Fundamental Theorems of Vector Calculus

We have studied the techniques for evaluating integrals over curves and surfaces. In the case of integrating over an interval on the real line, we were able to use the Fundamental Theorem of Calculus to simplify the integration process by evaluating an antiderivative of the function at the endpoints of the interval. Is there a process by which we can simplify a line integral or a surface integral to a simpler integral on the endpoints of the curve or the boundary of the surface?

1 Line Integrals

Suppose that g and G are real-valued continuous functions defined on a closed interval $[a, b]$, that G is differentiable on (a, b) and that $G' = g$. The Fundamental Theorem of Calculus states that

$$\int_a^b g(x)\,dx = G(b) - G(a).$$

Thus the value of the integral of g depends only on the value of G at the endpoints of the interval $[a, b]$. Since ∇f represents the derivative of f, we can ask if the integral of ∇f over a curve $\sigma(t)$ is completely determined by the values of f at the endpoints.

Theorem 1 (Line Integrals for Gradient Fields) *Suppose that $f\colon \mathbf{R}^n \to \mathbf{R}$ is continuously differentiable and that $\sigma\colon [a, b] \to \mathbf{R}^n$ is a piecewise continuously differentiable path. Then*

$$\int_\sigma \nabla f \cdot d\mathbf{s} = f(\sigma(b)) - f(\sigma(a)).$$

Any vector field $\mathbf{F}$ satisfying $\mathbf{F} = \nabla f$ is called a *gradient vector field.* This theorem is true then for any gradient vector field. Thus, for a gradient vector field the value of the line integral depends only on the endpoints of the path, $\sigma(a)$ and $\sigma(b)$, but not on the path itself. This leads us to say that the integral is *path independent.*

Recall that we had defined *simple* and *simple closed* curves earlier. Simple curves do not intersect themselves, except possibly at the endpoints, and simple closed curves are simple curves satisfying $\sigma(a) = \sigma(b)$. Note then that the integral of a gradient vector field over a simple closed curve must be zero, since the beginning and ending point are the same. Is it true then that if the integral of a vector field over a simple closed curve is zero, then the vector field will be a gradient field? This is answered in the following theorem.

Theorem 2 *Let* $\mathbf{F}$ *be a* C^1 *vector field defined on* $\mathbf{R}^3$ *(except possibly at a finite number of points). The following conditions on* $\mathbf{F}$ *are equivalent.*

a) For any oriented simple closed curve C, $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{s} = 0$.

b) For any two oriented simple curves C_1 *and* C_2 *which have the same endpoints,*

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{s} = \int_{C_2} \mathbf{F}\cdot d\mathbf{s}.$$

c) $\mathbf{F}$ *is a gradient vector field.*

d) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = 0$.

If $\mathbf{F}$ *satisfies any one (and hence all) of the above properties, then* $\mathbf{F}$ *is called a* ***conservative vector field****.*

For vector fields on $\mathbf{R}^2$ the curl does not exist. However, if we let $\mathbf{F}(x,y) = P(x,y)\boldsymbol{\imath} + Q(x,y)\boldsymbol{\jmath}$, then we can define an analogous map $\mathbf{F}(x,y,z) = P(x,y)\boldsymbol{\imath} + Q(x,y)\boldsymbol{\jmath}$ and

$$\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\boldsymbol{k}.$$

Sometimes the quantity $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$ is called the **scalar curl** of $\mathbf{F}$. In this case the condition $\operatorname{curl}\mathbf{F} = 0$ reduces to $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$.

Corollary 1 *If* $\mathbf{F}$ *is a* C^1 *vector field on* $\mathbf{R}^2$ *of the form above with* $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$*, then* $\mathbf{F}$ *is a gradient vector field.*

If you will recall one of the vector identities, we have $\operatorname{div}(\operatorname{curl}\mathbf{G}) = 0$ for any C^2 vector field $\mathbf{G}$. What about the converse? If $\operatorname{div}(\mathbf{F}) = 0$ is $\mathbf{F}$ the curl of a vector field $\mathbf{G}$?

Theorem 3 *If* $\mathbf{F}$ *is a* C^1 *vector field on* $\mathbf{R}^3$ *with* $\operatorname{div}\mathbf{F} = 0$*, then there exists a* C^2 *vector field* $\mathbf{G}$ *with* $\mathbf{F} = \operatorname{curl}\mathbf{G}$.

