CLASS NOTES
MATH 527 (SPRING 2011)
WEEK 3

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1. Mon, Jan. 31

CW complexes are well-behaved as topological spaces:

Theorem 1.1. *Let X be a CW complex. Then*

1. *The components of X are the path-components (Hatcher, A.4)*
2. *If K is a compact subset of X, then K meets only finitely many cells. (Hatcher, A.1)*
3. *X is Hausdorff (and even normal) (Hatcher, A.3)*

While we are discussing point-set issues, let me mention another important consideration. In algebra, given R-modules M, N, and P, there is a bijection

$$\mathrm{Hom}_R(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \underline{\mathbf{Hom}}(N, P)).$$

In topology, we similarly would like to have a bijection

$$\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, \mathrm{Map}(Y, Z)),$$

where $\mathrm{Map}(Y, Z)$ is the space of continuous maps $Y \longrightarrow Z$, equipped with the compact-open topology. But the canonical map

$$\mathrm{Hom}(X \times Y, Z) \longrightarrow \mathrm{Hom}(X, \mathrm{Map}(Y, Z))$$

is not surjective for <u>all</u> spaces X, Y, and Z. There are several ways to fix this problem, and the solution we shall take is to work with **compactly generated weak Hausdorff spaces.**

A space X is **weak Hausdorff** if the image of any compact Hausdorff space is closed in X. A weak Hausdorff space is **compactly generated** if a subset $C \subseteq X$ is closed if (and only if) for every continuous map $g : K \longrightarrow X$ with K compact, the subset $g^{-1}(C)$ is closed in K.

Any time from now on that we talk about spaces, we really mean compactly generated weak Hausdorff spaces. There are a couple more modifications that we need. One point is that if X and Y are compactly generated weak Hausdorff, then $X \times Y$ need not be. So we redefine the topology on $X \times Y$ by setting the closed subsets to be those satisfying the compactly generated condition. Similarly, the mapping space $\mathrm{Map}(X, Y)$ is not always compactly generated, so we similarly redefine the topology. It turns out that after these steps, these replacements work as desired, and we end up with a homeomorphism

$$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z)).$$

There is also a based variant of this. We write $\mathrm{Map}_*(X, Y)$ for the space of based maps. Then we have

$$\mathrm{Map}_*(X \wedge Y, Z) \cong \mathrm{Map}_*(X, \mathrm{Map}_*(Y, Z)).$$

See Chapter 5 of [May] or Neil Strickland's notes for more on compactly generated spaces.

Proposition 1.2. *If X and Y are CW complexes, then so is $X \times Y$. An n-cell of $X \times Y$ corresponds to a p-cell of X and a q-cell of Y, where $p + q = n$.*

Proof. The point is that we use the cube models for disks, then we have a homeomorphism

$$D^{p+q} = I^{p+q} \cong I^p \times I^q = D^p \times D^q,$$

and under this model we get

$$S^{p+q-1} = \partial D^{p+q} = \partial(D^p) \times D^q \cup D^p \times \partial(D^q) = S^{p-1} \times D^q \cup D^p \times S^{q-1}.$$

I should emphasize here that when we write product, we mean the product in the compactly generated sense. Otherwise, the topology on $X \times Y$ might not satisfy condition (3) from the definition of a CW complex. See [Hatcher, Theorem A.6]. ∎

There are two important generalizations: a **relative CW complex** (X, A) is defined in the same way, except that one starts with X_0 as the space A disjoint union a discrete set.

A **cell complex** is a space with an increasing filtration $X = \bigcup_n X_n$ as before, but there are now no conditions on the dimensions of the cells attached at stage n. For instance, X_1 might be obtained from X_0 by attaching a 0-cell and a 3-cell. There is also the notion of a relative cell complex (X, A).

