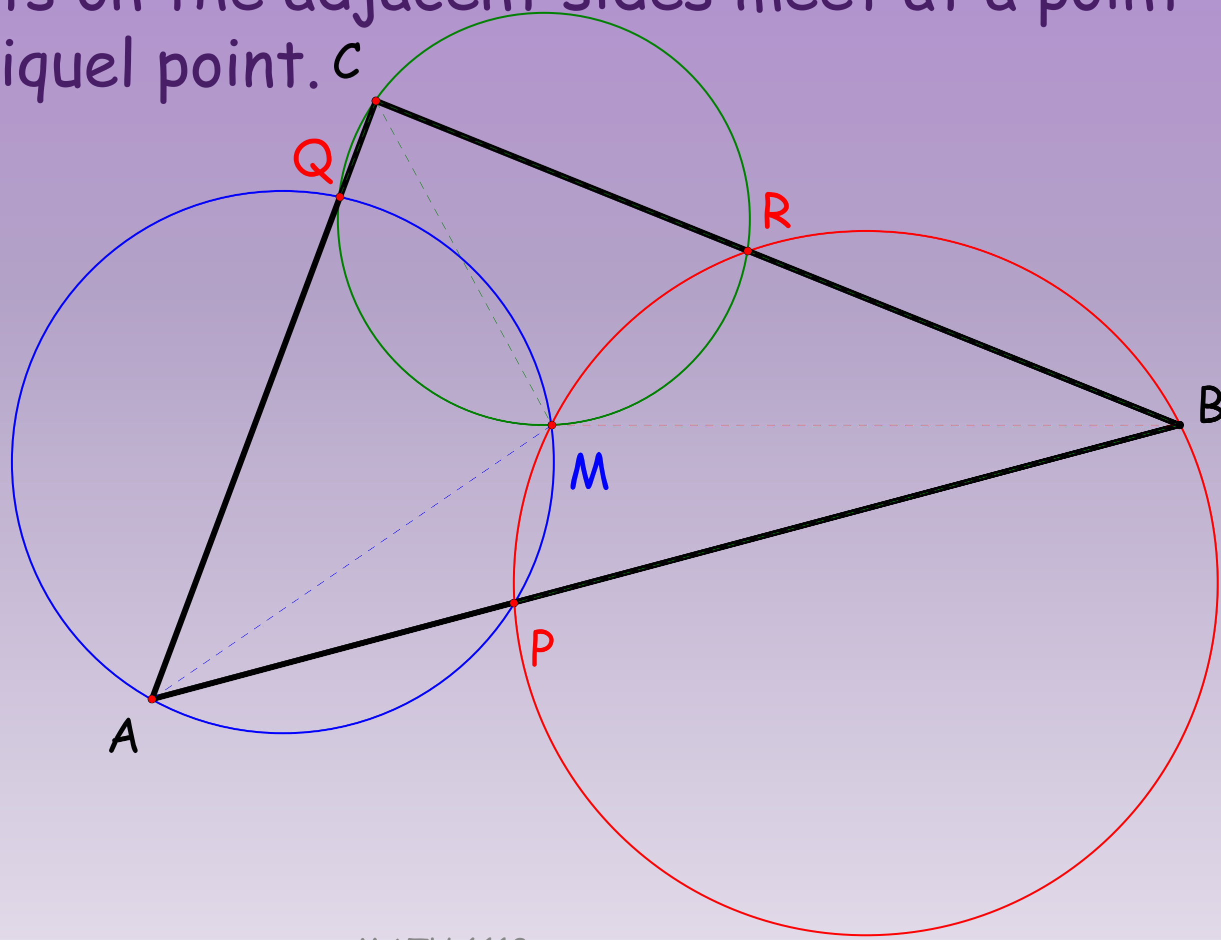
The Nagel Segment

1. The incenter (I), the Nagel point (N), the centroid (G) and the Spieker point (Sp) are collinear.
2. The Spieker point is the midpoint of the Nagel segment.
3. The centroid is one-third of the way from the incenter to the Nagel point.



Miquel's Theorem

If P, Q, and R are on BC, AC, and AB respectively, then the three circles determined by a vertex and the two points on the adjacent sides meet at a point called the Miquel point.