2 Classical Integration Theorems of Vector Calculus

We continue our look at the relationship between the concepts of integration and differentiation in vector calculus. The results in this section are contained in the theorems of Green, Gauss, and Stokes and are all variations of the same theme applied to different types of integration. Green's Theorem relates the path integral of a vector field along an oriented, simple closed curve in the xy-plane to the double integral of its derivative over the region enclosed by the curve. Gauss' Divergence Theorem extends this result to closed surfaces and Stokes' Theorem generalizes it to simple closed surfaces in space.

2.1 Green's Theorem

2.1.1 Type I, II, and III Regions

Let $f_1\colon [a,b] \to \mathbf{R}$ and $f_2\colon [a,b] \to \mathbf{R}$ be continuous functions satisfying $f_1(x) \leq f_2(x)$ for all $x \in [a,b]$. Let the region D be defined by

$$D = \{(x,y) \mid a \leq x \leq b,\ f_1(x) \leq y \leq f_2(x)\}\,.$$

This region is said to be of **type I**. The curves and straight line segments that bound the region taken together form the **boundary** of the D, denoted ∂D.

A region D is of **type II** if there are continuous functions g_1 and g_2 defined on $[c,d]$ so that

$$D = \{(x,y) \mid g_1(y) \leq x \leq g_2(y),\ c \leq y \leq d\}\,,$$

where $g_1(y) \leq g_2(y)$ for all $y \in [c,d]$.

A region is of **type III** if it is both type I and type II. We will refer to regions of type I, II, or III as **elementary regions**

2.1.2 Orientations on Boundary Curves

A simple closed curve C that is the boundary of an elementary region has two orientations. The counterclockwise orientation is the positive orientation, and the clockwise orientation is the negative orientation. We will denote the curve C with the positive orientation as C^+ and with the negative orientation as C^-.

We say that a region D in the plane is a *good* region if it is either an elementary region or can be divided into disjoint pieces $D_1, \ldots, D_n$ each of which is an elementary region. In the latter case

$$\iint_D f\,dA = \iint_{D_1} f\,dA + \cdots + \iint_{D_n} f\,dA,$$

for a real-valued function f. The boundary of D then consists of a finite number of simple closed curves, that are oriented by the following rule:

if we walk along the boundary curve in the positive direction, then the region is on our left.

As an example the annulus $D = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 4\}$ is not an elementary region. It can be divided into 4 elementary regions (in several ways). The boundary of D, ∂D consists of two circles, one of radius 1 and one of radius 2. The positive orientation on ∂D is given by taking the counterclockwise orientation on the outer circle and the clockwise orientation on the inner circle.

2.1.3 Green's Theorem

Theorem 4 *Let D be a good region in the plane and let $\sigma = \partial D$ be the positively-oriented boundary. If $\mathbf{F}(x, y) = P(x, y)\boldsymbol{\imath} + Q(x, y)\boldsymbol{\jmath}$ is a C^1 vector field on D, then*

$$\int_\sigma \mathbf{F} \cdot d\mathbf{s} = \iint_D \left(\frac{\partial Q(x, y)}{\partial x} - \frac{\partial P(x, y)}{\partial y} \right) dA.$$

Recall that there is a separate way to write the line integral in this case. We have a somewhat more familiar form of Green's Theorem:

$$\int_\sigma P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Example 1 Evaluate the path integral $\int_\sigma (y - \sin(x))dx + \cos(x)dy$ where σ is the triangle with vertices $(0, 0)$, $(\pi/2, 0)$, and $(\pi/2, 1)$. We will do it directly and using Green's Theorem.