Homotopy Extension Property

Definition 1.3. We say a map $A \longrightarrow X$ satisfies the **Homotopy Extension Property (HEP)** if, given any map $X \xrightarrow{f} Y$ and homotopy $h : A \times I \longrightarrow Y$ with $h_0 = f \circ i$, then there is an extension $\tilde{h} : X \times I \longrightarrow Y$ so that $\tilde{h} \circ (i \times \mathrm{id}) = h$ and $\tilde{h}_0 = f$. This can be represented by the following "lifting" diagram

$$\begin{array}{ccc} A & \xrightarrow{h} & Y^I \\ {\scriptstyle i}\downarrow & \overset{\tilde{h}}{\nearrow} & \downarrow{\scriptstyle ev_0} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

Another name for a map satisfying the HEP is (Hurewicz) cofibration.

There is a universal example of such a lifting diagram. The data of a map $X \xrightarrow{f} Y$ and a homotopy $h : A \times I \longrightarrow Y$ beginning at $f \circ i$ amounts to a single map from the space defined by the pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{\iota_0} & A \times I \\ {\scriptstyle i}\downarrow & & \downarrow \\ X & \longrightarrow & M(i) = X \cup_A A \times I. \end{array}$$

This is called the **mapping cylinder** of the map i. Any lifting diagram for the HEP factors as

$$\begin{array}{ccccc} A & \longrightarrow & M(i)^I & \longrightarrow & Y^I \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle ev_0} & & \downarrow{\scriptstyle ev_0} \\ X & \longrightarrow & M(i) & \longrightarrow & Y, \end{array}$$

so it is enough to find a lift in the case of $Y = M(i)$.

Proposition 1.4. *The map $i : A \longrightarrow X$ is a cofibration if and only if $M(i)$ is a retract of $X \times I$.*

Proposition 1.5. *If $i : A \longrightarrow X$ is a cofibration, then i is a closed inclusion (this uses that X is weak Hausdorff).*

Example 1.6. (1) The inclusion $\emptyset \hookrightarrow X$ is always a cofibration. Indeed, $Mi = X$, which is clearly a retract of $X \times I$.

(2) The inclusion $\{0\} \hookrightarrow I$ is a cofibration.
(3) The inclusion $\partial I \hookrightarrow I$ is a cofibration.
(4) The inclusion $S^{n-1} \hookrightarrow D^n$ is a cofibration

2. Wed, Feb. 2

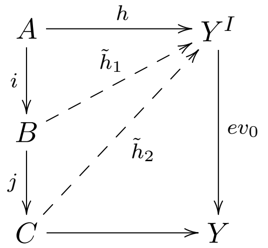
SNOW DAY!

3. Fri, Feb. 4

Proposition 3.1. *The class of cofibrations is closed under*

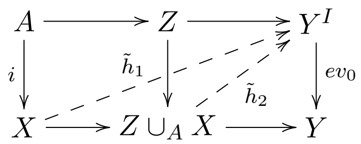
(1) *composition,*
(2) *pushouts,*
(3) *coproducts,*
(4) *retracts, and*
(5) *sequential colimits.*

Proof. (1) Suppose $A \xrightarrow{i} B \xrightarrow{j} C$ are both cofibrations, and consider a test diagram



The homotopy $\tilde{h}_1$ exists since i is a cofibration. We then form a lifting diagram with $\tilde{h}_1$ as the top horizontal arrow, and $\tilde{h}_2$ then exists because j is a cofibration.

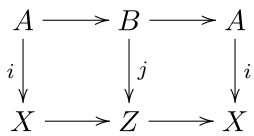
(2) Suppose that $i : A \longrightarrow X$ is a cofibration and $f : A \to Z$ is any map. We wish to show that the induced map $Z \longrightarrow Z \cup_A X$ is a cofibration. Consider the test diagram



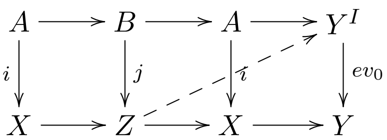
The homotopy $\tilde{h}_1$ exists since i is a cofibration. The lift $\tilde{h}_2$ then exists by the universal property of the pushout.

