CLASS NOTES
MATH 527 (SPRING 2011)
WEEK 10

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1. Mon, Mar. 28

At the end of class last time, we were discussing Eilenberg-Mac Lane spaces $K(G, n)$. Here is how one way to build these "by hand":

Let $n \geq 2$ and suppose G is abelian. Write G as a cokernel

$$\mathbb{Z}^{n_2} \longrightarrow \mathbb{Z}^{n_1} \longrightarrow G \longrightarrow 0$$

of a map of free abelian groups. We then define the n-skeleton to be $X_n = \bigvee_{n_1} S^n$. We define the $(n+1)$-skeleton as the cofiber of the map

$$\bigvee_{n_2} S^n \longrightarrow \bigvee_{n_1} S^n = X_n \longrightarrow X_{n+1}.$$

The cofiber X_{n+1} is then a "Moore space" of type $M(G, n)$, meaning that its only nontrivial reduced homology group is $\mathrm{H}_n(X_{n+1}) \cong G$. By the Hurewicz theorem, we also get $\pi_n(X_{n+1}) \cong G$. However, the cofiber X_{n+1} may have nontrivial homotopy in degrees $n+1$ and higher. We thus attach $n+2$-cells to kill π_{n+1}. The result may still have homotopy in degrees $n+2$ and higher, so we attach $n+3$-cells to kill π_{n+2}. Attaching cells to kill all higher homotopy groups, we arrive at a CW complex with the desired homotopy groups.

Although the above describes a construction of a space $K(G, n)$, the model for $K(G, n)$ specified above does not have as much structure as one might wish. There are constructions of spaces $K(G, n)$ which produce topological abelian groups. Here is one such, due to McCord:

Let A be an abelian group and let X be a space. Let $B(X, A)$ be the set of functions $X \longrightarrow A$ which are nonzero at only finitely points. This is an abelian group under pointwise addition. We topologize the space $B(X, A)$ as follows. For any $k \geq 1$, let $B_k(X, A)$ be the set of functions $X \longrightarrow A$ which are nonzero at at most k points. For $k = 1$, we have a surjection $X \times A \twoheadrightarrow B_1(X, A)$ which sends a pair (x, a) to the function $a_x : X \longrightarrow A$, where

$$a_x(y) = \begin{cases} a & y = x \\ 0 & y \neq x. \end{cases}$$

We topologize $B_1(X, A)$ as the quotient of $X \times A$. More generally, for any k, we have a surjection $(X \times A)^k \twoheadrightarrow B_k(X, A)$ defined by

$$((x_1, a_1), \ldots, (x_k, a_k)) \mapsto (a_1)_{x_1} + \cdots + (a_k)_{x_k}.$$

We topologize $B_k(X, A)$ as the above quotient space. The space $B(X, A)$ is the union of the spaces $B_k(X, A)$, and we give $B(X, A)$ the topology of the union.

We claim that $B(X, A)$ becomes a topological abelian group. That is, the addition and inverse maps are continuous. For the addition, it suffices to show that for any n and k, the addition $B_n(X, A) \times B_k(X, A) \longrightarrow B_{n+k}(X, A)$ is continuous. But we have the diagram

$$\begin{array}{ccc} B_n(X,A)\times B_k(X,A) & \longrightarrow & B_{n+k}(X,A) \\ \uparrow & & \uparrow \\ (X\times A)^n \times (X\times A)^k & \xrightarrow{\cong} & (X\times A)^{n+k} \end{array}$$

Continuity of the top horizontal arrow follows from the following

Lemma 1.1 (Point-set topological lemma, (Strickland, 2.20))**.** *If $f_1 : X_1 \longrightarrow Y_1$ and $f_2 : X_2 \longrightarrow Y_2$ are quotient maps, then so is the product $f_1 \times f_2$.*

Similarly, to see that the inverse is continuous, it is enough to see that the restriction $B_k(X, A) \longrightarrow B_k(X, A)$ is continuous. Again, we have the diagram

$$\begin{array}{ccc} B_k(X,A) & \longrightarrow & B_k(X,A) \\ \uparrow & & \uparrow \\ (X\times A)^k & \xrightarrow{\cong} & (X\times A)^k \end{array}$$

with both vertical maps quotient maps.

