MATH 6118-090
Non-Euclidean Geometry

Exercise Set #5
Solutions

A *parallelogram* is defined to be a quadrilateral in which the lines containing opposite sides are non-intersecting.

1. Prove that in Euclidean geometry, a quadrilateral is a parallelogram if and only if opposite sides are congruent. Show with a generic example that in hyperbolic geometry, the opposite sides of a parallelogram need not be congruent.

   $\angle ABD \cong \angle CDB$ *and* $\angle ADB \cong \angle DBD$ *by the Converse of the Alternate Interior Angle Theorem.* $DB \cong DB$ *so, by Angle-Side-Angle* $\Delta ABD \cong \Delta CDB$ *and it follows immediately that* $CD \cong AB$ *and* $BC \cong AD$.
   If opposite sides are congruent, then by SSS $\Delta ABD \cong \Delta CDB$, *thus* $\angle ABD \cong \angle CDB$ *and* $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ *by the Alternate Interior Angles Theorem. A similar proof shows that the other two sides are parallel.*
   A Saccheri quadrilateral is a parallelogram in $\mathbf{H}^2$ *whose base and summit are not congruent.*

NOTE: For the remainder of this problem, the geometry is *hyperbolic.*

2. Given $\square ABCD$ with opposite sides congruent, prove that opposite angles are congruent and that the lines containing opposite sides are hyperparallel. Such a quadrilateral is called a *symmetric parallelogram*.

   Given $AD \cong BC$ *and* $AB \cong DC$. $AC \cong AC$ *so that* $\Delta ACD \cong \Delta CAB$ *by Side-Side-Side Criterion. Thus,* $\angle D \cong \angle B$. *Using the diagonal BD we can prove similarly that* $\angle A \cong \angle C$.
   Let M be the midpoint of AC. Drop perpendiculars from M to $\overleftrightarrow{AB}$, *calling the foot* M_A, *and to* $\overleftrightarrow{CD}$, *calling the foot* M_C. *We know that* $MA \cong MC$. *Since* $\Delta ABC \cong \Delta CDA$, $\angle DCA \cong \angle BAC$. *Thus, by Hypotenuse-Angle* $\Delta MM_AA \cong \Delta MM_CC$ *and* $\angle M_CMC \cong \angle M_AMA$. *It then follows that* M_A, M_C, *and* M *are collinear and that* $\overleftrightarrow{AB}$ *and* $\overleftrightarrow{CD}$ *have a common perpendicular and are thus hyperparallel. A similar analysis shows that* $\overleftrightarrow{AD}$ *and* $\overleftrightarrow{BC}$ *are hyperparallel.*

3. For a symmetric parallelogram $\square ABCD$ prove that the diagonals have the same midpoint, *M*. Show that *M* is also the midpoint of the common perpendicular of both pairs of hyperparallel opposite sides.

   Let M be the midpoint of AC. Construct MB and MD. Since $\Delta ABC \cong \Delta CDA$, $\angle CAB \cong \angle ACD$ *and, hence,* $\Delta MAB \cong \Delta MCD$ *by Side-Angle-Side. Thus,* $BM \cong DM$ *and M is the midpoint of BD.*

From above we know that the common perpendicular to both pair of congruent sides passes through M. Also, we showed that $\Delta MM_A A \cong \Delta MM_C C$ *which implies that* $MM_A \cong MM_C$. *A similar argument shows that the same is true for the common perpendicular to* $\overleftrightarrow{BC}$ *and* $\overleftrightarrow{AD}$.

4. Show that the diagonals are perpendicular if and only if all four sides are congruent, and in that case, $\square ABCD$ has an inscribed circle with center M.

   ($\Leftarrow$) *Assume that all four sides are congruent. By SSS* $\Delta DMC \cong \Delta BMC$ *which implies that* $\angle DMC \cong \angle BMC$ *and since they are supplementary angles, they must be right angles. Thus,* $AD \perp BD$.
   ($\Rightarrow$) *Assume that the diagonals are perpendicular. Then since* $MD \cong MB$, $MC \cong MC$, *and* $\angle DMC \cong \angle BMC$ *(right angles), we have that* $\Delta DMC \cong \Delta BMC$. *It now follows that* $DC \cong BC$ *and hence* $AB \cong BC \cong CD \cong DA$.

5. Show that the diagonals are congruent if and only if all four angles are congruent; however in that case, show that all four sides need not be congruent.

   ($\Leftarrow$) *Assume that all four angles are congruent.* $\angle A \cong \angle B \cong \angle C \cong \angle D$. *Since* $AD \cong BC$ *and* $CD \cong CD$, $\Delta ACD \cong \Delta BCD$ *from which we have that* $BD \cong AC$.

   ($\Rightarrow$) $BD \cong AC$, $AD \cong BC$, *and* $CD \cong CD$ *so, by SSS* $\Delta ADC \cong \Delta BCD$ *and* $\angle D \cong \angle C$. *The rest follows simply.*

   Given a Lambert quadrilateral in which the sides adjacent to the acute angle are not congruent — such do exist — we take four copies and glue them together like so to get an equiangular symmetric quadrilateral in which all four sides are not congruent.