To orient the curve counterclockwise, so that the region in the triangle lies to the left of it, we will parametrize the three segments as follows:

$$\begin{aligned}
\sigma_1(t) &= (t, 0), \qquad t \in [0, \pi/2] \\
\sigma_2(t) &= (\pi/2, t), \qquad t \in [0, 1] \\
\sigma_3(t) &= (\pi/2, 1) + t(-\pi/2, -1) = (\pi/2(1 - t), 1 - t)
\end{aligned}$$

Then,

$$\begin{aligned}
\int_{\sigma_1} (y - \sin(x))dx + \cos(x)dy &= \int_0^{\pi/2} (0 - \sin(t)) \cdot 1 + \cos(t) \cdot 0dt \\
&= \int_0^{\pi/2} - \sin(t)\, dt = -1 \\
\int_{\sigma_2} (y - \sin(x))dx + \cos(x)dy &= \int_0^1 ((t-1) \cdot 0 + 0 \cdot 1)\, dt = 0 \\
\int_{\sigma_3} (y - \sin(x))dx + \cos(x)dy &= \int_0^1 \left[(1 - t - \sin(\frac{\pi}{2} - t\frac{\pi}{2}))(-\frac{\pi}{2}) + \cos(\frac{\pi}{2} - t\frac{\pi}{2})(-1) \right]\, dt \\
&= -\frac{\pi}{4} + 1 - \frac{2}{\pi} \\
\int_{\sigma} (y - \sin(x))dx + \cos(x)dy &= -1 + 0 - \frac{\pi}{4} + 1 - \frac{2}{\pi} \\
&= -\frac{\pi}{4} - \frac{2}{\pi}
\end{aligned}$$

Now, Green's Theorem avers that

$$\int_{\sigma} (y - \sin(x))dx + \cos(x)dy = \iint_D (-\sin(x) - 1)\, dA.$$

The region D is easily described by $D = \{(x, y) \mid 0 \leq x \leq \pi y/2,\ 0 \leq y \leq 1\}$. Thus,

$$\begin{aligned}
\iint_D (-\sin(x) - 1)\, dA &= \int_0^1 \int_0^{\pi y/2} (-\sin(x) - 1)\, dxdy \\
&= \int_0^1 (-\frac{\pi}{2} - \cos(\frac{\pi}{2}y) + \frac{\pi}{2}y)\, dy \\
&= -\frac{\pi}{4} - \frac{2}{\pi}
\end{aligned}$$

Example 2 Compute $\int_\sigma \mathbf{F} \cdot d\mathbf{s}$, if $\mathbf{F}(x, y) = xy^2\boldsymbol{\imath} - x^2y\boldsymbol{\jmath}$ and σ is the boundary of the region bounded by $x \geq 0$ and $0 \leq y \leq 1 - x^2$.

This region can be described as $D = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1 - x^2\}$. So,

$$\begin{aligned}
\int_\sigma \mathbf{F} \cdot d\mathbf{s} &= \iint_D \left(\frac{\partial}{\partial x}(-x^2y) -]\frac{\partial}{\partial y}(xy^2) \right)\, dA \\
&= \int_0^1 \int_0^{1-x^2} -4xy\, dydx \\
&= \int_0^1 -2x(1 - x^2)^2\, dx = -\frac{1}{3}
\end{aligned}$$

Theorem 5 (Area of a Region) *If C is a simple closed curve that bounds a region to which Green's Theorem applies, then the area of the region D bounded by $C = \partial D$ is*

$$a = \frac{1}{2}\int_{\partial D} x dy - y dx.$$

2.1.4 Double Integral of the Laplacian

Let D be a good redion and let $\sigma = \partial D$ be its positively-oriented smooth boundary curve. Assume that the function $f\colon D \to \mathbf{R}$ is of class C^2. Recall that the Laplacian of f is given by $\triangle f = \partial^2 f/\partial x^2 + \partial^2 f/\partial y^2$. Therefore,

$$\begin{aligned}
\iint_D \triangle f\, dA &= \iint_D \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dA \\
&= \iint_D \left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right) dA \\
&= \int_\sigma -\frac{\partial f}{\partial y}dx + \frac{\partial f}{\partial x}dy, \qquad \text{by Green's Theorem} \\
&= \int_a^b \left(-\frac{\partial f}{\partial y}\frac{dx}{dt} + \frac{\partial f}{\partial x}\frac{dy}{dt}\right) dt \quad \text{by the definition of line integral}
\end{aligned}$$

Now, look at this last integrand. We can write this as

$$-\frac{\partial f}{\partial y}\frac{dx}{dt} + \frac{\partial f}{\partial x}\frac{dy}{dt} = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle \cdot \langle \frac{dy}{dt}, -\frac{dx}{dt}\rangle.$$