Remark 1.2. There is another way to think about the above construction, as a "tensor product of functors". Let $\mathbb{F}$ denote the category of finite sets. Then any space X determines a contravariant functor $X^{(-)} : \mathbb{F} \longrightarrow \mathbf{Top}$ by $X^{(\mathbf{n})} = \mathrm{Map}(\mathbf{n}, X)$. If A is an abelian group, then A determines a covariant functor $A[-] : \mathbb{F} \longrightarrow \mathbf{Top}$. The (discrete) space $A[\mathbf{n}]$ is $\oplus_{\mathbf{n}} A$, and given a function $\phi : \mathbf{n} \longrightarrow \mathbf{k}$, the map $A[\phi]$ is defined by

$$A[\phi](a_1, \ldots, a_n) = (b_1, \ldots, b_k), \qquad b_i = \sum_{j\in\phi^{-1}(i)} a_j.$$

In general, given a contravariant functor $G : \mathbb{F}^{op} \longrightarrow \mathbf{Top}$ and a covariant functor $H : \mathbb{F} \longrightarrow \mathbf{Top}$, one can form their "tensor product" $G \otimes_{\mathbb{F}} H$ as the coequalizer (colimit) of the diagram

$$\coprod_{\phi:\mathbf{n}\longrightarrow\mathbf{k}} G(\mathbf{k}) \times H(\mathbf{n}) \rightrightarrows \coprod_{\mathbf{n}} G(\mathbf{n}) \times H(\mathbf{n}) \longrightarrow G \otimes_{\mathbb{F}} H$$

Taking $G = X^{(-)}$ and $H = A[-]$, we get a topological space which can be identified with the space $B(X, A)$ from above.

To see how the above coequalizer works, consider the functions $\phi : \mathbf{2} \longrightarrow \mathbf{1}$ and $\iota_1 : \mathbf{1} \hookrightarrow \mathbf{2}$. Then $X^\phi : X \longrightarrow X^2$ is the diagonal and $A[\phi] : A \oplus A \longrightarrow A$ is the addition. The coequalizer identifies a point of the form (x, x, a_1, a_2) in $X^2 \times A[2]$ with the point $(x, a_1 + a_2)$ in $X \times A[1]$. Similarly, the map $X^{\iota_1} : X^2 \longrightarrow X$ is the projection onto the first factor, and $A[\iota_1] : A \longrightarrow A \oplus A$ is the map $a \mapsto (a, 0)$. So the coequalizer will identify a point of the form $(x_1, x_2, a, 0)$ in $X^2 \times A[2]$ with the point (x_1, a) in $X \times A[1]$.

What does all of this have to do with Eilenberg-Mac Lane spaces? The claim is that $B(S^n, A)$ will be a model for $K(A, n)$. Actually, we modify the above construction slightly: if X is a based space, then we define $\tilde{B}(X, A) = B(X, A)/B(*, A)$ as the functions taking value 0 at the basepoint.

2. Wed., Mar. 30

Note that $\tilde{B}(X, A)$ is a covariant functor in the variable X. The cofiber sequence $S^{n-1} \hookrightarrow D^n \longrightarrow S^n$ of spaces gives a sequence of topological abelian groups

$$\tilde{B}(S^{n-1}, A) \xrightarrow{B(\iota)} \tilde{B}(D^n, A) \xrightarrow{B(q)} \tilde{B}(S^n, A).$$

Claim. $\tilde{B}(S^{n-1}, A)$ is the fiber of the map $B(q) : \tilde{B}(D^n, A) \longrightarrow \tilde{B}(S^n, A)$.

Proof. Suppose that $B(q)(\alpha) = B(q)(\beta)$. We can write $\alpha = \alpha_{n-1} + \alpha_n$, where α_{n-1} is supported on S^{n-1} and α_n is supported on the interior of D^n. There is an analogous decomposition $\beta = \beta_{n-1} + \beta_n$. The assumption that $B(q)(\alpha) = B(q)(\beta)$ means that $\alpha_n = \beta_n$. Thus $\beta - \alpha = \beta_{n-1} - \alpha_{n-1}$ comes from $\tilde{B}(S^{n-1}, A)$. ■

Even better, this map is a quasi-fibration, in the sense that the fiber is weakly equivalent to the homotopy fiber. One way to show this is to use the Dold-Thom technology (see Hatcher, Appendix 4K).

$\tilde{B}(X, A)$ preserves based homotopy equivalences, so $\tilde{B}(D^n, A)$ is contractible. It follows that $\tilde{B}(S^{n-1}, A) \simeq \Omega\tilde{B}(S^n, A)$.