Miquel's Theorem

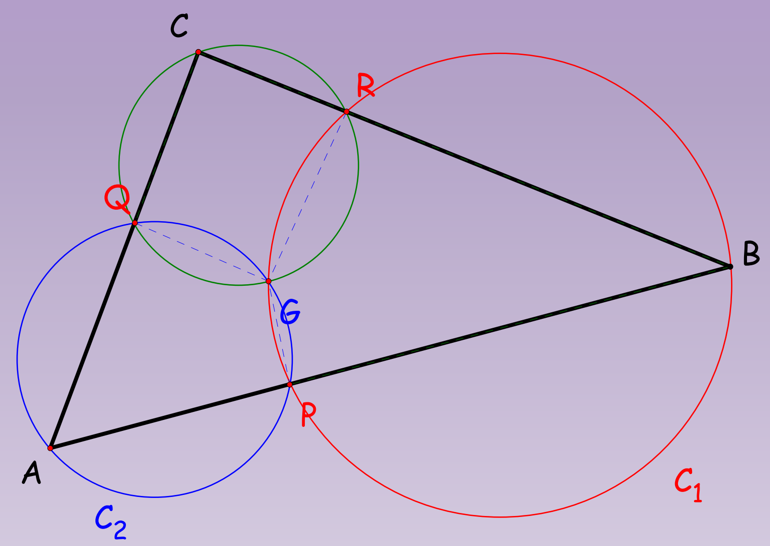
Let $\triangle ABC$ be our triangle and let P,Q, and R be the points on the sides of the triangle. Construct the circles of the theorem. Consider two of the circles, C_1 and C_2, that pass through P. They intersect at P, so they must intersect at a second point, call it G.

In circle C_2

$\measuredangle QGP + \measuredangle QAP = 180$

In circle C_1

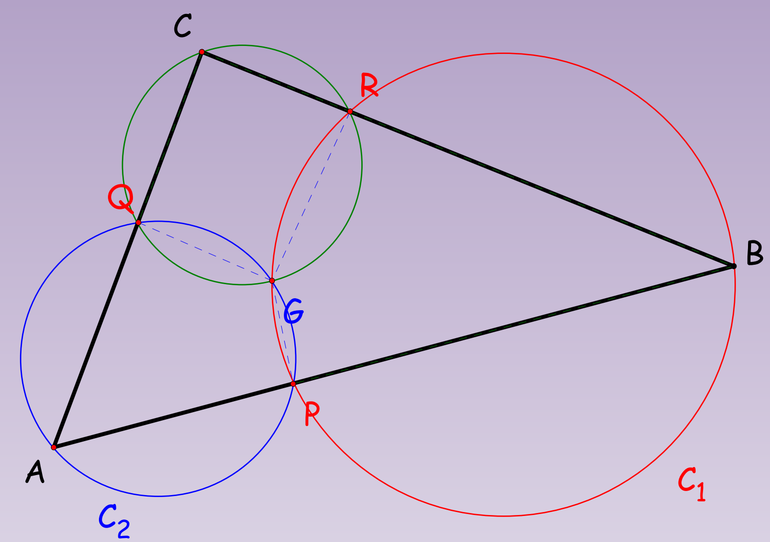
$\measuredangle RGP + \measuredangle RBP = 180$



