CLASS NOTES
MATH 527 (SPRING 2011)
WEEK 8

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1. Mon, Mar. 7

Matt Ando lectured today.

2. Wed, Mar. 9

Last time, we used Blakers-Massey to prove the Freudenthal theorem, which says that if X is well-pointed and n-connected, then the map suspension map $\pi_j(X) \longrightarrow \pi_{j+1}(\Sigma X)$ is an isomorphism for $j \leq 2n$ and a surjection for $j = 2n+1$. Of course, important examples come from taking X to be a sphere.

Example 2.1. (1) Take $X = S^2$. We know S^2 is 1-connected, so

$$\pi_j(S^2) \longrightarrow \pi_{j+1}(S^3)$$

is an isomorphism for $j \leq 2$ and a surjection for $j = 3$. Recall that we know $\pi_2(S^2) \cong \mathbb{Z}$ from the long exact sequence for the Hopf fibration η. The Freudenthal theorem then says that $\pi_3(S^3) \cong \mathbb{Z}$ and that $\pi_4(S^3)$ is cyclic (it turns out this is $\mathbb{Z}/2$).

(2) Taking $X = S^3$, since S^3 is 2-connected, we get isomorphisms

$$\pi_3(S^3) \cong \pi_4(S^4) \cong \mathbb{Z}, \qquad \pi_4(S^3) \cong \pi_5(S^4) \cong \mathbb{Z}/2$$

(3) As the connectivity of spheres improves as we suspend, for any fixed j and k, the sequence

$$\pi_j(S^k) \longrightarrow \pi_{j+1}(S^{k+1}) \longrightarrow \pi_{j+2}(S^{k+2}) \longrightarrow \dots$$

eventually stablizes to produce the **stable homotopy groups of spheres**. We write

$$\pi_n^s(S^0) = \operatorname*{colim}_k \pi_{n+k}(S^k).$$

This stabilizes at $\pi_n^s(S^0) = \pi_{2n+2}(S^{n+2})$. The first few stable homotopy groups of spheres are

$$\pi_0^s \cong \pi_2(S^2) \cong \mathbb{Z} \qquad \pi_1^s \cong \pi_4(S^3) \cong \mathbb{Z}/2 \qquad \pi_2^s \cong \pi_6(S^4) \cong \mathbb{Z}/2$$
$$\pi_3^s \cong \pi_8(S^5) \cong \mathbb{Z}/24 \qquad \pi_4^s \cong \pi_{10}(S^6) \cong 0 \qquad \pi_5^s \cong 0$$
$$\pi_6^s \cong \mathbb{Z}/2 \qquad \pi_7^s \cong \mathbb{Z}/240 \qquad \pi_8^s \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

(4) One can also talk of stable homotopy groups for arbitrary based spaces. If X is a based space, then we define the nth stable homotopy group of X to be

$$\pi_n^s(X) = \operatorname*{colim}_k \pi_{k+n}(\Sigma^k X).$$

These stabilize as above.

We will return to stable homotopy groups when we discuss the **stable homotopy category of spectra**.

The Blakers-Massey theorem also allows us to compare wedges with products:

Proposition 2.2. *Let X be p-connected and Y be q-connected well-pointed space ($p, q \geq 2$). Then the inclusion $X \vee Y \hookrightarrow X \times Y$ is $p+q$-connected.*

Proof. Since X and Y are well-pointed, the square

$$\begin{array}{ccc} * & \longrightarrow & Y \\ \downarrow & & \downarrow i_Y \\ X & \xrightarrow{i_X} & X \vee Y \end{array}$$

is a homotopy pushout square. Homotopy excision tells us that the homotopy pullback $X \times^h_{X\vee Y} Y$ is $(p+q-1)$-connected. As we have discussed, this is equivalent to the statement that the map

$$F(* \to X) \longrightarrow F(Y \xrightarrow{i_Y} X \vee Y)$$

is $(p+q-1)$-connected. Note that the first fiber is just $\Omega(X)$. So the maps

$$\pi_i(X) \cong \pi_{i-1}(\Omega(X)) \longrightarrow \pi_{i-1}(F(i_Y)) \cong \pi_i(X \vee Y, Y)$$

are isomorphisms for $i \leq p+q-1$ and surjective for $i = p+q$.

