CLASS NOTES
MATH 527 (SPRING 2011)
WEEK 12

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1. Mon, Apr. 11

Last time, we proved Moore's result that topological abelian monoids split as products of Eilenberg-Mac Lane spaces. It is worth emphasizing that all of the hypotheses are needed. For instance, S^3 is a non-abelian topological group. Let's show that it has some nontrivial k-invariants.

The Postnikov tower for S^3 begins

$$\begin{tikzcd}
& P_4(S^3) \arrow[d] & \\
S^3 \arrow[ur] \arrow[r] \arrow[dr] & P_3(S^3) = K(Z,3) \arrow[r, "k_3"] \arrow[d] & K(\mathbb{Z}/2.5) \\
& P_1(S^3) = P_2(S^3) = * &
\end{tikzcd}$$

It turns out that $\mathrm{H}^5(K(\mathbb{Z},3);\mathbb{Z}/2) \cong \mathbb{Z}/2$, so the k-invariant k_3 is either the nontrivial class or the trivial class. Suppose $k_3 = 0$. If yes, then

$$P_4(S^3) \simeq K(\mathbb{Z},3) \times K(\mathbb{Z}/2,4).$$

Let's consider homology. $P_4(S^3)$ is built from S^3 by attaching cells of dimension 6 and greater. It follows that $\mathrm{H}_4(P_4(S^3)) \cong \mathrm{H}_4(S^3) = 0$. But

$$\mathrm{H}_4(K(\mathbb{Z},3) \times K(\mathbb{Z}/2,4)) \cong \mathbb{Z}/2 \neq 0.$$

It follows that k_3 must be nontrivial.

Localization and Completion

We now turn to a convenient calculational framework. Most computations in homotopy theory these days are done "one prime at a time".

Let's begin with localization of abelian groups. Let p be a fixed prime (we allow $p = 0$). We say an abelian group A is **p-local** if it is a $\mathbb{Z}_{(p)}$-module. This is equivalent to requiring that, for all primes $q \neq p$, multiplication by q, $A \xrightarrow{q\cdot} A$ is an isomorphism.

Example 1.1. The basic examples are $A = \mathbb{Z}_{(p)}$ and $A = \mathbb{F}_p$.

Definition 1.2. A **localization** at p of an abelian group A is a universal map $A \xrightarrow{\phi} A_{(p)}$ such that $A_{(p)}$ is p-local.

The map $A \longrightarrow A \otimes \mathbb{Z}_{(p)}$ satisfies the universal property and is thus **the** localization.

Let BA denote a space of the homotopy type $BA \simeq K(A,1)$.

Theorem 1.3. *The localization map $\phi : A \longrightarrow A_{(p)}$ induces an isomorphism*

$$\mathrm{H}_*(BA;\mathbb{Z}_{(p)}) \xrightarrow{\cong} \mathrm{H}_*(BA_{(p)};\mathbb{Z}_{(p)}).$$

If A is p-local then

$$\tilde{\mathrm{H}}_*(BA;\mathbb{Z}) \xrightarrow{\cong} \tilde{\mathrm{H}}_*(BA;\mathbb{Z}_{(p)})$$

is an isomorphism.

Proof. We will show this for $A = \mathbb{Z}/q$ or $A = \mathbb{Z}$.

Let's start with $A = \mathbb{Z}$. Then $BA \simeq S^1$. What is $BA_{(p)}$? Here is one construction of $A_{(p)}$ for any A. Enumerate the primes q_i different from p. Define $r_1 = q_1$, $r_2 = r_1 \cdot q_1 q_2$, $r_3 = r_2 \cdot q_1 q_2 q_3$, etc. Then

$$A_{(p)} \cong \operatorname*{colim}_n (A \xrightarrow{r_1} A \xrightarrow{r_2} A \xrightarrow{r_3} A \xrightarrow{r_4} \dots).$$

In analogy, we define

$$Y = \operatorname*{hocolim}_n (S^1 \xrightarrow{r_1} S^1 \xrightarrow{r_2} S^1 \xrightarrow{r_3} S^1 \xrightarrow{r_4} \dots).$$

