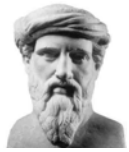
MATH 6118



Rules of the Game

Everyone is a fool for at least five minutes a day; WISDOM consists of not exceeding the limit

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The Power of Deduction

Theorem: There exist two irrational numbers a and b so that a^b is rational.

We know that √2 is irrational. Consider the number

$$a = \sqrt{2}^{\sqrt{2}}$$

A is either rational or irrational. If it is rational, we are done. So assume that it is irrational. Let b = √2. Then

$$a^b = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^{2} = 2$$

which is clearly rational.

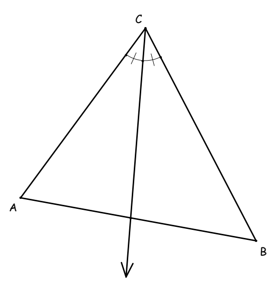
All power corrupts, but
we still need electricity.

Absolute Power Corrupts Absolutely

Theorem: Every triangle is isosceles.

Given ΔABC with $AC \neq BC$.

1. Construct the bisector of $\angle C$. Call it l. l is not perpendicular to AB.

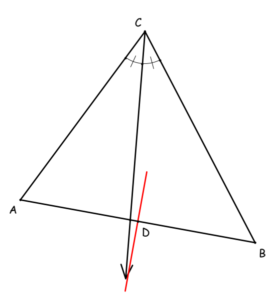


Every Triangle is Isosceles

Theorem: Every triangle is isosceles.

Given ΔABC with $AC \neq BC$.

1. Construct the bisector of $\angle C$. Call it l. l is not perpendicular to AB.
2. Construct the perpendicular bisector of AB. Call it m, and let $D = AB \cap m$

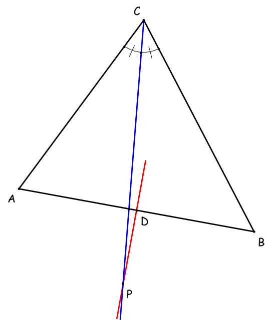


Every Triangle is Isosceles

Theorem: Every triangle is isosceles.

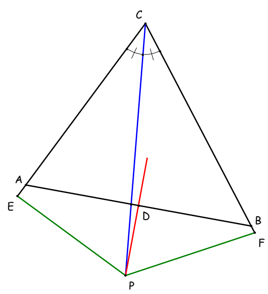
Given ΔABC with AC ≠ BC.

1. Construct the bisector of ∠C. Call it l. l is not perpendicular to AB.
2. Construct the perpendicular bisector of AB. Call it m, and let D = AB ∩ m.
3. l ∩ m ≠ ∅. Call it P.



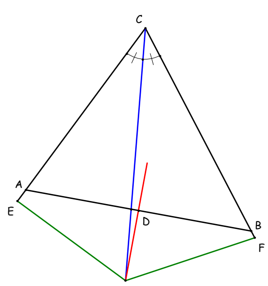
Every Triangle is Isosceles

1. Construct the bisector of ∠C. Call it l. l is not perpendicular to AB.
2. Construct the perpendicular bisector of AB. Call it m, and let D = AB ∩ m.
3. l ∩ m ≠ ∅. Call it P.
4. Drop a perpendicular from P to BC and from P to AC.



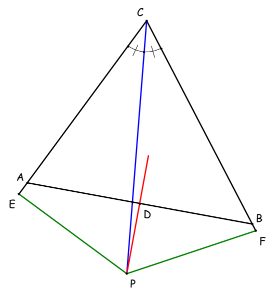
Every Triangle is Isosceles

1. Construct the bisector of ∠C. Call it l. l is not perpendicular to AB.
2. Construct the perpendicular bisector of AB. Call it m, and let D = AB ∩ m.
3. l ∩ m ≠ ∅. Call it P.
4. Drop a perpendicular from P to BC and from P to AC.
5. ∠ACP ≅ ∠ BCP.