Consider now the long exact sequence for the pair $(X \vee Y, Y)$:

$$\ldots \longrightarrow \pi_i(Y) \longrightarrow \pi_i(X \vee Y) \longrightarrow \pi_i(X \vee Y, Y) \longrightarrow \ldots.$$

The map $X \vee Y \longrightarrow Y$ which collapses X to a point gives a splitting to i_Y, and we conclude that

$$\pi_i(X \vee Y) \cong \pi_i(Y) \oplus \pi_i(X \vee Y, Y).$$

Putting this together, we learn that

$$\pi_i(X \vee Y) \cong \pi_i(Y) \oplus \pi_i(X)$$

for $i \leq p+q-1$. To see that $\pi_{p+q}(X \vee Y) \longrightarrow \pi_{p+q}(X \times Y)$, note that the inclusions i_X and i_Y induce a splitting for this map, so that in fact $\pi_i(X \vee Y) \longrightarrow \pi_i(X \times Y)$ is onto for *every* i. ∎

Corollary 2.3. *Let $n \geq 2$. Then $\pi_n(S^n \vee S^n) \cong \mathbb{Z} \oplus \mathbb{Z}$.*

We will use this to prove the Hurewicz theorem. Let $\alpha : S^n \longrightarrow X$ be any map. Passing to homology gives a map

$$\mathbb{Z} \cong \mathrm{H}_n(S^n) \xrightarrow{H(\alpha)} \mathrm{H}_n(X),$$

and we define $h(\alpha)$ to be the image of 1 under this map.

Lemma 2.4. *The resulting map $h : \pi_n(S^n) \longrightarrow \mathrm{H}_n(X)$ is a well-defined homomorphism.*

Proof. Well-definedness is the statement that homotopic maps induce the same map on homology. To see this is a homomorphism, recall that if $\alpha, \beta \in \pi_n(S^n)$, then their sum is defined as the class of

$$S^n \xrightarrow{p} S^n \vee S^n \xrightarrow{\alpha\vee\beta} X \vee X \xrightarrow{\nabla} X.$$

The conclusion now follows, using that $\tilde{\mathrm{H}}_n(Y \vee Z) \cong \tilde{\mathrm{H}}_n(Y) \oplus \tilde{\mathrm{H}}_n(Z)$. ∎

3. Fri, Mar. 11

Theorem 3.1. *Let X be $n-1$-connected. Then the Hurewicz map is an isomorphism*

$$\pi_n(X) \xrightarrow{\cong} \mathrm{H}_n(X)$$

if $n \geq 2$. If $n = 1$, this is the abelianization map.

Proof. The $n=1$ statement should be familiar from 525, so we assume $n \geq 2$.

As homology takes weak equivalences to isomorphisms, we may assume X is CW. Since it is $n-1$-connected, we may assume it has a single 0-cell and no cells in dimensions between 0 and n. Furthermore, the (cellular) homology group H_n and the homotopy group π_n of X only depends on the $n+1$-skeleton X_{n+1}. So we assume that X is a cofiber of a map

$$A = \bigvee S^n \longrightarrow B = \bigvee S^n \longrightarrow X.$$

We then have a diagram

$$\begin{array}{ccccccc}
\pi_n(A) & \longrightarrow & \pi_n(B) & \longrightarrow & \pi_n(X) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{H}_n(A) & \longrightarrow & \mathrm{H}_n(B) & \longrightarrow & \mathrm{H}_n(X) & \longrightarrow & 0
\end{array}$$

The bottom row is exact because $A \xrightarrow{\alpha} B \xrightarrow{\beta} X$ is a cofiber sequence. The Blakers-Massey theorem now tells us that since the maps $A \longrightarrow *$ and $A \xrightarrow{\alpha} B$ are n-connected and $n-1$-connected, respectively, the map $A \longrightarrow F(\beta)$ is $2n-2$-connected. As $n \geq 2$, $n \leq 2n-2$ and we get $\pi_n(A) \twoheadrightarrow \pi_n(F)$. It follows that the top row above is exact.

