HOMEWORK IV
MATH 527
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Problem 1. (i) Show that if $E \xrightarrow{p} B$ is a fibration and Z is any space, then $\mathrm{Map}(Z, E) \longrightarrow \mathrm{Map}(Z, B)$ is also a fibration.

(ii) Show that if $A \xrightarrow{j} X$ is a cofibration and Z is any space, then $\mathrm{Map}(X, Z) \longrightarrow \mathrm{Map}(A, Z)$ is a fibration.

(iii) Show that if $A \xrightarrow{j} X$ is a cofibration and $E \xrightarrow{p} B$ is a fibration, then the natural map

$$\mathrm{Map}(X, E) \longrightarrow \mathrm{Map}(A, E) \times_{\mathrm{Map}(A,B)} \mathrm{Map}(X, B)$$

is a fibration.

Hint: The key is to show that one can find a lift in a diagram of the form $\begin{array}{ccc} M(j) & \longrightarrow & E \\ \downarrow & & \downarrow \\ X \times I & \longrightarrow & B \end{array}$.

This can be done by establishing that the inclusion $M(j) \hookrightarrow X \times I$ is a retract of the inclusion $i_0 : X \times I \hookrightarrow (X \times I) \times I$. One argument for the latter fact is as follows.

We have previously shown that if $j : A \longrightarrow X$ is a cofibration, then the inclusion $M(j) \hookrightarrow X \times I$ admits a retraction $r : X \times I \longrightarrow M(j)$. This can be improved: define a homotopy $h : X \times I \times I \longrightarrow X \times I$ by the formula

$$h(x, t, s) = (r_1(x, t(1 - s)), st + (1 - s)r_2(x, t))$$

(r_1 and r_2 are the two components of the map r). Then h defines a homotopy from r to the identity, rel $M(j)$. Thus $M(j)$ is a deformation retract of $X \times I$.

By Theorem 6.4 of [May], there is a continuous $u : X \longrightarrow I$ such that $u^{-1}(0) = A$. Define $v : X \times I \longrightarrow I$ by $v(x, t) = t \cdot u(x)$, and note that $v^{-1}(0) = M(j)$. Define now a new homotopy $\Phi : X \times I \times I \longrightarrow X \times I$ by

$$\Phi(x, t, s) = \begin{cases} h(x, t, s \cdot v(x, t)^{-1}) & s < v(x, t) \\ h(x, t, 1) = (x, t) & s \geq v(x, t) \end{cases}$$

(you should convince yourself that Φ is continuous). Finally, define $\Lambda : X \times I \longrightarrow X \times I \times I$ by $\Lambda(x, t) = (x, t, v(x, t))$. Check that the following diagram is a retract diagram

$$\begin{array}{ccccc} M(j) & \longrightarrow & X \times I & \xrightarrow{r} & M(j) \\ \downarrow & & \downarrow{\scriptstyle i_0} & & \downarrow \\ X \times I & \xrightarrow{\Lambda} & X \times I \times I & \xrightarrow{\Phi} & X \times I. \end{array}$$

Problem 2. (The Hopf invariant) Let $k \geq 2$ and let $f : S^{2k-1} \longrightarrow S^k$ be a map. Then the cofiber $C(f)$ has a natural CW structure with cells in dimensions 0, k, and $2k$. As $k \geq 2$, the cellular chain complex has trivial differentials, and the cohomology of $C(f)$ is $\mathbb{Z}$ in dimensions 0, k, and $2k$. Let x be the generator of $\mathrm{H}^k(C(f))$ corresponding to the top cell of S^k, and let y be the

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generator of $\mathrm{H}^{2k}(C(f))$ coresponding to the cell attached via f. Then $x^2 = h(f)y$ for some integer $h(f)$, which is called the **Hopf invariant** of the map f.

(i) Show that if k is odd, then $h(f) = 0$.

(ii) Let $\eta : S^3 \longrightarrow S^2$ be the Hopf map. Show that $h(\eta) = 1$.

(iii) Let $k = 2n$ and let $\iota : S^{2n} \longrightarrow S^{2n}$ be the identity map. Show that the Whitehead product $[\iota, \iota] \in \pi_{4n-1}(S^{2n})$ has Hopf invariant 2. (Hint: Use the diagram

$$\begin{array}{ccccc} S^{2n} \vee S^{2n} & \longrightarrow & S^{2n} \times S^{2n} & \longrightarrow & S^{4n} \\ \downarrow & & \downarrow & & \downarrow \\ S^{2n} & \longrightarrow & C([\iota, \iota]) & \longrightarrow & S^{4n} \end{array}$$

to compute $h([\iota, \iota])$.

By part (iii) and the fact that the Hopf invariant is a homomorphism (see Hatcher, 4B.1), for each n, the map $h : \pi_{4n-1}(S^{2n}) \longrightarrow \mathbb{Z}$ is either surjective or has image $2\mathbb{Z}$. In either case, we have a surjective homomorphism from $\pi_{4n-1}(S^{2n})$ onto an infinite cyclic group, and such a map always has a section. We conclude that $\pi_{4n-1}(S^{2n})$ always has a summand of $\mathbb{Z}$. J. P. Serre proved that the complement is finite and that every other homotopy group of spheres is finite. Frank Adams proved that maps of Hopf invariant one exist only for $k = 2n = 2$, 4, or 8 (examples are the Hopf maps η, ν, and σ).

Problem 3. Let $\eta \in \pi_3(S^2)$ be the Hopf map and let $\iota \in \pi_2(S^2)$ be correspond to the identity of S^2. Show that for any integers $a, b \in \mathbb{Z}$ we have

$$(a\iota) \circ (b\eta) = a^2 b\eta.$$

This shows that the map $\pi_2(S^2) \longrightarrow \pi_3(S^2)$ induced by precomposition with η is not a homomorphism.

Hint: Consider a commutative diagram

$$\begin{array}{ccccc} S^2 & \longrightarrow & C(b\eta) & \longrightarrow & S^4 \\ \downarrow{\scriptstyle a\iota} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ S^2 & \longrightarrow & C(b\eta \circ a\iota) & \longrightarrow & S^4 \end{array}$$

and the Hopf invariants.