Lemma 2.1. *If X is path-connected, then $B(X, A)$ and $\tilde{B}(X, A)$ are path-connected.*

The space $\tilde{B}(S^0, A)$ is homeomorphic to A, and by induction we now deduce that $\tilde{B}(S^n, A)$ is a $K(A, n)$.

Eilenberg-Mac Lane spaces and cohomology

By the Hurewicz theorem, we know that $\mathrm{H}_n(K(A, n); \mathbb{Z}) \cong A$. The universal coefficients theorem gives, for any X, a split exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(\mathrm{H}_{n-1}(X), A) \longrightarrow \mathrm{H}^n(X; A) \longrightarrow \mathrm{Hom}(\mathrm{H}_n(X), A) \longrightarrow 0.$$

Taking $X = K(A, n)$, then $\mathrm{H}_{n-1}(X)$ vanishes (if $n \geq 2$), so that

$$\mathrm{H}^n(K(A, n); A) \cong \mathrm{Hom}(\mathrm{H}_n(K(A, n)), A) \cong \mathrm{Hom}(A, A).$$

In particular, the identity map of A corresponds to a distinguished element $u \in \mathrm{H}^n(K(A, n); A) \cong \tilde{\mathrm{H}}^n(K(A, n); A)$.

Let X be a based space. Then any based map $f : X \longrightarrow K(A, n)$ induces a map

$$f^* : \tilde{\mathrm{H}}^n(K(A, n); A) \longrightarrow \tilde{\mathrm{H}}^n(X; A).$$

Proposition 2.2. *For any based CW complex X, the map $[X, K(A, n)] \longrightarrow \tilde{\mathrm{H}}^n(X; A)$ sending f to $f^*(u)$ is a bijection (actually, an isomorphism of abelian groups).*

Proof. We will write $Y = K(A, n)$. Consider the commutative square, in which the horizontal maps are induced by the inclusion $X_{n+1} \hookrightarrow X$ of the $(n + 1)$-skeleton of X:

$$\begin{tikzcd} {[X, Y]} \arrow[r] \arrow[d] & {[X_{n+1}, Y]} \arrow[d] \\ \mathrm{H}^n(X; A) \arrow[r] & \mathrm{H}^n(X_{n+1}; A). \end{tikzcd}$$

The bottom horizontal map is an isomorphism (this is clear from the cellular cochain complexes). We claim that the top horizontal map is also a bijection.

For surjectivity, suppose given a map $f_{n+1} : X_{n+1} \longrightarrow Y$. Can we extend f_{n+1} to a map $f_{n+2} : X_{n+2} \longrightarrow Y$? On each $(n + 2)$-cell, we already have a map to Y defined on the boundary of the cell, and we need to know if this map on the boundary is null. But $\pi_{n+1}(Y) = 0$, so such a map is necessarily null, and we can define the extension f_{n+2} on each cell. Clearly, the same argument works to extend f_{n+1} all the way up the skeletal filtration of X.

Similarly, suppose we have a map $f : X \longrightarrow Y$ such that the restriction to X_{n+1} is null. Can we extend the null homotopy to a null homotopy of f on X? We first consider and extension to X_{n+2}. Again, it suffices to define the null homotopy on each cell. Glueing the map f_{n+2} on a given cell α to the null homotopy h on the boundary $\partial\alpha$ defines a map $S^{n+2} \longrightarrow Y$. Since $\pi_{n+2}(Y) = 0$, this extends to a map $D^{n+2} \longrightarrow Y$, i.e. a null homotopy of the map on the $n + 2$-cell. Similarly, we can extend the null homotopy over all higher cells, proving injectivity.

We have thus reduced to the statement in the case that $X = X_{n+1}$ is of dimension $n+1$. The sorts of arguments we have just employed are closely related to "Obstruction Theory", which we will need in the remainder of the proof.

The first result from obstruction theory concerns the extension of a map $f_n : X_n \longrightarrow Y$ to a map $f_{n+1} : X_{n+1} \longrightarrow Y$. As we have just seen, what is needed is, for each attaching map $\alpha : S^n \longrightarrow X_n$ for an $(n+1)$-cell of X, a null homotopy for $f \circ \alpha : S^n \longrightarrow Y$. Here is another interpretation of this statement: composition with f defines a map (homomorphism) $C_{n+1}(X) \longrightarrow \pi_n(Y)$, i.e. an $(n+1)$-cochain $c(f) \in C^{n+1}(X; \pi_n(Y))$. The map f_n extends to f_{n+1} precisely when the cochain $c(f)$ vanishes.