Lemma 3.2. *The Hurewicz map is an isomorphism for any wedge $\bigvee S^n$.*

Proof. We showed last time that $\pi_n(S^n \vee S^n) \cong \mathbb{Z}$, where generators are given by the two natural inclusions $S^n \hookrightarrow S^n \vee S^n$, and it follows that the Hurewicz map is an isomorphism. by induction, we get the same result for any finite wedge. For an infinite wedge, we have

$$\mathrm{H}_n(\bigvee S^n) \cong \bigoplus \mathrm{H}_n(S^n) \quad (n \geq 1)$$

Since S^n is compact, any map $S^n \longrightarrow \bigvee S^n$ must have image contained in a finite wedge. ∎

∎

We discuss a few more consequences of Homotopy Excision. Recall that, given a map $f : A \longrightarrow X$, we previously defined a map $\varphi : F(f) \longrightarrow \Omega C(f)$. This induces a map

$$\pi_i(X, A) \cong \pi_{i-1}(F(f)) \longrightarrow \pi_i(C(f)),$$

and the latter may be reinterpreted as $\pi_i(X/A)$ if f is a cofibration.

Proposition 3.3. *Suppose that A is m-connected and that $f : A \longrightarrow X$ is an n-equivalence. Then the map*

$$F(f) \longrightarrow \Omega C(f)$$

is an $m+n$-equivalence.

Proof. Consider the homotopy pushout square

$$\begin{array}{ccc}
A & \longrightarrow & CA \\
{\scriptstyle f}\downarrow & & \downarrow \\
X & \longrightarrow & C(f).
\end{array}$$

The inclusion $A \hookrightarrow CA$ is an $m+1$-equivalence, and f is an n-equivalence. By Blakers-Massey, it follows that induced map

$$F(f) \longrightarrow F(CA \longrightarrow C(f)) \simeq \Omega C(f)$$

is an $m+n$-equivalence. ∎

Remark 3.4. Another way to interpret the above result in the case that f is a cofibration is that

$$\pi_i(X, A) \longrightarrow \pi_i(C(f)) \cong \pi_i(X/A)$$

is an isomorphism for $i < m + n + 1$ and a surjection for $i = m + n + 1$.

As we have said previously, the analogue of the map $F(f) \longrightarrow \Omega C(f)$ in spectra is always an equivalence.

We use the above result to prove a result of Whitehead.

Theorem 3.5 (Whitehead)**.** *Let $f : X \longrightarrow Y$ be a map between simply connected spaces such that $\mathrm{H}_*(f)$ is an isomorphism. Then f is a weak homotopy equivalence.*

Proof. The cofiber sequence $X \xrightarrow{f} Y \longrightarrow C(f)$ induces a long exact sequence in homology, and we deduce that the homology groups of $C(f)$ vanish. Note that since A and X are 1-connected, it follows that $F(f)$ is 0-connected. By the previous result, $\pi_0(F(f)) \cong \pi_1(C(f))$, so $C(f)$ is simply connected. Since it also has no homology, the Hurewicz theorem implies that $C(f)$ is weakly contractible ($C(f) \longrightarrow *$ is a weak equivalence).

Since f is a 1-equivalence and X is 1-connected, the previous result tells us that

$$\pi_1(F(f)) \cong \pi_2(C(f)) \cong 0,$$

so that f is in fact a 2-equivalence. By induction, the same argument will show that f is an n-equivalence for every n, i.e. a weak equivalence. ∎

Remark 3.6. There is a more general form of this result in which one assumes only that the spaces X and Y are *simple*.

If $\pi_1(X)$ is nonabelian, this result can easily fail. For instance, there are spaces with nontrivial fundamental group but trivial homology. In this case, the fundamental group must be a nontrivial group with trivial abelianization. In other words, the group must be generated by commutators. Any finite simple group, like A_5, gives such an example.

Remark 3.7. It is not enough to simply know that two spaces are simply connected and have the same homology. For instance, consider $X = S^2 \vee S^4$ and $Y = \mathbb{CP}^2$. The homology of both spaces is $\mathbb{Z}$ in degrees 0, 2, and 4. But there is no map between them inducing a homology isomorphism.

There is a natural map $S^2 \cong \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2$ given an isomorphism on H_2, and the quotient map $\mathbb{CP}^2 \longrightarrow \mathbb{CP}^2/\mathbb{CP}^1 \cong S^4$ gives an H_4-isomorphism. But there is no map $S^4 \longrightarrow \mathbb{CP}^2$ inducing the H_4-iso. Indeed, the fiber sequence $S^1 \longrightarrow S^5 \longrightarrow \mathbb{CP}^2$ shows that $\pi_4(\mathbb{CP}^2) \cong 0$ and $\pi_3(\mathbb{CP}^2) \cong 0$, so that no such map can exist by Whitehead. Note also that if we consider cohomology, then it is clear that there can be no map inducing a H^*-isomorphism since Y has nontrivial cup products, whereas X does not.