One can then use the Van Kampen theorem to deduce that $\pi_1(Y) \cong \pi_1(Y)_{(p)}$. Also, $\pi_n(Y) = 0$ for $n > 1$ since any map $S^n \longrightarrow Y$ must land in a *finite* stage of the homotopy colimit. So $Y \simeq BA_{(p)}$. Finally, $\mathrm{H}(-;\mathbb{Z}_{(p)})$ commutes with (homotopy) colimits, so the map on homology is

$$\tilde{\mathrm{H}}_*(BA;\mathbb{Z}_{(p)}) \longrightarrow \tilde{\mathrm{H}}_*(\operatorname*{hocolim}_n BA;\mathbb{Z}_{(p)}) \cong \operatorname*{colim}_n \tilde{\mathrm{H}}_*(BA;\mathbb{Z}_{(p)}).$$

But the maps $r_n : \tilde{\mathrm{H}}_1(BA;\mathbb{Z}_{(p)}) \longrightarrow \tilde{\mathrm{H}}_1(BA;\mathbb{Z}_{(p)})$ are all isomorphisms, so we are done.

Now let $A = \mathbb{Z}/q$. The statement is clear if $q = p$, so assume $q \neq p$. Then $A_{(p)} = 0$, so we will need to show that $\tilde{\mathrm{H}}_*(BA;\mathbb{Z}_{(p)}) = 0$. If $q = 2$, then $BA \simeq \mathbb{RP}^\infty$ and

$$\tilde{\mathrm{H}}_n(BA;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2 & n \text{ odd} \\ 0 & \text{else.} \end{cases}$$

Similarly, if q is odd, BA is a Lens space, and

$$\tilde{\mathrm{H}}_n(BA;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/q & n \text{ odd} \\ 0 & \text{else.} \end{cases}$$

Universal coefficients now says that

$$\tilde{\mathrm{H}}_n(BA;\mathbb{Z}_{(p)}) \cong \tilde{\mathrm{H}}_n(BA;\mathbb{Z}) \otimes \mathbb{Z}_{(p)} \oplus \mathrm{Tor}(\tilde{\mathrm{H}}_{n-1}(BA;\mathbb{Z}), \mathbb{Z}_{(p)}).$$

The Tor group vanishes since $\mathbb{Z}_{(p)}$ is flat. Again, we win since $\mathbb{Z}/q \otimes \mathbb{Z}_{(p)} \cong 0$ if $q \neq p$.

For the second statement, if $A = \mathbb{Z}/p$, then all nonzero groups $\tilde{\mathrm{H}}_*(BA;\mathbb{Z})$ are p-local, so we are done. If $A = \mathbb{Z}_{(p)}$, the above computation shows that

$$\tilde{\mathrm{H}}_*(B\mathbb{Z}_{(p)};\mathbb{Z}) \cong \tilde{\mathrm{H}}_*(B\mathbb{Z};\mathbb{Z}) \otimes \mathbb{Z}_{(p)},$$

so $\tilde{\mathrm{H}}_*(B\mathbb{Z}_{(p)};\mathbb{Z})$ is already p-local. ∎

The following statement now follows from universal coefficients.

Corollary 1.4. *The induced map*

$$\mathrm{H}^*(BA_{(p)};M) \longrightarrow \mathrm{H}^*(BA;M)$$

is an isomorphism for all p-local M.

2. Wed., Apr. 13

Last time we discussed localization of abelian groups and the classifying spaces of p-local abelian groups. Again, p is a fixed prime.

Definition 2.1. (1) We say that a map of space $f : X \longrightarrow Y$ is a **p-equivalence** if

$$f_* : \mathrm{H}_*(X; \mathbb{Z}_{(p)}) \xrightarrow{\cong} \mathrm{H}_*(Y; \mathbb{Z}_{(p)})$$

is an isomorphism.

(2) We say Z is **p-local** if for all p-equivalences $f : X \longrightarrow Y$, the induced map

$$f^* : [Y, Z] \longrightarrow [X, Z]$$

is a bijection.

(3) A **p-localization** of X is a map $X \xrightarrow{\phi} X_{(p)}$ that is a p-equivalence and such that $X_{(p)}$ is p-local.