(3) Exercise

(4) Suppose that $i : A \longrightarrow X$ is a retract of $j : B \longrightarrow Z$; that is, we have a diagram



in which both horizontal compositions are the identity. Consider the test diagram



The displayed lift exists because j is a cofibration, and the desired lift is obtained by composing with the given map $X \longrightarrow Z$.

(5) Exercise ■

Corollary 3.2. *If $A \hookrightarrow X$ is a relative CW complex, the inclusion is a cofibration.*

Proposition 3.3. *If $A \hookrightarrow X$ is a cofibration and Z is any space, then $A \times Z \hookrightarrow X \times Z$ is a cofibration.*

Proof. The two following lifting diagrams are equivalent, and we know there is a lift in the second:

$$\begin{array}{ccc} A \times Z & \longrightarrow & Y^I \\ \downarrow & & \downarrow \\ X \times Z & \longrightarrow & Y \end{array} \qquad \begin{array}{ccc} A & \longrightarrow & (Y^Z)^I \\ \downarrow & \nearrow & \downarrow \\ X & \longrightarrow & Y^Z. \end{array}$$

■

Remark 3.4. There is also a notion of based cofibration, in which one starts with a test diagram of based maps (including a based homotopy on A) and asks for a based homotopy $\tilde{h}$.

Proposition 3.5. *If a based map $A \longrightarrow X$ is an unbased cofibration, then it is a based cofibration.*

Proof. Given a based lifting diagram, we have a lift $\tilde{h} : X \longrightarrow Y \times I$ if we forget about basepoints. But this lift must be a based homotopy, since the basepoint is in A, and the initial homotopy on A was assumed to be based. ■

We say a based space (X, x) is **non-degenerately based** (or well-pointed) if the inclusion of the basepoint is an unbased cofibration (note that it is vacuously a based cofibration). Given any based space (X, x), one may "attach a whisker" to force X to be well-pointed. That is, form the space $X' = X \vee I$, with new basepoint at the endpoint 1 of the interval (we glue I to X at the endpoint 0). Then it is easy to show that $* \to X'$ is a cofibration. This is a special case of the following result:

Proposition 3.6. *(Replacing a map by a cofibration) Any map $f : X \longrightarrow Y$ factors as a composition $X \xrightarrow{i} M(f) \xrightarrow{p} Y$, where i is a cofibration and p is a homotopy equivalence.*

Proof. The map i includes X at time 1, and the map p is defined by $p(x,t) = f(x)$ for $t > 0$ and $p(y, 0) = y$.

To see that i is a cofibration, we need to provide a retraction to the inclusion $M(i) \hookrightarrow M(f) \times I$. The map $r : M(f) \times I \longrightarrow M(i)$ defined by

$$r(x,s,t) = \begin{cases} (x, s(1+t), 0) & s \leq 1/(1+t) \\ (x, 1, s(1+t)-1) & s \geq 1/(1+t) \end{cases}$$

does the trick.

We now check that p is a homotopy equivalence. Define $q : Y \longrightarrow M(f)$ to be the inclusion at time 0. Then $p \circ q = \mathrm{id}_Y$ and $q \circ p(x,t) = (f(x), 0)$. A homotopy $h : q \circ p \simeq \mathrm{id}_M$ is given by

$$h(x,t,s) = (x, ts) \qquad h(y,s) = y.$$

■

Proposition 3.7. *If X is non-degenerately based and Y is a path-connected based space, then any map $f : X \longrightarrow Y$ is homotopic to a based map.*

Proof. Let $h : * \longrightarrow Y^I$ specify a path from $f(*)$ to the basepoint y. By the HEP for the inclusion $* \hookrightarrow X$, this extends to a homotopy $\tilde{h} : X \longrightarrow Y^I$, and the map $\tilde{h}(0,1) : X \longrightarrow Y$ is based by construction. ■