The first term in the dot product is the gradient of the function f. The second term in the product is our outward pointing normal vector,$\mathbf{N}$, to the curve $\sigma(t) = (x(t), y(t))$, because

$$\langle \frac{dx}{dt}, \frac{dy}{dt}\rangle \cdot \langle \frac{dy}{dt}, -\frac{dx}{dt}\rangle = 0.$$

Note that $\|\mathbf{N}\| = \|\sigma'(t)\|$. It now follows that

$$\begin{aligned}
\iint_D \triangle f\, dA &= \int_a^b \nabla f \cdot \mathbf{N}\, dt = \int_a^b \left(\nabla f \cdot \frac{\mathbf{N}}{\|\mathbf{N}\|}\right) \|\mathbf{N}\|\, dt \\
&= \int_a^b \left(\nabla f \cdot \frac{\mathbf{N}}{\|\mathbf{N}\|}\right) \|\sigma'\|\, dt
\end{aligned}$$

Now, let $\mathbf{n} = \dfrac{\mathbf{N}}{\|\mathbf{N}\|}$ denote the outward-pointing **unit normal**. Recall that from the definition of the line integral $\int_c g\, ds = \int_a^b g\|c'\|\, dt$, so reading this backward and applying it to our last equation above, we get

$$\iint_D \triangle f\, dA = \int_\sigma \nabla f \cdot \mathbf{n}\, ds = \int_\sigma D_{\mathbf{n}} f\, ds.$$

Here $D_{\mathbf{n}} f$ denotes the directional derivative of f in the normla direction. This directional derivative is called the *normal derivative* of f and is denoted by $\partial f/\partial n$. Hence,

$$\iint_D \triangle f \, dA = \int_{\sigma=\partial D} \frac{\partial f}{\partial n} \, ds.$$

2.2 Divergence Theorem

Gauss' Divergence Theorem is like Green's Theorem in that it relates an integral over a closed geometric object (a closed surface) to an integral over the region enclosed by it.

Just as in the case of the plane, we will define "good" regions in three-space. Elementary (3D) regions are regions in $\mathbf{R}^3$ bounded by surfaces that are graphs of real-valued functions of two variables. Depending on which of the variables are involved the regions are called type I(3D), type II(3D0, or type III(3D). A region is of type I(3D) if its top and bottom sides are graphs of continuous functions $f_1(x, y)$ and $f_2(x, y)$. A region is of type II(3D) if the back and front sides are graphs of continuous functions $g_1(y, z)$ and $g_2(y, z)$, and of type III(3D) if its left and right sides are graphs of continuous functions $h_1(x, z)$ and $h_2(x, z)$. A region is of type IV(3D) if it is of type I(3D), type II(3D), and type III(3D).

A region $V \subset \mathbf{R}^3$ is called *good (3D)* if it is either type IV(3D) or can be described as the union of type IV(3D) regions. How would you describe the region between two cubes, one inside the other?

A boundary ∂V of a good (3D) region is either a closed surface or a union of closed surfaces. It can be oriented in two ways: either by choosing an outward normal or an inward normal. We define the *positive* orientation as the choice of an outward normal.

Theorem 6 (Gauss' Divergence Theorem) *Let V be a good (3D) region in $\mathbf{R}^3$ and let ∂V be its positively oriented boundary. Assume that $\mathbf{F}$ is a C^1 vector field on V. Then*

$$\iint_{S=\partial V} \mathbf{F} \cdot d\mathbf{S} = \iiint_V \operatorname{div}(\mathbf{F}) \, dV.$$

Example 3 Verify this theorem for the vector field $\mathbf{F} = x^2\boldsymbol{\imath} + y^2\boldsymbol{\jmath} + z^2\boldsymbol{k}$, where S is the surface of the cylinder $x^2 + y^2 = 4$, $0 \le z \le 5$, together with the top disk $\{(x, y) \mid x^2 + y^2 \le 4, z = 5\}$ and the bottom disk $\{(x, y) \mid x^2 + y^2 \le 4, z = 0\}$, oriented by the outward normal.

To compute $\iint_S \mathbf{F} \cdot \mathbf{S}$ directly we need to parametrize the top disk, the bottom disk and the cylinder with the outward pointing normal.

