CLASS NOTES
MATH 527 (SPRING 2011)
WEEK 9

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1. Mon, Mar. 7

Today, we will prove the Homotopy Excision theorem.

Theorem 1.1 (Blakers-Massey Homotopy Excision Theorem)**.** *Let $(X; A, B)$ be an excisive triad with $C = A \cap B$ nonempty. Then, if the inclusions $C \hookrightarrow A$ and $C \hookrightarrow B$ are p-connected and q-connected, respectively, then the square*

$$\begin{array}{ccc} C & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & X \end{array}$$

is $(p+q-1)$-cartesian, meaning that the map $C \longrightarrow B \times_X^h A$ is a $(p+q-1)$-equivalence.

Proof. Note that the projection map $B \times_B^h C \longrightarrow C$ is a homotopy equivalence. It thus suffices to show that the map

$$B \times_B^h C \longrightarrow B \times_X^h A$$

is a $(p+q-1)$-equivalence. We will show that the relative homotopy group

$$\pi_n(B \times_X^h A, B \times_B^h C)$$

vanishes for $n < p+q$. Suppose given a map of triples

$$(I^n, \partial I^n, J^n) \longrightarrow (B \times_X^h A, B \times_B^h C, c_0)$$

(c_0 is an arbitrary basepoint for C). It suffices to show that this can be deformed, through maps of the above form, into a map sending all of I^n into $B \times_B^h C$.

The above data corresponds to a map of triples

$$(I^n \times I, I^n \times \{0\} \cup \partial I^n \times I, J^n \times I) \xrightarrow{F,F'} (X, B, b_0)$$

such that $F(I^n \times \{1\}) \subseteq A$. We need to deform this so that $F(I^n \times I) \subseteq B$. We will use the following technical result to do so. First, subdivide the cube $I^n \times I$ into subcubes D such that for each D, either $F(D) \subseteq A$ or $F(D) \subseteq B$. This can be done by the Lebesgue Lemma, since $\{A, B\}$ is an open cover for X.

Proposition 1.2. *There is a homotopy $\Phi : I^{n+1} \times I \longrightarrow X$ through maps of the above form such that $\Phi_0 = F$ and such that for any subcube $D \subseteq I^{n+1}$,*

1. *If $F(D) \subseteq A$, then $\Phi(D) \subseteq A$, and similarly for B.*
2. *If $F(D) \subseteq C$, then Φ is constant on D.*
3. *If $F(D) \subseteq A$, then $\Phi_1^{-1}(A \setminus C) \cap D \subseteq M_{p+1}(D)$*
4. *If $F(D) \subseteq B$, then $\Phi_1^{-1}(B \setminus C) \cap D \subseteq L_{q+1}(D)$,*

where $M_n(D) \subseteq D$ is the subset consisting of points (x_i) such that at least n of the coordinates satisfy $\frac{1}{2} < x_i \leq 1$ (after identifying the cube with I^j). Similarly, points in $L_n(D)$ satisfy $0 \leq x_i < \frac{1}{2}$ for at least n coordinates.

We will use this result to prove the theorem and return to prove this technical result afterwards. Let us write $G = \Phi_1$. Consider now the projection $P : I^n \times I \longrightarrow I^n$.

Claim: The intersection $PG^{-1}(A \setminus C) \cap PG^{-1}(B \setminus C)$ is empty.

To see this, let $y = P(z) \in PG^{-1}(A \setminus C)$. Suppose that $z \in G^{-1}(A \setminus C) \cap D$ for some D. Then $z \in M_{p+1}(D)$, so $y \in M_p(P(D))$. Similarly, if $y \in PG^{-1}(B \setminus C)$ then $y = P(z')$, $z' \in D'$, and $y \in L_q(P(D'))$. Thus $y \in M_p(P(D) \cap P(D')) \cap L_q(P(D) \cap P(D'))$, which is impossible if $n < p+q$.

By the claim, the closed subsets $PG^{-1}(A \setminus C)$ and $PG^{-1}(B \setminus C)$ of I^n are disjoint, so by the Urysohn lemma, there exists a Urysohn function $u : I^n \longrightarrow I$ such that

$$u(PG^{-1}(A \setminus C)) \equiv 0, \qquad u(PG^{-1}(B \setminus C)) \equiv 1.$$

We now define a homotopy $\Psi : I^n \times I \times I \longrightarrow X$ with $\Psi_0 = G$ by the formula

$$\Psi((x,t),s) = G(x, (1-s)t + stu(x)).$$

We claim that this is again a homotopy through maps of the correct form and that $H = \Psi_1$ has image in B.

For instance, we check that $\Psi_s(x,1) \in A$ for every $x \in I^n$. If $x \in PG^{-1}(B \setminus C)$, then $u(x) = 1$, so $\Psi_s(x,1) = G(x, 1-s+s) = G(x,1) \in A$. On the other hand, if $x \notin PG^{-1}(B \setminus C$, then $\Psi_s(x,1) = G(x, 1-s+s \cdot u(x)) \notin B \setminus C = X \setminus A$, so $\Psi_s(x,1) \in A$.