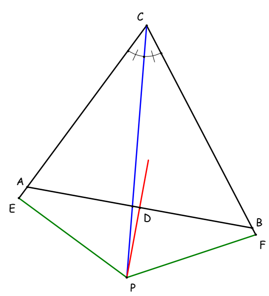
Every Triangle is Isosceles

2. Construct the perpendicular bisector of AB. Call it m, and let D = AB ∩ m.
3. l ∩ m ≠ ∅. Call it P.
4. Drop a perpendicular from P to BC and from P to AC.
5. ∠ACP ≅ ∠ BCP.
6. ∠CEP ≅ ∠ CFP, as they are both right angles..



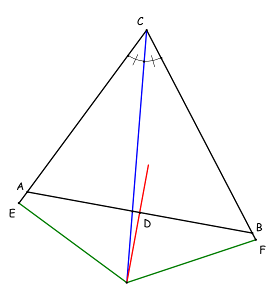
Every Triangle is Isosceles

2. Construct the perpendicular bisector of AB. Call it m, and let D = AB ∩ m.
3. l ∩ m ≠ ∅. Call it P.
4. Drop a perpendicular from P to BC and from P to AC.
5. ∠ACP ≅ ∠ BCP.
6. ∠CEP ≅ ∠ CFP.
7. ΔCEP ≅ ΔCFP by AAS.



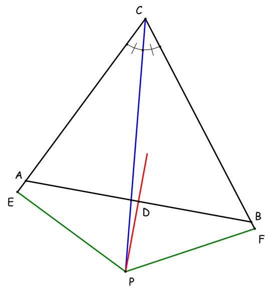
Every Triangle is Isosceles

3. l ∩ m ≠ ∅. Call it P.
4. Drop a perpendicular from P to BC and from P to AC.
5. ∠ACP ≅ ∠ BCP.
6. ∠CEP ≅ ∠ CFP.
7. ΔCEP ≅ ΔCFP.
8. EP ≅ FP and CE ≅ CF by CPCTC.



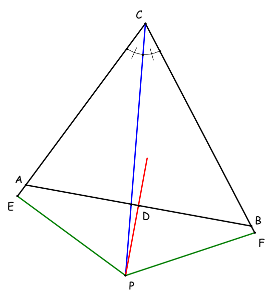
Every Triangle is Isosceles

3. $l \cap m \neq \varnothing$. Call it P.
4. Drop a perpendicular from P to BC and from P to AC.
5. $\angle ACP \cong \angle BCP$.
6. $\angle CEP \cong \angle CFP$.
7. $\Delta CEP \cong \Delta CFP$.
8. $EP \cong FP$ and $CE \cong CF$.
9. $\angle AEP \cong \angle BFP$ as they are both right angles.



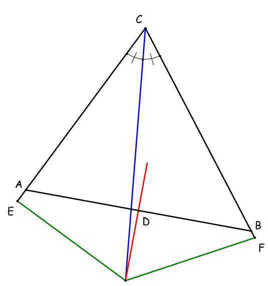
Every Triangle is Isosceles

3. $l \cap m \neq \varnothing$. Call it P.
4. Drop a perpendicular from P to BC and from P to AC.
5. $\angle ACP \cong \angle BCP$.
6. $\angle CEP \cong \angle CFP$.
7. $\Delta CEP \cong \Delta CFP$.
8. $EP \cong FP$ and $CE \cong CF$.
9. $\angle AEP \cong \angle BFP$.
10. CLAIM: $EA \cong FB$.



Every Triangle is Isosceles

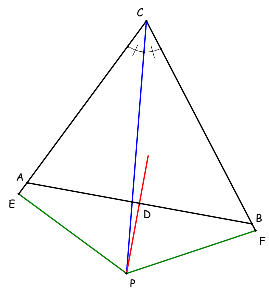
10. $EA \cong FB$.
    - i. Suppose $EA > BF$.
    - ii. There is a point A' on EA, different from A so that $EA' \cong FB$ and $\angle A'EP$ is a right angle.
    - iii. $\angle AA'P$ is obtuse by the Exterior Angle Theorem
    - iv. $\Delta FPB \cong \Delta EPA'$ by SAS
    - v. $PA' \cong PB$
    - vi. $PA' \cong PA$
    - vii. This contradicts assumption so $EA \cong FB$



Every Triangle is Isosceles

4. Drop a perpendicular from P to BC and from P to AC.
5. $\angle ACP \cong \angle BCP$.
6. $\angle CEP \cong \angle CFP$.
7. $\Delta CEP \cong \Delta CFP$.
8. $EP \cong FP$ and $CE \cong CF$.
9. $\angle AEP \cong \angle BFP$.
10. $EA \cong FB$.
11. $AC \cong BC$ since $CE \cong CF$ and $EA \cup AC = CE$ and $FB \cup BC = CF$

Therefore ΔABC is isosceles

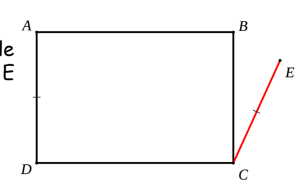


What is wrong with this "proof"?