Top disk S_1: Use the parametrization $\Phi_1(u, v) = (u, v, 5)$, where $u^2 + v^2 \le 4$. Then $\mathbf{T}_u = (1, 0, 0)$ and $\mathbf{T}_v = (0, 1, 0)$, and $\mathbf{N} = \mathbf{T}_u \times \mathbf{T}_v = (0, 0, 1)$.

$$\begin{aligned} \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \int\int_{u^2+v^2 \le 4} (u^2, v^2, 25) \cdot (0, 0, 1) \, dA \\ &= \int\int_{u^2+v^2 \le 4} 25 \, dA = 25 \times \text{area of disk of radius } 4 = 100\pi. \end{aligned}$$

Bottom disk S_2: Use the parametrization $\Phi_2(u,v) = (u,v,0)$, where $u^2+v^2 \leq 4$. The tangents and the normal for the bottom are the same as the top, but because we must choose the outward pointing normal, we must take $\mathbf{N}_2 = (0,0,-1)$.

$$\begin{aligned}\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} &= \int\int_{u^2+v^2\leq 4} (u^2,v^2,0)\cdot(0,0,-1)\,dA \\ &= \int\int_{u^2+v^2\leq 4} 0\,dA = 0.\end{aligned}$$

Cylinder S_3: Parametrize this surface as

$$\Phi_3(u,v) = (2\cos(u), 2\sin(u), v), \quad 0 \leq u \leq 2\pi,\ 0 \leq v \leq 5.$$

Now, $\mathbf{T}_u = (-2\sin(u), 2\cos(u), 0)$ and $\mathbf{T}_v = (0,0,1)$. Then $\mathbf{N}_3 = (2\cos(u), 2\sin(u), 0)$ which points outward. Thus,

$$\begin{aligned}\iint_{S_3} \mathbf{F}\cdot d\mathbf{S} &= \int_0^5\int_0^{2\pi} (4\sin^2(u), 4\cos^2(u), v^2)\cdot(2\cos(u), 2\sin(u), 0)\,dudv \\ &= \int_0^5\int_0^{2\pi} (8\sin^2(u)\cos(u) + 8\cos^2(u)\sin(u))dudv \\ &= 8\int_0^5 0\,dv = 0\end{aligned}$$

Thus, the surface integral is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = 100\pi.$$

By the Divergence Theorem, we should get the same result from

$$\iiint_V \operatorname{div}(\mathbf{F})\,dV.$$

Where V is the solid cylinder. Since $\operatorname{div}(\mathbf{F}) = 2x+2y+2z$, we get

$$\begin{aligned}\iiint_V \operatorname{div}(\mathbf{F})dV &= \iiint 2z+2y+2z\,dxdydz \\ &= \int_0^{2\pi}\int_0^2\int_0^5 (2r\cos(\theta) + 2r\sin(\theta) + 2z)\,rdzdrd\theta \\ &\quad \text{by changing to cylindrical coordinates} \\ &= \int_0^{2\pi}\int_0^2 10r^2\cos(\theta) + 10r^2\sin(\theta) + 25r\,drd\theta \\ &= \int_0^{2\pi} \frac{80}{3}\cos(\theta) + \frac{80}{3}\sin(\theta) + 50\,d\theta \\ &= 100\pi\end{aligned}$$

2.3 Stokes' Theorem

Stokes' Theorem is similar in spirit to Green's Theorem. It relates the path integral of a vector field around a closed curve σ in $\mathbf{R}^3$ to an integral over a surface S whose boundary is σ.

Let's take a look at the case where the surface, S, is the graph of a function $z = f(x, y)$, where $(x, y) \in D$. Assume that the region D is a region to which Green's Theorem applies. The boundary ∂D of D is a simple closed curve oriented positively. Parametrize the surface S as

$$\Phi(u, v) = (u, v, f(u, v)), \quad (u, v) \in D,$$

and choose the upward pointing normal $\mathbf{N} = (-\partial f/\partial x, -\partial f/\partial y, 1)$ as the orientation of S. The positive orientation of the boundary ∂S of S is defined by *lifting* the positive orientation of ∂D. This is done as parametrizing the ∂D as $\sigma(t) = (x(t), y(t))$, $t \in [a, b]$. The boundary curve of S is then given by

$$\tau(t) = (x(t), y(t), f(\sigma(t))), \quad t \int [a, b].$$

The orientation on σ defines the positive orientation of the boundary curve of S.