The argument that $H = \Psi_1$ has image in B is similar. If $x \in PG^{-1}(A \setminus C$, then

$$H(x,t) = G(x, t \cdot u(x)) = G(x,0) \in B.$$

On the other hand, if $x \notin PG^{-1}(A \setminus C)$, then

$$H(x,t) = G(x, t \cdot u(x)) \notin A \setminus C = X \setminus B, \qquad \Longrightarrow \; H(x,t) \in B.$$

This finishes the proof of the Blakers-Massey theorem. ■

2. Wed, Mar. 16

Last time, we proved the Blakers-Massey Theorem, assuming the following "technical" result. Recall that we had an excisive triad $(X; A, B)$ with $C = A \cap B$ nonempty, that the inclusions $C \hookrightarrow A$ and $C \hookrightarrow B$ were p-connected and q-connected, respectively. We had a map $F : I^{n+1} \longrightarrow X$ and a subdivision of I^{n+1} into subcubes C such that the image of any C under F is entirely contained in A or in B. For simplicity, we will write n rather than $n+1$.

Proposition 2.1. *There is a homotopy $\Phi : I^n \times I \longrightarrow X$ such that $\Phi_0 = F$ and such that for any subcube $D \subseteq I^n$,*

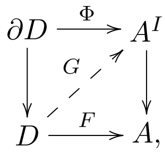
1. *If $F(D) \subseteq A$, then $\Phi(D) \subseteq A$, and similarly for B.*
2. *If $F(D) \subseteq C$, then Φ is constant on D.*
3. *If $F(D) \subseteq A$, then $\Phi_1^{-1}(A \setminus C) \cap D \subseteq M_{p+1}(D)$*
4. *If $F(D) \subseteq B$, then $\Phi_1^{-1}(B \setminus C) \cap D \subseteq L_{q+1}(D)$,*

where $M_k(D) \subseteq D$ is the subset consisting of points (x_i) such that at least k of the coordinates satisfy $\frac{1}{2} < x_i \leq 1$ (after identifying the cube with I^j). Similarly, points in $L_k(D)$ satisfy $0 \leq x_i < \frac{1}{2}$ for at least k coordinates.

Proof. The homotopy will be built inductively over the faces of cubes (induction on dimension).

First, suppose that $D = \{d\}$ is a 0-dimensional cube. If $F(d) \in C$, then we define Φ to be constant ($\Phi(d,t) = F(d)\ \forall t$). If $F(d) \in A \setminus C$, then there is a path in A from $F(d)$ to a point in $C = B \cap A$, since the pair (A, C) is 0-connected (meaning that $\pi_0(C) \twoheadrightarrow \pi_0(B)$). We use this path for the homotopy Φ on $D = \{d\}$. This makes (1) and (3) hold since $\Phi_1^{-1}(A \setminus C) \cap D$ is empty. We do the same if $F(d) \in B \setminus C$.

Suppose now that we have a k-dimensional cube D, and that we have the homotopy Φ defined on all faces of D, satisfying (1)-(4) above. We then extend this homotopy to D, using that the inclusion $\partial D \hookrightarrow D$ is a cofibration. If $F(D) \subseteq C$, we must take the constant homotopy. If this is not the case, suppose without loss of generality that $F(D) \subseteq A$. Then since the homotopy Φ defined on the faces satisfies (1), we have a square



and so we get a homotopy G defined on all of D and satisfying (1). Unfortunately, property (3) may not yet hold, so we need to modify this construction. There are two cases.

Suppose first that $k = \dim D \leq p$. Recall that the inclusion $C \hookrightarrow A$ is assumed to be p-connected. Then there is a homotopy $H : D \times I \longrightarrow A$, rel ∂D with $H_0 = G_1$ and $H_1(D) \subseteq C$. This holds because $\pi_k(A, C) = 0$ since $k \leq p$. Now the concatenation $H * G$ is a homotopy Φ on D satisfying (3), since again $\Phi_1^{-1}(A \setminus C)$ is empty.