Remark 2.2. The definition of p-equivalence is equivalent to requiring that $\mathrm{H}^*(Y; M) \longrightarrow \mathrm{H}^*(X; M)$ is an isomorphism for all p-local M.

In fact, more generally, for any PID R, a map $f : X \longrightarrow Y$ induces an isomorphism $\mathrm{H}_*(X; R) \longrightarrow \mathrm{H}_*(Y; R)$ if and only if it induces isomorphisms $\mathrm{H}^*(Y; M) \longrightarrow \mathrm{H}^*(X; M)$ for all R-modules M. The forward implication is a simple application of universal coefficients. For the reverse implication, let $C = \mathrm{cofib}(f)$ and consider the R-module $M = \mathrm{H}_n(C; R)$. Then universal coefficients gives an isomorphism

$$\mathrm{H}^n(C; M) \cong \mathrm{Hom}(H_n(C; R), M) \oplus \mathrm{Ext}^1(\mathrm{H}_{n-1}(C; R), M).$$

Assuming that f induces a cohomology isomorphism, it follows that $\mathrm{H}^n(C; M) = 0$. In particular,

$$\mathrm{Hom}(H_n(C; R), M) = \mathrm{Hom}(H_n(C; R), \mathrm{H}_n(C; R)) = 0,$$

so $\mathrm{H}_n(C; R)$ must be 0. Since this is valid for all n, we may deduce that $\mathrm{H}_*(X; R) \longrightarrow \mathrm{H}_*(Y; R)$ is an isomorphism.

Remark 2.3. It follows from the definition that the p-localization of a space is unique (up to homotopy). For suppose $\phi_1 : X \longrightarrow X_{(p)}$ and $\phi_2 : X \longrightarrow X'_{(p)}$ are both p-localizations. Then, since $X'_{(p)}$ is p-local, the p-equivalence ϕ_1 induces a bijection

$$\phi_1^* : [X_{(p)}, X'_{(p)}] \xrightarrow{\cong} [X, X'_{(p)}].$$

Thus there must be a map $\psi : X_{(p)} \longrightarrow X'_{(p)}$ such that $\psi \circ \phi_1 \simeq \phi_2$. Reversing the roles of $X_{(p)}$ and $X'_{(p)}$, we get a map in the other direction, and the fact that $X_{(p)}$ and $X'_{(p)}$ are p-local allows us to deduce that they are homotopy inverses.

Proposition 2.4. *An abelian group A is p-local if and only if the space BA is p-local.*

Proof. ($\Rightarrow$) Let $f : X \longrightarrow Y$ be a p-equivalence. Then, by the definition of p-equivalence, the bottom horizontal map in the diagram

$$\begin{tikzcd}
{[Y, BA]} \arrow[r] \arrow[d, "\cong"'] & {[X, BA]} \arrow[d, "\cong"] \\
\mathrm{H}^1(Y; A) \arrow[r, "\cong"] & \mathrm{H}^1(X; A)
\end{tikzcd}$$

is an isomorphism. It follows that BA is p-local.

($\Leftarrow$) Clearly the identity map $\mathrm{id} : BA \longrightarrow BA$ is a p-localization if BA is already p-local. Now let $\varphi : A \longrightarrow A_{(p)}$ be an algebraic p-localization. Then we saw last time that $BA \longrightarrow B(A_{(p)})$ is a

p-equivalence. By the implication already shown, we know $B(A_{(p)})$ is p-local, so $BA \longrightarrow B(A_{(p)})$ is also a p-localization. It follows that $BA \simeq B(A_{(p)})$, so that $A \cong A_{(p)}$. ■

More generally, the following is true.

Theorem 2.5. *For any abelian group A and $n \geq 0$, the map $K(A, n) \longrightarrow K(A_{(p)}, n)$ is a p-localization.*

So far, we have only given the construction of $X_{(p)}$ in limited cases, namely when $X = BA$.

Theorem 2.6. *Any simple space X admits a p-localization.*

Proof. The main idea is to replace X be a (principal) Postnikov tower. We will show by induction that the nth stage $P_n(X)$ admits a p-localization.