Corruption II

Every right angle has the same measure as an obtuse angle.

Proof: Construct rectangle ABCD and choose a point E not on the rectangle so that $AD \cong CD$.

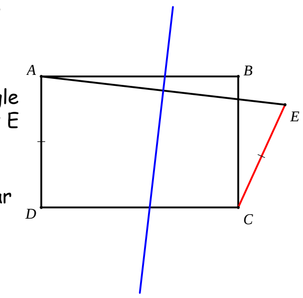


Corruption II

Every right angle has the same measure as an obtuse angle.

Proof: Construct rectangle ABCD and choose a point E not on the rectangle so that AD ≅ CD.

Let l be the perpendicular bisector of AE.



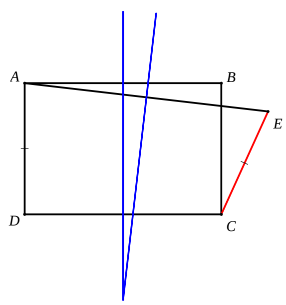
Corruption II

Every right angle has the same measure as an obtuse angle.

Proof: Construct rectangle ABCD and choose a point E not on the rectangle so that AD ≅ CD.

Let l be the perpendicular bisector of AE.

Let m be the perpendicular bisector of CD.



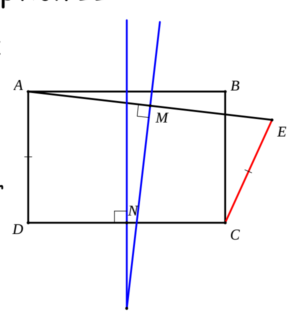
Corruption II

Proof: Construct rectangle ABCD and choose a point E not on the rectangle so that AD ≅ CD.

Let l be the perpendicular bisector of AE.

Let m be the perpendicular bisector of CD.

Let P = l ∩ m. Let M denote the midpoint of AE and N denote the midpoint of CD



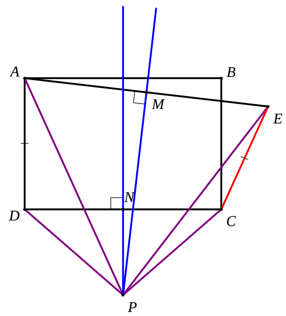
Corruption II

Let l be the perpendicular bisector of AE.

Let m be the perpendicular bisector of CD.

Let P = l ∩ m. Let M denote the midpoint of AE and N denote the midpoint of CD.

Construct segments DP, EP, AP, and CP.



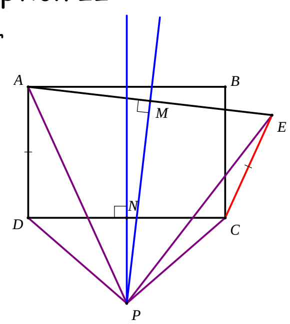
Corruption II

Let m be the perpendicular bisector of CD.

Let P = l ∩ m. Let M denote the midpoint of AE and N denote the midpoint of CD.

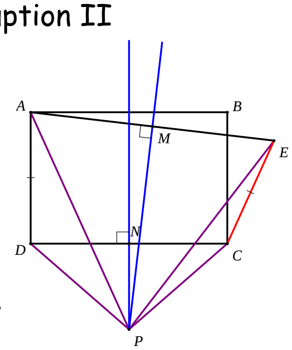
Construct segments DP, EP, AP, and CP.