The other case is that $k = \dim D > p$. We need to deform $g = G_1$ so that the preimage of $A \setminus C$ in D will be in $M_{p+1}(D)$. Consider the map $h : I^k \longrightarrow I^k$ as in the picture. This is a map that is the identity on ∂I^k and that takes $M_k(D)$ isomorphically to I^k. Furthermore, $h \simeq \mathrm{id}$ rel ∂I^k. It follows that $gh \simeq g$ rel ∂I^k, and we claim that this is the desired homotopy. Suppose that $gh(z) \in A \setminus C$. If $z \in M_k(D)$, then certainly $M_k(D) \subseteq M_{p+1}(D)$ since $k \geq p+1$, so we are done. On the other hand, if $z \notin M_k(D)$, then $h(z)$ is in ∂I^k. But then $h(z)$ lies on a face of the cube and $h(z) \in g^{-1}(A \setminus C)$, so it follows that $h(z)$ is in M_{p+1} of the fact. Because of the definition of h, it follows that z is also in $M_{p+1}(D)$. ■

Eilenberg-Mac Lane spaces

Given a group G and $n \geq 0$, an Eilenberg-Mac Lane space $K(G, n)$ is a CW complex satisfying

$$\pi_i(K(G,n)) \cong \begin{cases} G & i = n \\ 0 & i \neq n \end{cases}$$

Note that if $n \geq 2$, then G must be abelian. For $n = 0$, we can just take the discrete space G itself.

By the Hurewicz theorem, $K(G, n)$ has no homology in degrees $< n$, and $\mathrm{H}_n(K(G, n); \mathbb{Z}) \cong G$ if $n > 1$. By the Universal Coefficient Theorem,

$$\mathrm{H}^n(K(G,n); A) \cong \mathrm{Hom}(G, A).$$

For example, we have

$$\mathrm{H}^n(K(\mathbb{Z},n);\mathbb{Z}) \cong \mathbb{Z}, \qquad \mathrm{H}^n(K(\mathbb{Z}/m,n);\mathbb{Z}/m) \cong \mathbb{Z}/m.$$

Examples

(1) S^1 is a $K(\mathbb{Z}, 1)$.
(2) Since the homotopy groups of a product are the product of homotopy groups, it follows that $K(A, n) \times K(B, n)$ is a $K(A \oplus B, n)$. For example, the torus is a $K(\mathbb{Z} \oplus \mathbb{Z}, 1)$.
(3) The wedge $S^1 \vee S^1$ is a $K(F_2, 1)$.
(4) The space $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$. Recall that for each n we have a fiber sequence

$$S^1 \hookrightarrow S^{2n+1} \longrightarrow \mathbb{CP}^n.$$

The long exact sequence in homotopy gives $\pi_2(\mathbb{CP}^n) \cong \mathbb{Z}$ and $\pi_i(\mathbb{CP}^n) = 0$ for $2 < i < 2n+1$. This implies that the infinite union $\mathbb{CP}^\infty$ has $\pi_2(\mathbb{CP}^\infty) \cong \mathbb{Z}$ and no other nontrivial homotopy groups.

(5) The space $\mathbb{RP}^\infty$ is a $K(\mathbb{Z}/2,1)$. Again, this comes from considering the fiber sequences

$$S^0 = \mathbb{Z}/2 \hookrightarrow S^n \longrightarrow \mathbb{RP}^n.$$

The universal cover S^n of $\mathbb{RP}^n$ is a double cover, so $\pi_1(\mathbb{RP}^n)$ has order two, and the higher homotopy groups of $\mathbb{RP}^n$ agree with those of S^n.

(6) The spaces $K(\mathbb{Z}/p,1)$ for p odd are known as Lens spaces. They can be constructed in a similar manner. Recall that S^1 acts on S^{2n+1} as complex multiplication. The group $\mathbb{Z}/p$ sits insides S^1 as the pth roots of unity. Taking the quotient of this action on S^{2n+1} gives a space with fundamental group $\mathbb{Z}/p$, and as we take a colimit over n we get an Eilenberg-Mac Lane space.

(7) For any group (or monoid) G, there is a "classifying space" BG, which is a $K(G,1)$. Moreover, there are constructions of BG which are functorial with respect to *group homomorphisms*. Recall that the multiplication map $G \times G \longrightarrow G$ is a homomorphism if and only if G is abelian. So for an abelian group G, the classifying space BG inherits a multiplication and becomes an abelian monoid. We can then iterate B to get $B^n G = K(G,n)$.

The last example describes one way of producing Eilenberg-Mac Lane spaces, given the construction B. One can also construct $K(G,n)$'s by hand, as follows. Let $n \geq 2$ and suppose G is abelian. Write G as a cokernel

$$\mathbb{Z}^{n_2} \longrightarrow \mathbb{Z}^{n_1} \longrightarrow G \longrightarrow 0$$

of a map of free abelian groups. We then define the n-skeleton to be $X_n = \bigvee_{n_1} S^n$. We define the $(n+1)$-skeleton as the cofiber of the map

$$\bigvee_{n_2} S^n \longrightarrow \bigvee_{n_1} S^n = X_n \longrightarrow X_{n+1}.$$

The cofiber X_{n+1} may have nontrivial homotopy in degrees $n+1$ and higher. We thus attach $n+2$-cells to kill π_{n+1}. The result may still have homotopy in degrees $n+2$ and higher, so we attach $n+3$-cells to kill π_{n+2}. Attaching cells to kill all higher homotopy groups, we arrive at a CW complex with the desired homotopy groups.