Miquel's Theorem

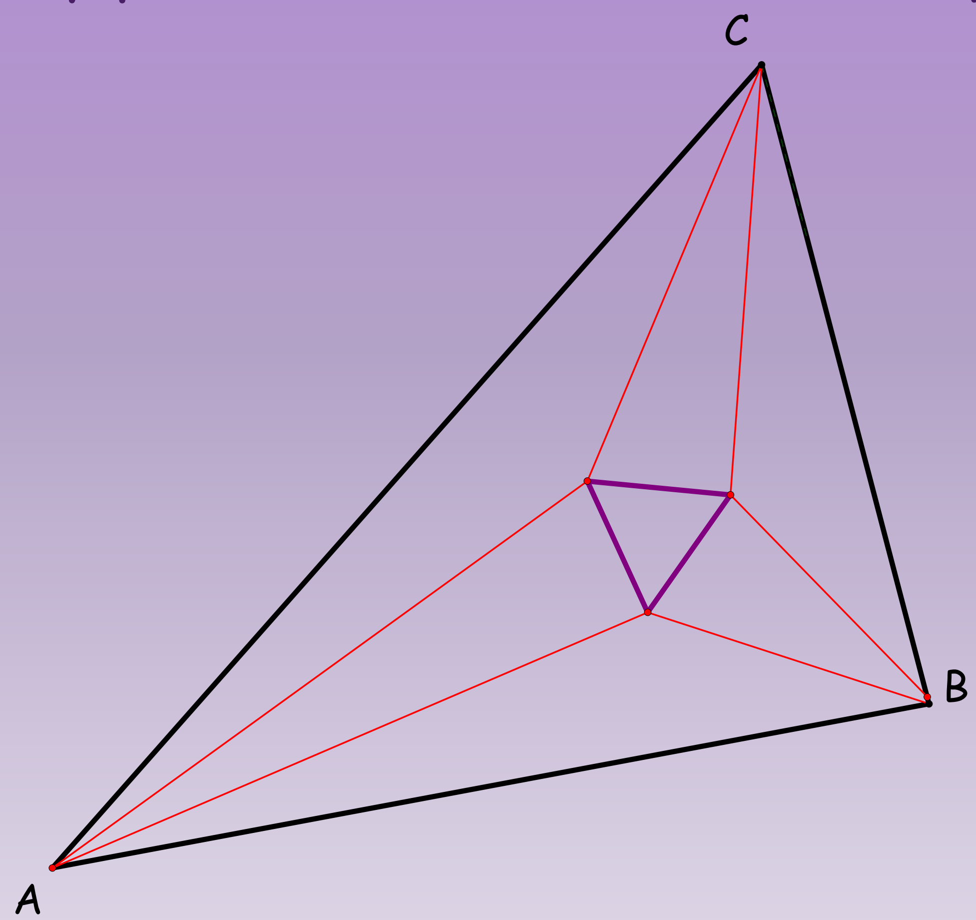
$$\measuredangle QGP + \measuredangle QGR + \measuredangle RGP = 360$$
$$(180 - \measuredangle A) + \measuredangle QGR + (180 - \measuredangle B) = 360$$
$$\measuredangle QGR = \measuredangle A + \measuredangle B$$
$$= 180 - \measuredangle C$$

Thus, ∡QGR and ∡C are supplementary and so Q, G, R, and C are concyclic. These circle then intersect in one point.



Morley's Theorem

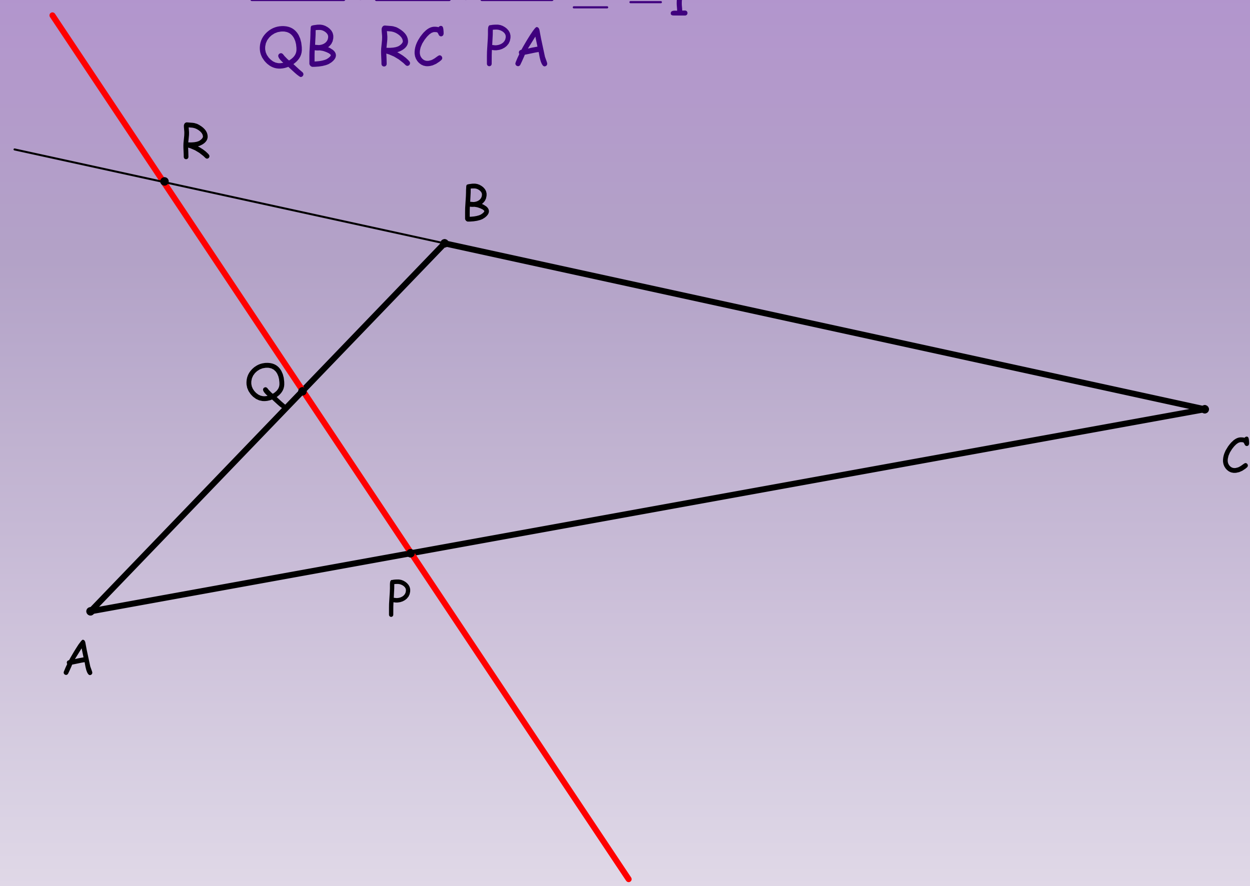
The adjacent trisectors of the angles of a triangle are concurrent by pairs at the vertices of an equilateral triangle.