1. AP ≅ EP
2. DP ≅ CP
3. AD ≅ CE by construction



Corruption II

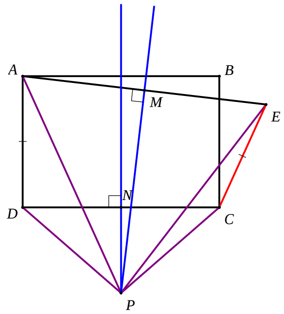
1. AP ≅ EP
2. DP ≅ CP
3. AD ≅ CE
4. ΔECP ≅ ΔADP by SSS
5. ∠ECP ≅ ∠ ADP
6. ΔDNP ≅ ΔCNP by SSS
7. ∠DCP ≅ ∠ CDP
8. ∠ECP = ∠ADP + ∠ECD and ∠ADP = ∠CDP + ∠ADC
9. ∠ECD ≅ ∠ADC



Corruption II

3. AD ≅ CE
4. ΔECP ≅ ΔADP by SSS
5. ∠ECP ≅ ∠ ADP
6. ΔDNP ≅ ΔCNP by SSS
7. ∠DCP ≅ ∠ CDP
8. ∠ECP = ∠ADP + ∠ECD and

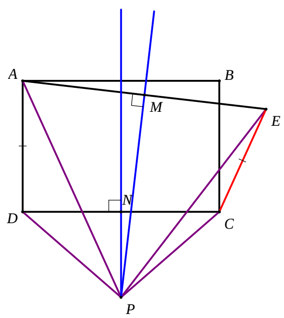
   ∠ADP = ∠CDP + ∠ADC
9. ∠ECD ≅ ∠ADC
10. ∠ECD > ∠ BCD and is obtuse.



Corruption II

5. ∠ECP ≅ ∠ ADP
6. ΔDNP ≅ ΔCNP by SSS
7. ∠DCP ≅ ∠ CDP
8. ∠ECP = ∠ADP + ∠ECD and

   ∠ADP = ∠CDP + ∠ADC
9. ∠ECD ≅ ∠ADC
10. ∠ECD > ∠ BCD and is obtuse.
11. ∠ACD is a right angle
12. By #9 we are done.



What is wrong with this "proof"?

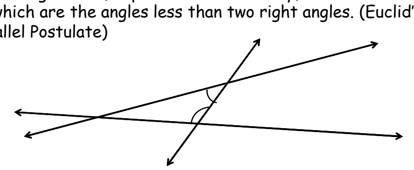
Nothing is impossible for anyone impervious to reason.

Axiom Systems

Euclid's Axioms

Let the following be postulated

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles. (Euclid's Parallel Postulate)



Euclid's Common Notions

1. Things that are equal to the same thing are also equal to one another.
2. It equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

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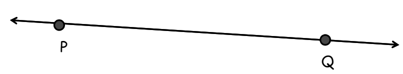
Euclid's Axioms – Modern Version

1. For every point P and every point Q not equal to P there exists a unique line that passes through P and Q.



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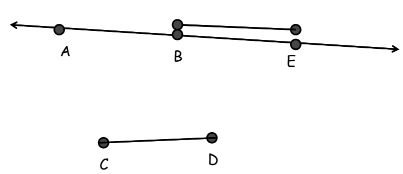
Euclid's Axioms – Modern Version

1. For every point P and every point Q not equal to P there exists a unique line that passes through P and Q.
2. For every segment AB and every segment CD there exists a unique point E on line AB such that B is between A and E and segment CD is congruent to BE.



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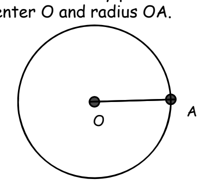
Euclid's Axioms – Modern Version

1. For every point P and every point Q not equal to P there exists a unique line that passes through P and Q.
2. For every segment AB and every segment CD there exists a unique point E on line AB such that B is between A and E and segment CD is congruent to BE.
3. For every point O and every point A not equal to O there is a circle with center O and radius OA.



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Euclid's Axioms – Modern Version

1. For every point P and every point Q not equal to P there exists a unique line that passes through P and Q.
2. For every segment AB and every segment CD there exists a unique point E on line AB such that B is between A and E and segment CD is congruent to BE.
3. For every point O and every point A not equal to O there is a circle with center O and radius OA.
4. All right angles are congruent to one another.