At the bottom (assuming X connected), we have $X \longrightarrow K(\pi_1(X), 1)$. We have just shown that $K(\pi_1(X)_{(p)}, 1)$ is a model for $K(\pi_1(X), 1)_{(p)}$. Assume thus that we have already built a localization $P_n(X)_{(p)}$. Consider the following diagram:

$$\begin{tikzcd}
K(\pi_{n+1}X, n+1) \arrow[r, "\Omega\phi_n"] \arrow[d] & K(\pi_{n+1}(X)_{(p)}, n+1) \arrow[d, dashed] \\
P_{n+1}(X) \arrow[r, dashed, "\Phi_{n+1}"] \arrow[d] & P_{n+1}(X)_{(p)} \arrow[d, dashed] \\
P_n(X) \arrow[r, "\Phi_n"] \arrow[d, "k_n"'] & P_n(X)_{(p)} \arrow[d, dashed, "(k_n)_{(p)}"] \\
K(\pi_{n+1}(X), n+2) \arrow[r, "\phi_n"'] & K(\pi_{n+1}(X)_{(p)}, n+2)
\end{tikzcd}$$

The map $(k_n)_{(p)}$ exists since Φ_n is a p-equivalence and since $K(\pi_{n+1}(X)_{(p)}, n+2)$ is p-local. The space $P_{n+1}(X)_{(p)}$ is then defined to be the fiber of $(k_n)_{(p)}$.

It remains to show that Φ_{n+1} is a p-localization. The fact that Φ_{n+1} is a p-equivalence is an easy spectral sequence argument, given that Φ_n and ϕ_n are p-equivalences. The fact that $P_{n+1}(X)_{(p)}$ is p-local follows from a "dual Whitehead" result. Recall that we can think of Postnikov towers as cocellular objects, built using homotopy limits out of cocells (Eilenberg-Mac Lane spaces). The analogue of a weak equivalence in this context (determined by mapping cells in) is an H^*-isomorphism (determined by mapping to cocells). Since we are using only p-local cocells (that is, $K(A, n)$ with A p-local), the relevant H^*-isomorphisms are precisely the p-equivalences. Dualizing the arguments used to prove the Whitehead theorem for CW complexes (that is, one should prove a coHELP theorem, etc.) one gets precisely the statement that cocellular objects are p-local.

By induction, we have now produced a p-localization $P_n(X) \longrightarrow P_n(X)_{(p)}$ in such a way that the $P_n(X)_{(p)}$ assemble into a tower, and we define $X_{(p)} := \operatorname{holim}_n P_n(X)_{(p)}$. Again, the dual Whitehead theorem says that this is indeed a p-local space. That the map $\mathrm{H}_n(X; \mathbb{Z}_{(p)}) \longrightarrow \mathrm{H}_n(X_{(p)}; \mathbb{Z}_{(p)})$ is an isomorphism follows from the fact that, for $i \geq n$, the map

$$\mathrm{H}_n(P_i(X); \mathbb{Z}_{(p)}) \longrightarrow \mathrm{H}_n(P_i(X)_{(p)}; \mathbb{Z}_{(p)})$$

is an isomorphism. ■

It turns out that there is also a very convenient description of p-local spaces, given by the following result.

Theorem 2.7. *Let X be simple. Then the following are equivalent*

(1) *X is p-local*

(2) $\pi_n(X)$ *is p-local for all* n
(3) $\tilde{\mathrm{H}}_n(X;\mathbb{Z})$ *is p-local for all* n.

Proof. Today, we will only have time to show (1) $\iff$ (2). Given our construction of localizations using Postnikov towers, this equivalence follows from the following:

Lemma 2.8. *X is p-local* $\iff$ $X \xrightarrow{\sim} X_{(p)}$.

Proof. ($\Rightarrow$) X and $X_{(p)}$ are both p-localizations, so $X \simeq X_{(p)}$ by the uniqueness of localizations.