Menelaus's Theorem

The three points P, Q, and R one the sides AC, AB, and BC, respectively, of $\triangle ABC$ are collinear if and only if

$$\frac{AQ}{QB}\cdot\frac{BR}{RC}\cdot\frac{CP}{PA}=-1$$



Menelaus's Theorem

Assume P, Q, and R are collinear.

From the vertices drop perpendiculars to the line.

$\triangle CH_cR \sim \triangle BH_bR$, $\triangle CH_cP \sim \triangle AH_aP$, $\triangle AH_aQ \sim \triangle BH_bQ$.

Therefore

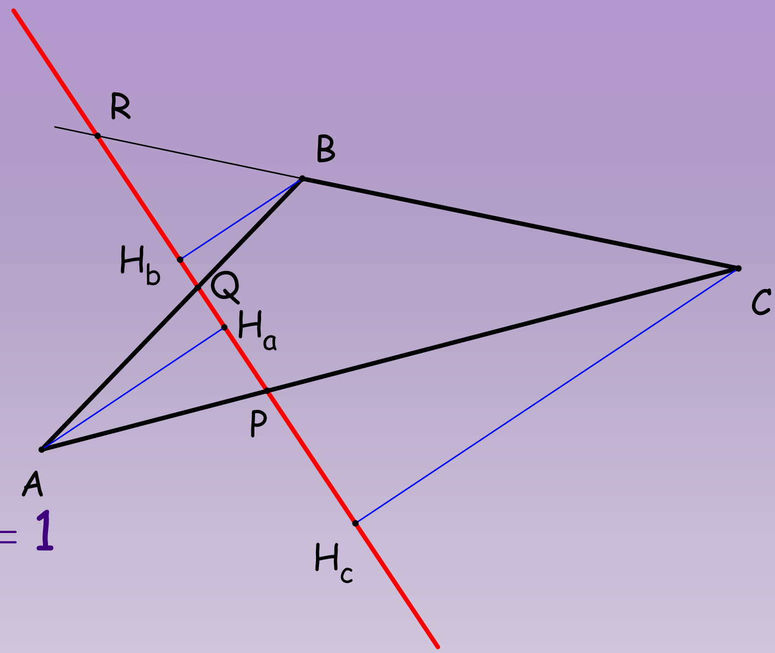
$BR/CR = BH_b/CH_c$,

$CP/AP = CH_c/AH_a$,

$AQ/BQ = AH_a/BH_b$.

Therefore,

$$\frac{AQ}{QB}\cdot\frac{BR}{RC}\cdot\frac{CP}{PA} = \frac{AH_a}{BH_b}\cdot\frac{BH_b}{CH_c}\cdot\frac{CH_c}{AH_a} = 1$$



BR/RC is a negative ratio if we take direction into account. This gives us our negative.

Menelaus's Theorem

For the reverse implication, assume that we have three points such that $AQ/QB \cdot BR/RC \cdot CP/PA = 1$. Assume that the points are not collinear. Pick up any two. Say P and Q. Draw the line PQ and find its intersection R' with BC. Then

$AQ/QB \cdot BR'/R'C \cdot CP/PA = 1$.

Therefore $BR'/R'C = BR/RC$, from which $R' = R$.