A COUNTING PROOF FOR WHEN 2 IS A QUADRATIC RESIDUE

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Abstract. Using the group consisting of the eight Möbius transformations x, $-x$, $1/x$, $-1/x$, $(x-1)/(x+1)$, $(x+1)/(1-x)$, $(x+1)/(x-1)$, and $(1-x)/(x+1)$ we present an enumerative proof of the classical result for when the element 2 is a quadratic residue in the finite field F_q.

Recall that a nonzero element x in a field F is a *quadratic residue* if it is a square, that is, we can write $x = y^2$ where $y \in F$.

Assume that q is an odd prime power and let F_q be the finite field of q elements. The classical result that -1 is a quadratic residue in F_q if and only if $q \equiv 1 \bmod 4$ can be proved by partitioning the nonzero elements of the field into orbits of the form $\{x, -x, -1/x, 1/x\}$. Note that one orbit is $\{1, -1\}$. If $\alpha^2 = -1$ has a solution, then $\{\alpha, -\alpha\}$ is also an orbit. The remaining orbits all have cardinality 4. Thus by counting the nonzero elements of the field modulo 4, we obtain that $q \equiv 1 \bmod 4$, implying that $q - 1 \equiv 0 \equiv |\{1, -1\}| + |\{\alpha, -\alpha\}| \bmod 4$ and hence that the orbit $\{\alpha, -\alpha\}$ exists, that is, -1 is a quadratic residue. Similarly, $q \equiv 3 \bmod 4$ implies that there is no such orbit and hence -1 is not a quadratic residue. See [1, Theorem 2.2.7].

We present a similar argument for when the element 2 is a quadratic residue. We use a larger set of rational functions and we have four different types of orbits.

Theorem 1. *Let q be an odd prime power. Then the element 2 is a quadratic residue in the finite field F_q if and only if $q \equiv \pm 1 \bmod 8$.*

Proof. Consider the eight rational functions x, $-x$, $1/x$, $-1/x$, $(x-1)/(x+1)$, $(x+1)/(1-x)$, $(x+1)/(x-1)$, and $(1-x)/(x+1)$. Note that they form a group G under composition. These rational functions are Möbius transformations and act naturally on the field F_q with the point at infinity adjoined, that is, on $F_q \cup \{\infty\}$. The orbits of this action are as follows. First there is the orbit $\{0, \pm 1, \infty\}$. In fact, the group permutes these elements as the vertices of a square, showing that the group is isomorphic to the symmetric group of a square. Assuming that 2 is a quadratic residue in the field F_q, we have the orbit $B = \{\pm 1 \pm \sqrt{2}\}$ of size 4. Next, assuming that -1 is a quadratic residue, we have the orbit $C = \{\pm i\}$ of size 2. Finally, the remaining orbits all have size 8.

We now have four cases. In each case, it is enough to count the $q - 3$ elements in $F_q - \{0, \pm 1\}$ modulo 8, hence only keeping track if the orbits B and C occur.

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– If -1 and 2 are both quadratic residues, then both B and C occur, yielding $q-3 \equiv 4+2 \bmod 8$, that is, $q \equiv 1 \bmod 8$.
– If -1 and 2 are both not quadratic residues, then all orbits have size 8, yielding $q-3 \equiv 0 \bmod 8$, that is, $q \equiv 3 \bmod 8$.
– If -1 is a quadratic residue and 2 is not, then only C occurs, yielding $q-3 \equiv 2 \bmod 8$, that is, $q \equiv 5 \bmod 8$.
– Finally, if 2 is a quadratic residue and -1 is not, then only B occurs, yielding $q-3 \equiv 4 \bmod 8$, that is, $q \equiv 7 \bmod 8$. □

A similar proof can be obtained by using the order 6 group $H = \{x, 1-x, 1/(1-x), x/(x-1), (x-1)/x, 1/x\}$. When $q \equiv 3 \bmod 4$, the result follows by counting the number of quadratic residues in orbits of H. Similarly, when $q \equiv 1 \bmod 4$, the result follows by counting the number of quadratic nonresidues.

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References

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