($\Leftarrow$) If $Y \longrightarrow Z$ is a p-equivalence, then all maps in the following diagram are known to be bijections save for the top horizontal one

$$\begin{tikzcd} {[Z,X]} \arrow[r] \arrow[d, "\cong"'] & {[Y,X]} \arrow[d, "\cong"] \\ {[Z,X_{(p)}]} \arrow[r, "\cong"] & {[Y,X_{(p)}]}. \end{tikzcd}$$

■

We will show (1) $\iff$ (3) next time. ■

3. Fri, Apr. 15

Last time, we proved the equivalence (1) $\iff$ (2) of the following theorem.

Theorem 3.1. *Let X be simple. Then the following are equivalent*

(1) X *is p-local*
(2) $\pi_n(X)$ *is p-local for all* n
(3) $\tilde{\mathrm{H}}_n(X;\mathbb{Z})$ *is p-local for all* n.

Now it's time to finish the job.

Proof. (1) $\Rightarrow$ (3) Assume X is p-local. We will prove by induction on j that $\tilde{\mathrm{H}}_n(P_jX;\mathbb{Z})$ is p-local for all n.

When $j = 1$, we have $P_1X = K(\pi_1(X), 1)$, and we have already shown that $\tilde{\mathrm{H}}_n(K(\pi_1(X), 1);\mathbb{Z})$ is p-local if $\pi_1(X)$ is p-local. Now assume that $\tilde{\mathrm{H}}_n(P_jX;\mathbb{Z})$ is p-local for all n. The Serre spectral sequence for the fiber sequence

$$K(\pi_{j+1}(X), j+1) \longrightarrow P_{j+1}X \longrightarrow P_jX$$

has E_2-term $\mathrm{H}_p(P_j(X);\mathrm{H}_q(K(\pi_{j+1}(X), j+1);\mathbb{Z}))$ and converges to $\mathrm{H}_{p+q}(P_{j+1}X;\mathbb{Z})$.This maps to the spectral sequence for computing the homology with coefficients in $\mathbb{Z}_{(p)}$, and the map on E_2-terms is an isomorphism by assumption. We conclude that the map

$$\tilde{\mathrm{H}}_*(P_{j+1}(X);\mathbb{Z}) \longrightarrow \tilde{\mathrm{H}}_*(P_{j+1}(X);\mathbb{Z}_{(p)})$$

is an isomorphism.

(3) $\Rightarrow$ (1). We wish to show that the map $X \longrightarrow X_{(p)}$ is an equivalence. But all maps in the following diagram except the top horizontal one are known to be isomorphisms:

$$\begin{tikzcd} \tilde{\mathrm{H}}_*(X;\mathbb{Z}) \arrow[r] \arrow[d, "\cong"'] & \tilde{\mathrm{H}}_*(X_{(p)};\mathbb{Z}) \arrow[d, "\cong"] \\ \tilde{\mathrm{H}}_*(X;\mathbb{Z}_{(p)}) \arrow[r, "\cong"] & \tilde{\mathrm{H}}_*(X_{(p)};\mathbb{Z}_{(p)}) \end{tikzcd}$$

Since X is simple, it follows by the Whitehead theorem that $X \xrightarrow{\sim} X_{(p)}$. ■

When $p = 0$, p-local homotopy theory is known as "rational homotopy theory". We will use this to prove Serre's theorem on the finiteness of homotopy groups of spheres. We begin with the following computation.

Proposition 3.2. *Let $z_n \in \mathrm{H}^n((K(\mathbb{Q},n);\mathbb{Q})$ be the canonical class, corresponding to the identity $K(\mathbb{Q},n) \longrightarrow K(\mathbb{Q},n)$. Then $\mathrm{H}^*(K(\mathbb{Q},n);\mathbb{Q})$ is the free graded-commutative $\mathbb{Q}$-algebra on the class z_n. When n is even, this means the polynomial algebra $\mathbb{Q}[z_n]$, and when n is odd, it is the exterior algebra $\mathbb{Q}[z_n]/z_n^2$.*

Proof. We will prove this by induction. Let $n = 1$. Then we know $S^1 = K(\mathbb{Z},1) \longrightarrow K(\mathbb{Q},1)$ is rationalization, so

$$\mathrm{H}(K(\mathbb{Q},1);\mathbb{Q}) \cong \mathrm{H}^*(S^1;\mathbb{Q}) \cong \mathbb{Q}[z_1]/z_1^2.$$