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Euclid's Axioms – Modern Version

1. For every point P and every point Q not equal to P there exists a unique line that passes through P and Q.
2. For every segment AB and every segment CD there exists a unique point E on line AB such that B is between A and E and segment CD is congruent to BE.
3. For every point O and every point A not equal to O there is a circle with center O and radius OA.
4. All right angles are congruent to one another.
5. For every line l and for every point P not on l there exists a unique line m through P that is parallel to l. (Playfair's Postulate)

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Euclid's Axioms – Problems therein

1. How do we know points exist? It is never stated in any postulate.
2. Euclid takes betweenness and line separation for granted, never stating the properties that he uses in any axioms or postulates.
3. The proof that Euclid gives for SAS is faulty. He assumes that certain motions are possible without affirming them in postulates or axioms.
4. Euclid assumes all of the continuity properties that he needs for granted. For example, he assumes that if two circles are sufficiently close then they have to intersect in two points.

Alternative Axiom Systems

Tarski's Axioms:
http://education.uncc.edu/droyster/courses/Spring08/axioms/Tarski.htm
(PDF file)
Hilbert's Axioms:
http://education.uncc.edu/droyster/courses/Spring08/axioms/Hilbert.htm
(PDF file)
Birkoff's Axioms:
http://education.uncc.edu/droyster/courses/Spring08/axioms/Birkhoff.htm
(PDF file)
SMSG Axioms:
http://education.uncc.edu/droyster/courses/Spring08/axioms/SMSG.htm
(PDF file)

SMSG Axiom System

Undefined Terms:
Point, line, plane

- **Postulate 1.** (*Line Uniqueness*) Given any two distinct points there is exactly one line that contains them.
- **Postulate 2.** (*Distance Postulate*) To every pair of distinct points there corresponds a unique positive number. This number is called the distance between the two points.
- **Postulate 3.** (*Ruler Postulate*) The points of a line can be placed in a correspondence with the real numbers such that:
- To every point of the line there corresponds exactly one real number.
- To every real number there corresponds exactly one point of the line.
- The distance between two distinct points is the absolute value of the difference of the corresponding real numbers.
- **Postulate 4.** (*Ruler Placement Postulate*) Given two points *P* and *Q* of a line, the coordinate system can be chosen in such a way that the coordinate of *P* is zero and the coordinate of *Q* is positive.
- **Postulate 5.** (*Existence of Points*)
- Every plane contains at least three non-collinear points.
- Space contains at least four non-coplanar points.

SMSG Axiom System

- **Postulate 6.** (*Points on a Line Lie in a Plane*) If two points lie in a plane, then the line containing these points lies in the same plane.
- **Postulate 7.** (*Plane Uniqueness*) Any three points lie in at least one plane, and any three non-collinear points lie in exactly one plane.
- **Postulate 8.** (*Plane Intersection*) If two planes intersect, then that intersection is a line.
- Definition: A set of points is convex if whenever two points are in the set the line segment containing the two points is in the set.

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SMSG Axiom System

- **Postulate 9.** (*Plane Separation Postulate*) Given a line and a plane containing it, the points of the plane that do not lie on the line form two sets such that:
  - each of the sets is convex;
  - if *P* is in one set and *Q* is in the other, then segment *PQ* intersects the line.
- **Postulate 10.** (*Space Separation Postulate*) The points of space that do not lie in a given plane form two sets such that:
  - Each of the sets is convex.
  - If *P* is in one set and *Q* is in the other, then segment *PQ* intersects the plane.
- **Postulate 11.** (*Angle Measurement Postulate*) To every angle there corresponds a real number between 0° and 180°.
- **Postulate 12.** (*Angle Construction Postulate*) Let $\overrightarrow{AB}$ be a ray on the edge of the half-plane H. For every r between 0 and 180, there is exactly one ray $\overrightarrow{AP}$ with P in H such that $m\angle PAB=r$.

