HOMEWORK III
MATH 527
SPRING 2011

BERTRAND GUILLOU

Problem 1. (The Hopf fibration) Let $S^3 \subseteq \mathbb{C}^2 \cong \mathbb{R}^4$ be the unit sphere. Stereographic projection provides an identification $S^2 \cong \mathbb{CP}^1$. The composition $S^3 \hookrightarrow \mathbb{C}^2 - \{0\} \longrightarrow \mathbb{CP}^1$, where the second map is the natural quotient map, is called the "Hopf map" and is usually denoted by η.

(i) Show that $\eta^{-1}(1) \cong S^1$.

(ii) Let $F(\eta)$ be the homotopy fiber of $\eta : S^3 \longrightarrow S^2$. There is a natural map $\eta^{-1}(1) \longrightarrow F(\eta)$ which assigns to any point x the pair (c_1, x), where c_1 is the constant path at 1 in S^2. Show that this map is a homotopy equivalence.

(iii) Use the long exact sequence in homotopy for the map η to show that $\pi_3(S^2) \cong \mathbb{Z}$, generated by the element $[\eta]$, and that $\pi_n(S^3) \cong \pi_n(S^2)$ for $n \geq 3$. You may assume that $\pi_3(S^3) \cong \mathbb{Z} \cong \pi_2(S^2)$ and that S^3 and S^2 have no nontrivial lower homotopy groups.

Problem 2. (Whitehead products) For each $n \geq 0$, equip S^n with the CW structure having one cell in dimension 0 and one cell in dimension n. Then the product $S^p \times S^q$ has four cells, and if $p, q \geq 1$ the attaching map for the top cell takes the form $S^{p+q-1} \longrightarrow S^p \vee S^q$. For any based space X, the resulting map

$$\pi_p(X) \times \pi_q(X) \longrightarrow \pi_{p+q-1}(X)$$

is called the Whitehead product.

(i) When $p = q = 1$, the Whitehead product takes the form $\pi_1(X) \times \pi_1(X) \longrightarrow \pi_1(X)$. What is this map? More generally, what is another description of this map when $p = 1$ (but q is arbitrary)?

(ii) Show that a path-connected H-space has trivial Whitehead products.

Problem 3. Recall that if X is a based space, then ΩX denotes the space of based loops in X.

(i) Show that concatenation of loops gives ΩX the structure of a homotopy associative H-space

(ii) (Moore loops) Let $\Omega_M(X)$ denote the space of "Moore" based loops. Such a loop consists of a pair (γ, r), where $r \geq 0$ and γ is a based loop in X thought of as a map $\gamma : [0, r] \longrightarrow X$. Note that if $r = 0$, then γ is necessarily the constant loop. The operation

$$(\beta, r) * (\gamma, s) = (\beta * \gamma, r + s)$$

defines a *strictly associative* operation on the space of Moore loops.

Show that the projection map $\Omega_M(X) \longrightarrow \Omega(X)$ which drops the parameter r is a homotopy equivalence.

Problem 4. Let $i : A \longrightarrow X$ be a based map. Recall that the map $p : F(i) \longrightarrow A$ is defined by $p(\gamma, a) = a$. Show that $\Omega X \simeq F(F(i) \xrightarrow{p} A)$.

Date: February 20, 2011.