Now consider the case of a parametrized surface S given by $\Phi\colon D \to \mathbf{R}^3$, where D is an elementary region in the plane. Consider the boundary curve $\sigma = \partial D$. We are tempted to define the boundary of S as the image of σ under Φ, i.e. $\Phi(\sigma(t))$. However, this will not work.

Why not? Consider the parametrization of the sphere that we have been using. First, the sphere does not have a boundary! Yet the region D does. Thus, the image of the boundary of D does not go to the "boundary" of the sphere.

Again, parametrize the cylinder by

$$\Phi(u, v) = (\cos(u), \sin(u), v), \quad 0 \le u \le 2\pi,\ 0 \le v \le 1.$$

The image of ∂D under this map is the top and bottom of the cylinder connected with a vertical segment. This is not what we would consider the boundary of a cylinder.

What is happening in both of these cases is that there are different points in D being mapped to the same point on S, i.e., Φ is not one-to-one. If we require Φ to be one-to-one on the region D, then the boundary of S is $\Phi(\partial D)$, and we are in the same situation as above for the graph of a function.

Theorem 7 (Stokes' Theorem) *Let S be an oriented surface parametrized by a one-to-one parametrization $\Phi\colon D \to \mathbf{R}^3$ and let ∂S be its positively-oriented piecewise smooth boundary curve. If $\mathbf{F}$ is a C^1 vector field on S, then*

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

Example 4 Evaluate the line integral $\int_\sigma \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F} = 2yz\boldsymbol{\imath} + xz\boldsymbol{\jmath} + xy\boldsymbol{k}$ and σ is the intersection of the cylinder $x^2 + y^2 = 1$ and the parabolic sheet $z = y^2$.

By Stokes' Theorem

$$\int_\sigma \mathbf{F} \cdot d\mathbf{s} = \iint_S \operatorname{curl}(F) \cdot d\mathbf{S}$$

Parametrize the surface S by

$$\Phi(u, v) = (u, v, v^2), \quad u^2 + v^2 \leq 1.$$

Then $\mathbf{T}_u = (1, 0, 0)$, $\mathbf{T}_v = (0, 1, 2v)$, and $\mathbf{N} = \mathbf{T}_u \times \mathbf{T}_v = (0, -2v, 1)$. The normal $\mathbf{N}$ points upward, and with the given orientation of σ, the surface S lies on its left. Now, $\operatorname{curl}(\mathbf{F}) = (0, y, -z)$ and so

$$\begin{aligned}
\int_\sigma \mathbf{F} \cdot d\mathbf{s} &= \iint_S \operatorname{curl}(F) \cdot d\mathbf{S} = \iint_S \operatorname{curl}(F) \cdot \mathbf{N}\, dS \\
&= \iint_{\{u^2+v^2 \leq 1\}} (0, v, -v^2) \cdot (0, -2v, 1)\, dA = \iint_{\{u^2+v^2 \leq 1\}} (-3v^2)\, dA \\
&= \int_0^{2\pi} \int_0^1 (-3r^2 \sin^2(\theta))\, r dr d\theta \qquad \text{by passing to polar coordinates} \\
&= -\frac{3\pi}{4}
\end{aligned}$$

Stokes' Theorem states that, in order to compute the surface integral $\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S}$, all we really need are the values of $\mathbf{F}$ on the boundary of S, and nowhere else. Therefore, as long as two surfaces S_1 and S_2 have the same boundary with the orientation requirement satisfied, then $\iint_{S_1} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \iint_{S_2} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S}$ for any C^1 vector field $\mathbf{F}$. Consequently, in computing the path integral around a simple closed curve using Stokes' Theorem, we are free to choose any surface that is bounded by the given curve.

This leads us to the following result.

Theorem 8 (Surface Independence) *Let* $\mathbf{G}$ *be a vector field defined on a region* $R \subset \mathbf{R}^3$. *If either*

a) $\mathbf{G} = \operatorname{curl}(\mathbf{F})$ *for some* $\mathbf{F}$ *or*

b) $\operatorname{div}(\mathbf{G}) = 0$ *and* R *is all of* $\mathbf{R}^3$,

then

$$\iint_{S_1} \mathbf{G} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{G} \cdot d\mathbf{S}$$

whenever S_1 *and* S_2 *are two oriented surfaces in* R *such that* $\partial S_1 = \partial S_2$.