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SMSG Axiom System

- **Postulate 13.** (*Angle Addition Postulate*) If D is a point in the interior of $\angle BAC$, then $m\angle BAC = m\angle BAD + m\angle DAC$
- **Postulate 14.** (*Supplement Postulate*) If two angles form a linear pair, then they are supplementary.
- **Postulate 15.** (*SAS Postulate*) Given a one-to-one correspondence between two triangles (or between a triangle and itself). If two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.
- **Postulate 16.** (*Parallel Postulate*) Through a given external point there is at most one line parallel to a given line.
- **Postulate 17.** (*Area of Polygonal Region*) To every polygonal region there corresponds a unique positive real number called the area.
- **Postulate 18.** (*Area of Congruent Triangles*) If two triangles are congruent, then the triangular regions have the same area.

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SMSG Axiom System

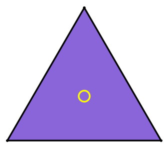
- **Postulate 19.** (*Summation of Areas of Regions*) Suppose that the region *R* is the union of two regions R_1 and R_2. If R_1 and R_2 intersect at most in a finite number of segments and points, then the area of *R* is the sum of the areas of R_1 and R_2.
- **Postulate 20.** (*Area of a Rectangle*) The area of a rectangle is the product of the length of its base and the length of its altitude.
- **Postulate 21.** (*Volume of Rectangular Parallelpiped*) The volume of a rectangular parallelpiped is equal to the product of the length of its altitude and the area of its base.
- **Postulate 22.** (*Cavalieri's Principle*) Given two solids and a plane. If for every plane that intersects the solids and is parallel to the given plane the two intersections determine regions that have the same area, then the two solids have the same volume.

Euclidean Geometry

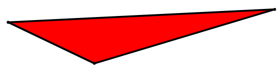
Advanced Euclidean Geometry

Triangles and their Centers

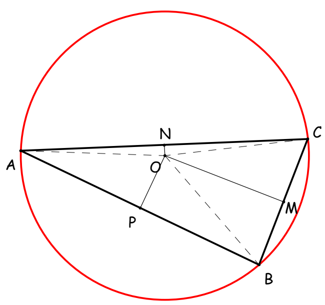
What is the CENTER of a triangle?



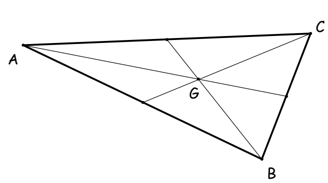
What if it is not equilateral?



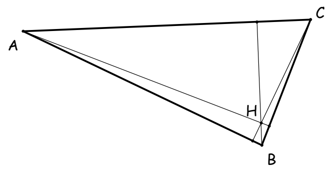
Circumcenter



Centroid



Orthocenter



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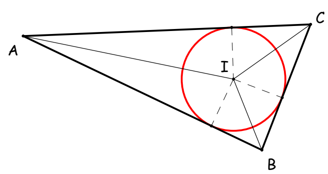
Incenter



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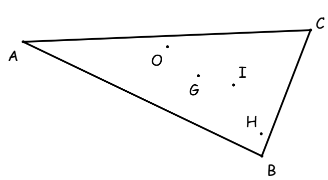
The 4 Centers so far



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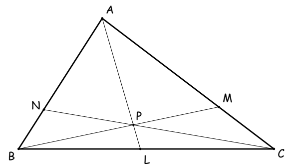
Ceva's Theorem

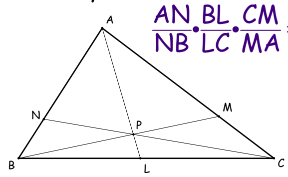
Let P be a point and let △ABC be any triangle. The rays from each of the vertices and intersecting opposite sides are called the cevians of the triangle with respect to P.



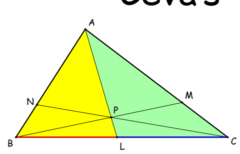
Ceva's Theorem

The three line containing the vertices A, B, and C of △ABC and intersecting opposite sides at points L, M, and N, respectively, are concurrent if and only if

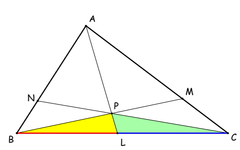
$$\frac{AN}{NB} \bullet \frac{BL}{LC} \bullet \frac{CM}{MA} = 1$$