Certainly, if X is $(n+1)$-dimensional, the cochain $c(f)$ is a cocycle, but this is in fact true in general.

Proposition 2.3. *There is a map $g : X_{n+1} \longrightarrow Y$ with $g|_{X_{n-1}} = f_{n-1}$ if and only if the class $[c(f)]$ vanishes in $\mathrm{H}^{n+1}(X; \pi_n(Y))$.*

We will give a proof of this statement next time. ■

3. Fri, Apr. 1

We began to prove the following result:

Proposition 3.1. *For any based CW complex X, the map $[X, K(A,n)] \longrightarrow \tilde{\mathrm{H}}^n(X; A)$ sending f to $f^*(u)$ is a bijection (actually, an isomorphism of abelian groups).*

Proof. Last time, we were able to reduce this to the case of $X = X_{n+1}$. Consider the cofiber sequence $\bigvee S^n \longrightarrow X_n \longrightarrow X_{n+1}$. This induces a commutative diagram

$$\begin{tikzcd}
0 = [\bigvee S^{n+1}, Y] \arrow[r] \arrow[d] & {[X_{n+1}, Y]} \arrow[r] \arrow[d] & {[X_n, Y]} \arrow[r] \arrow[d] & {[\bigvee S^n, Y]} \arrow[d, "\cong"] \\
0 = \tilde{\mathrm{H}}^n(\bigvee S^{n+1}; A) \arrow[r] & \tilde{\mathrm{H}}^n(X_{n+1}; A) \arrow[r] & \tilde{\mathrm{H}}(X_n; A) \arrow[r] & \tilde{\mathrm{H}}^n(\bigvee S^n; A).
\end{tikzcd}$$

If we can show that $[X_n, Y] \longrightarrow \tilde{\mathrm{H}}^n(X_n; A)$ is an isomorphism, this will finish the proof. We will do this by identifying both terms with a third set. Let $[\![X_n, Y]\!]$ denote the set of homotopy classes, rel X_{n-2}, of maps $f_n : X_n \longrightarrow Y$ such that f_{n-1} is constant at the basepoint of Y.

There is a natural map $[\![X_n, Y]\!] \longrightarrow [X_n, Y]$ which takes the class of f to the class of f. We first establish that this natural map is surjective. Let $f : X_n \longrightarrow Y$. Since we may take a CW model for Y in which the $(n-1)$-skeleton is a point, the restriction of f to X_{n-1} is null. Now we can use the homotopy extension property with respect to the inclusion $X_{n-1} \hookrightarrow X_n$ to extend the null homotopy of f_{n-1} to a homotopy $f \simeq g$ with g_{n-1} constant.

Injectivity of the map $[\![X_n, Y]\!] \longrightarrow [X_n, Y]$ takes a little more work, but we have already seen the idea when we proved that a homotopy equivalence between nondegenerately based spaces is a based homotopy equivalence. Suppose given a map $f : X_n \longrightarrow Y$ such that $f_{n-1} = *$ and such that f is null. Let $h : f \simeq *$ be the given null homotopy. We wish to find a null homotopy rel X_{n-2}.

Let us work for the moment at the level of the $(n-2)$-skeleton X_{n-2}. We have the given homotopy $h : X_{n-2} \times I \longrightarrow Y$, which is a possibly nonconstant homotopy between constant maps. Define a map $X_{n-2} \times \partial I^2 \longrightarrow Y$, where the map h is used on $I \times \{0\}$, and the constant homotopy is used on the rest of the boundary. Does this map extend to a map $H : X_{n-2} \times I^2 \longrightarrow Y$? We are trying to extend a map from an $(n-1)$-dimensional complex to an n-dimensional complex, and the obstruction cochain lives in the group $C^n(X_{n-2} \times I^2, \pi_{n-1}(Y))$. The obstruction vanishes because $\pi_{n-1}(Y) = 0$.