Assume now that $K(\mathbb{Q},n)$ has cohomology algebra as described as in the statement. Consider the path-loop fiber sequence

$$K(\mathbb{Q},n) \longrightarrow PK(\mathbb{Q},n+1) \longrightarrow K(\mathbb{Q},n+1).$$

We treat the n even and n odd cases separately.

n odd: Since $K(\mathbb{Q},n)$ has no cohomology in degrees $0 < i < n$ and since $PK(\mathbb{Q},n+1)$ has no cohomology at all, we see that $d_{n+1}(z_n) = z_{n+1}$. The class $z_n z_{n+1}$ cannot survive to E_∞ and it cannot be the target of any differential. The only possibility is that $d_{n+1}(z_n z_{n+1})$ is nonzero. The Leibniz rule tells us that

$$d_{n+1}(z_n z_{n+1}) = d(z_n)z_{n+1} + (-1)^n z_n d(z_{n+1}) = z_{n+1}^2.$$

More generally, $d_{n+1}(z_n z_{n+1}^j) = (z_{n+1})^{j+1}$, so $(z_{n+1})^{j+1}$ must be nonzero for every j. Moreover, there is no room for more classes in $\mathrm{H}^*(K(\mathbb{Q},n+1);\mathbb{Q})$, as they would survive to E_∞.

n even: Again, we see that $d_{n+1}(z_n) = z_{n+1}$. But this time, $d_{n+1}(z_n^2) = z_n z_{n+1}$, so $z_n z_{n+1}$ does not support any differentials. More generally, $d_{n+1}(z_n^j) = z_n^{j-1} z_{n+1}$. It follows that E_{n+2} is concentrated on the bottom row. This row cannot have any classes, and we conclude that $K(\mathbb{Q},n+1)$ must have nontrivial cohomology only in degree $n+1$. ∎

Theorem 3.3 (Serre)**.** *The homotopy groups $\pi_k(S^n)$ are finite except for $\pi_n(S^n) \cong \mathbb{Z}$, $n \geq 1$, and $\pi_{4n-1}(S^{2n}) \cong \mathbb{Z} \oplus$ finite, $n \geq 1$.*

Proof. Let $\alpha : S^n \longrightarrow K(\mathbb{Z},n)$ correspond to a generator. The fiber $S^n\langle n\rangle$ is the "n-connected cover of S^n". It has the same homotopy groups as S^n in degrees $> n$ but no homotopy in degrees $\leq n$. Consider the diagram

$$\begin{tikzcd}
K(\mathbb{Z},n-1) \arrow[r] \arrow[d] & S^n\langle n\rangle \arrow[r] \arrow[d, dashed] & S^n \arrow[r, "\alpha"] \arrow[d] & K(\mathbb{Z},n) \arrow[d] \\
K(\mathbb{Q},n-1) \arrow[r] & F \arrow[r] & S^n_{(0)} \arrow[r, dashed, "\alpha_{(0)}"] & K(\mathbb{Q},n).
\end{tikzcd}$$

The map $\alpha_{(0)}$ exists because $S^n \longrightarrow S^n_{(0)}$ is a rational equivalence and $K(\mathbb{Q},n)$ is rational (i.e. 0-local). The space F is defined to be the fiber of $\alpha_{(0)}$. The same techniques we have used before show that $F \simeq S^n\langle n\rangle_{(0)}$.

We have already seen that when n is odd, the map $S^n \longrightarrow K(\mathbb{Q},n)$ is rationalization. Thus $\alpha_{(0)}$ is an equivalence, and $F \simeq *$. It follows that for all $k > n$, $\pi_k(S^n) \otimes \mathbb{Q} = 0$. This tells us that $\pi_k(S^n)$ is a torsion group. To conclude that it is finite, we must cite another result of Serre.

Theorem 3.4 (Serre)**.** *Let X be simply connected. Then $\pi_n(X)$ is finitely generated for all n if and only if $\mathrm{H}_n(X;\mathbb{Z})$ is finitely generated for all n.*

We will handle the n even case next time. ∎