Ceva's Theorem

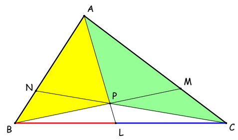


$$\frac{K(\triangle ABL)}{K(\triangle ACL)} = \frac{BL}{LC}$$



$$\frac{K(\triangle PBL)}{K(\triangle PCL)} = \frac{BL}{LC}$$

Ceva's Theorem



$$\frac{BL}{LC}=\frac{K(\triangle ABL)-K(\triangle PBL)}{K(\triangle ACL)-K(\triangle PCL)}=\frac{K(\triangle ABP)}{K(\triangle ACP)}$$

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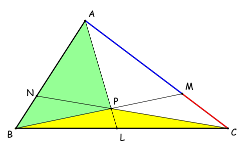
Ceva's Theorem



$$\frac{CM}{MA}=\frac{K(\triangle BMC)-K(\triangle PMC)}{K(\triangle BMA)-K(\triangle PMA)}=\frac{K(\triangle BCP)}{K(\triangle BAP)}$$

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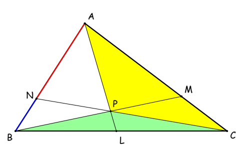
Ceva's Theorem



$$\frac{AN}{NB}=\frac{K(\triangle ACN)-K(\triangle APN)}{K(\triangle BCN)-K(\triangle BPN)}=\frac{K(\triangle ACP)}{K(\triangle BCP)}$$

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Ceva's Theorem

$$\frac{AN}{NB}\bullet\frac{BL}{LC}\bullet\frac{CM}{MA}=\frac{K(\triangle ACP)}{K(\triangle BCP)}\bullet\frac{K(\triangle ABP)}{K(\triangle ACP)}\bullet\frac{K(\triangle BCP)}{K(\triangle ABP)}=1$$

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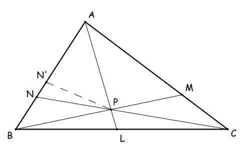
Ceva's Theorem

Now assume that

$$\frac{AN}{NB}\bullet\frac{BL}{LC}\bullet\frac{CM}{MA}=1$$

Let BM and AL intersect at P and construct CP intersecting AB at N', N' different from N.



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Ceva's Theorem

Then AL, BM, and CN' are concurrent and

$$\frac{AN'}{N'B}\bullet\frac{BL}{LC}\bullet\frac{CM}{MA}=1$$

From our hypothesis it follows that

$$\frac{AN'}{N'B}=\frac{AN}{NB}$$

So N and N' must coincide.

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Medians & Centroid

The medians intersect at a common point G.

Proof: Since L, M, and N are midpoints, we have that AN=NB, BL=LC, and CM=MA. Therefore,

$$\frac{AN}{NB}\bullet\frac{BL}{LC}\bullet\frac{CM}{MA}=\frac{AN}{AN}\bullet\frac{BL}{BL}\bullet\frac{CM}{CM}=1$$

Altitudes & Orthocenter

The altitudes intersect at a common point O.

$$\triangle ANC \sim \triangle AMB \Rightarrow \frac{AN}{MA}=\frac{AC}{AB}$$
$$\triangle BLA \sim \triangle BNC \Rightarrow \frac{BL}{NB}=\frac{AB}{BC}$$
$$\triangle CMB \sim \triangle CLA \Rightarrow \frac{CM}{LC}=\frac{BC}{AC}$$
$$\frac{AN}{MA}\bullet\frac{BL}{NB}\bullet\frac{CM}{LC}=\frac{AC}{AB}\bullet\frac{AB}{BC}\bullet\frac{BC}{AC}=1$$

Angle Bisectors & Incenter

The angle bisectors intersect at a common point I.

$$\frac{AB}{AC}=\frac{BL}{LC}$$
$$\frac{BC}{BA}=\frac{CM}{AM}$$
$$\frac{CA}{CB}=\frac{AN}{BN}$$
$$\frac{AN}{MA}\bullet\frac{BL}{NB}\bullet\frac{CM}{LC}=\frac{AB}{AC}\bullet\frac{BC}{BA}\bullet\frac{CA}{CB}=1$$

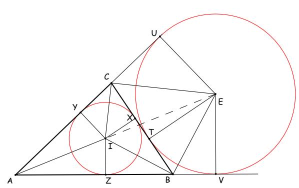
Inradius & Area

If the inradius is r and the semiperimeter is s.

$$K(\triangle ABC) = rs$$

Heron's Formula

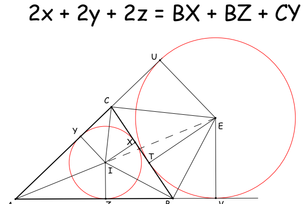
By External Tangents Theorem AZ=AY, BZ=BX, BV=BT, CX=CY, and CU=CT.