We may use the homotopy extension property to obtain a lift in the diagram

$$\begin{tikzcd} X_{n-2} \times I \arrow[r, "H"] \arrow[d] & Y^I \arrow[d, "ev"] \\ X_n \times I \arrow[r, "h"] \arrow[ur, dashed, "\tilde{H}"] & Y \end{tikzcd}$$

Then the concatenation $\tilde{H}(1,t) * \tilde{H}(s,1) * \tilde{H}(0,t)$ is a null homotopy for f that restricts to the constant homotopy on X_{n-2}. This establishes the isomorphism $[\![X_n, Y]\!] \cong [X_n, Y]$.

It remains to compare $[\![X_n, Y]\!]$ to $\tilde{\mathrm{H}}^n(X_n; A)$. Note that since X_n is n-dimensional, every n-cochain is a cocycle. To proceed further, we will need to return once again to obstruction theory.

The second result from obstruction theory concerns a pair of maps $f, g : W_n \longrightarrow Z$ such that $f_{n-1} = g_{n-1}$. On an n-cell of W_n, the maps f and g specify two maps $D^n \longrightarrow Z$ which agree on the boundary. Glueing these together defines a map $S^n \longrightarrow Z$. This map $C_n(W) \longrightarrow \pi_n(Z)$ is the "difference cochain" $d(f,g) \in C^n(W; \pi_n(Z))$. We will need the following facts about the difference cochain construction:

- $\partial d(f,g) = c(f) - c(g)$
- $d(f,g) + d(g,h) = d(f,h)$. It is clear that the maps f and g are homotopic rel W_{n-1} if and only if this difference cochain vanishes, but once again there is a cohomological criterion:

  Proposition 3.2. *Suppose given maps $f, g : W \longrightarrow Z$ such that $f_{n-1} = g_{n-1}$. Then $f_n \simeq g_n$ rel W_{n-2} if and only if $[d(f,g)] = 0$ in $\tilde{\mathrm{H}}^n(W; \pi_n(Z))$.*

Returning to the proof, to a map $f : X_n \longrightarrow Y$ which is constant on X_{n-1}, we associate the class $d(f, *) \in C^n(X_n; \pi_n(Y))$. Suppose that $[d(f,*)] = [d(g,*)]$ for some other $g \in [\![X_n, Y]\!]$. By the above additivity result, this is equivalent to the statement $[d(f,g)] = 0$. But this is precisely the obstruction to the existence of a homotopy $f \simeq g$ rel W_{n-2}, so we have injectivity.

Let $v \in \tilde{\mathrm{H}}^n(X_n; A)$, and let z be a representing cocycle for the class v. We want to write $v = [d(f,*)]$ for some $f : X_n \longrightarrow Y$ which is constant on X_{n-1}. But the quotient X_n/X_{n-1} is a wedge $\bigvee S^n$ indexed by the n-cells of X, and we take the map $\bigvee S^n \longrightarrow Y$ specified by $z(\alpha)$ on the summand corresponding to an n-cell α. By construction, $d(f,*) = z$, so we are done. ■

To remove a bit of the mystery, let's prove the first obstruction theory result from last time:

Proposition 3.3. *Let Y be a connected simple space, and suppose given a map $f : X_n \longrightarrow Y$. There is a map $g : X_{n+1} \longrightarrow Y$ with $g|_{X_{n-1}} = f_{n-1}$ if and only if the class $[c(f)]$ vanishes in $\mathrm{H}^{n+1}(X; \pi_n(Y))$.*

The simple assumption is there to make sure we don't have to worry about choices of basepoints for Y.

Proof. ($\Rightarrow$) Suppose that we have $g : X_{n+1} \longrightarrow Y$. Thus the cochain $c(g_n)$ vanishes. But $f_{n-1} = g_{n-1}$, so we may form $d(f,g) \in C^n(X; \pi_n(Y))$ with $\partial d(f,g) = c(f) - c(g) = c(f)$. Thus the cocycle $c(f)$ is a coboundary, so the class $[c(f)]$ vanishes.

($\Leftarrow$) Suppose that there is $d \in C^n(X; \pi_n(Y))$ with $\partial(d) = c(f)$. Then we build a map $g : X_n \longrightarrow Y$ such that $g_{n-1} = f_{n-1}$ and $d = d(f,g)$. On each n-cell $\Phi : D^n \longrightarrow X_n$, the map g is defined so that the resulting map $S^n \xrightarrow{f \cup g} Y$ is (a representative for) $d(\Phi) \in \pi_n(Y)$. Then $c(f) = \partial d = \partial d(f,g) = c(f) - c(g)$, so $c(g) = 0$. Thus g extends to X_{n+1}. ■