Let
x = BX
y = CY
z = AZ
u = CU
v = BV

Heron's Formula

1. x + y + z = s, the semiperimeter.
   Proof:
   2x + 2y + 2z = BX + BZ + CY + CX + AZ + AY = p

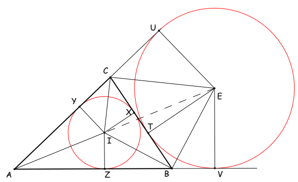


Heron's Formula

2. x = s - b

Proof:

b = AC = y + z = s – x and we are done.



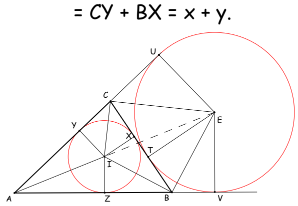
Similarly
y = s – c
z = s - a

Heron's Formula

3. u + v = x + y

Proof:

u + v = CU + BV = CT + BT = CX + BX
= CY + BX = x + y.

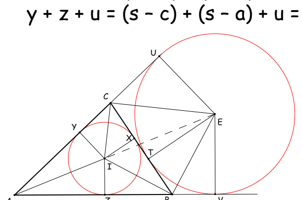


Heron's Formula

4. x + z + v = y + z + u

Proof:

x + z + v = (s – b) + (s – a) + v = c + v = AV
y + z + u = (s – c) + (s – a) + u = b + u = AU



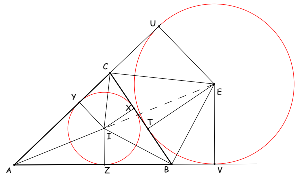
But
AV = AU

Heron’s Formula

5. u = x and v = y.

Proof:

u + v = x + y so x + v = y + u or x – y = u – v. Solving those equations gives us u = x and v = y.

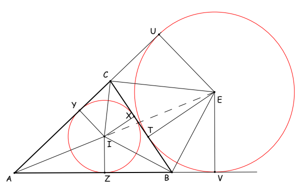


Heron’s Formula

6. AV = s.

Proof:

AV = AB + BV
= AZ + ZB + BV
= z + x + v
= x + y + z = s

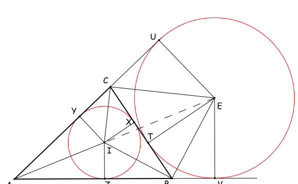


Heron’s Formula

7. EV/s = r/(s - a).

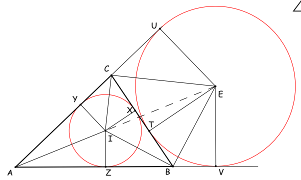
Proof:

Note that ΔEVA ~ ΔIZA
So EV/AV = IZ/AZ
EV/s = r/z = r/(s – a)



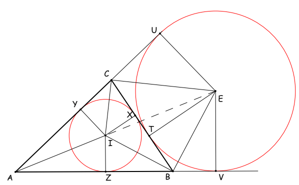
Heron's Formula

8. ∠ZBI + ∠EBV = 90 .
   Proof: ∠ZBI + ∠IBX + ∠XBE + ∠EBV = 180
   ∠ZBI = ∠IBX and ∠XBE = ∠EBV, so
   ∠ZBI + ∠EBV = 90
   Note then that
   ΔZBI ~ ΔVEB



Heron's Formula

9. (s – b)/r = EV/(s – c) .
   Proof: This follows directly from the similarity.



Heron's Formula

10. $K(\triangle ABC) = \sqrt{s(s-a)(s-b)(s-c)}$
    Proof: K = sr
    = EV(s – a)
    EV = (s – b)(s – c)/r
    K = (s – a)(s – b)(s – c)/r
    K = [(s – a)(s – b)(s – c)]/[K/s]
    K = [s(s – a)(s – b)(s – c)]/K
    K^2 = s(s – a)(s – b)(s – c)