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On Posets and Hopf Algebras

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We generalize the notion of the rank-generating function of a graded poset. Namely, by enumerating different chains in a poset, we can assign a quasi-symmetric function to the poset. This map is a Hopf algebra homomorphism between the reduced incidence Hopf algebra of posets and the Hopf algebra of quasi-symmetric functions. This work implies that the zeta polynomial of a poset may be viewed in terms Hopf algebras. In the last sections of the paper we generalize the reduced incidence Hopf algebra of posets to the Hopf algebra of hierarchical simplicial complexes. © 1996 Academic Press, Inc.

INTRODUCTION

Joni and Rota established in [5] that the incidence coalgebra and reduced incidence coalgebra of posets are much more basic structures than the incidence algebra and reduced incidence algebra. In fact, they showed that the reduced incidence coalgebra naturally extends to a bialgebra, with the Cartesian product as the product of the bialgebra. Schmitt [10] continued this work by showing that the reduced incidence bialgebra of posets can be extended to a Hopf algebra structure. That is, there is an endofunction of the bialgebra, called the antipode, that fulfills a certain defining relation. One can view the antipode as a generalization of the Möbius function, which has played an important role in the theory of posets since Rota's seminal work [9]. Moreover, Schmitt gives a closed formula for the antipode which is a generalization of Philip Hall's formula for the Möbius function.

In Sections 8 and 9 we extend the reduced incidence Hopf algebra of posets to a Hopf algebra of hierarchical simplicial complexes. To construct the reduced incidence Hopf algebra of posets we only need to know the chains of the posets and the order of the elements within each chain. Recall that the chains of a poset P form a simplicial complex $\Delta(P)$, called the

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order complex of P. A simplicial complex does not carry any information about an order on each face. Thus we define a *hierarchical simplicial complex* to be a simplicial complex Δ, each face of which is enriched with a linear order such that the orders are compatible. Examples of hierarchical simplicial complexes can be found in Section 8. Using these objects we may define the reduced incidence Hopf algebra of hierarchical simplicial complexes, which we will denote with Υ.

The linear space of quasi-symmetric functions was shown by Gessel [4] to have a Hopf algebra structure. By exploring this Hopf algebra, we generalize the notion of the rank-generating function of a graded poset. We introduce the *F-quasi-symmetric function* $F(P)$. This function encodes the flag f-vector of the graded poset in a very pleasant way. Not only is the map $P \mapsto F(P)$ an algebra homomorphism from the reduced incidence Hopf algebra to the Hopf algebra of quasi-symmetric functions, but it is also a coalgebra homomorphism. In fact, it is a Hopf algebra homomorphism. This observation captures an essential part of the structure of posets. The map F may also be extended to the reduced incidence Hopf algebra of hierarchical simplicial complexes Υ; see Section 10.

Our work implies that the zeta polynomial $Z(P; x)$ of a poset P may be viewed in terms of Hopf algebras. Indeed, the zeta polynomial of a poset is a Hopf algebra homomorphism. This is easily seen by composing two Hopf algebra homomorphisms: one from the reduced incidence Hopf algebra to the quasi-symmetric functions, and one from the quasi-symmetric functions to polynomials in one variable. One may also express the characteristic polynomial $\chi(P; q)$ in terms of the F-quasi-symmetric function $F(P)$.

Malvenuto and Reutenauer [7, 8] have studied the structure of the Hopf algebra of quasi-symmetric functions in detail. In fact, we obtain independently the explicit formula of the antipode of a monomial quasi-symmetric function. (See Proposition 3.4.)

In Section 7 we use our techniques to study Eulerian posets. We show an identity that holds for the F-quasi-symmetric function of a Eulerian poset. This identity is equivalent to a statement about the F-quasi-symmetric function of the antipode of a poset. Moreover, this is also equivalent to the statement that the rank-selected Möbius invariant $\beta(S)$ is equal to $\beta(\overline{S})$, where $\overline{S}$ is the complement set of S. This last condition is well known for Eulerian posets. As a corollary we obtain that $Z(P; -x) = (-1)^{\rho(P)} \cdot Z(P; x)$ for Eulerian posets.

2. Definitions

We begin by recalling some basic facts about posets. The reader is referred to [12] for more details. All the posets considered in this paper have a finite number of elements, a minimum element $\hat{0}$ and a maximum element $\hat{1}$. For two elements x and y in a poset P, such that $x \leqslant y$, define the interval $[x, y] = \{z \in P : x \leqslant z \leqslant y\}$. We will consider $[x, y]$ as a poset, which inherits its order relation from P. Observe that $[x, y]$ is a poset with minimum element x and maximum element y. For $x, y \in P$ such that $x \leqslant y$, we may define the *Möbius function* $\mu(x, y)$ recursively by

$$\mu(x, y) = \begin{cases} -\sum_{x \leqslant z < y} \mu(x, z), & \text{if} \quad x < y, \\ 1, & \text{if} \quad x = y. \end{cases}$$

We write $\mu(P) = \mu(\hat{0}, \hat{1})$. An element y in P *covers* another element x in P if $x < y$ and there is no $z \in P$ such that $x < z < y$. Recall that a poset P, not necessarily with a $\hat{0}$ and $\hat{1}$, is *ranked* if there exists a rank function $\rho: P \to \mathbb{Z}$ such that $\rho(x) + 1 = \rho(y)$ if y covers x. A poset P is *graded* if it is ranked and $\rho(\hat{0}) = 0$. Observe that this implies that the rank function ρ maps the poset into $\mathbb{N}$, the nonnegative integers. We will write $\rho(x, y) = \rho(y) - \rho(x)$ for $x \leqslant y$, and $\rho(P) = \rho(\hat{0}, \hat{1})$. A poset P is *Eulerian* if for all $x, y \in P$ such that $x \leqslant y$, the Möbius function is given by $\mu(x, y) = (-1)^{\rho(x, y)}$. The *dual* poset P^* is defined by reversing the order relation of P. That is, $x \leqslant_{P^*} y$ if and only if $y \leqslant_P x$. The *Cartesian product* of two posets P and Q is the set $P \times Q$ with the order relation $(x, y) \leqslant_{P \times Q} (z, w)$ if $x \leqslant_P z$ and $y \leqslant_Q w$. Two important posets are the Boolean algebra of rank m, which we denote by B_m, and the chain of length m, denoted by C_m. Note that the ranks of these posets are $\rho(B_m) = \rho(C_m) = m$. Also, the family of Boolean algebras is multiplicative; that is, $B_m \times B_r = B_{m+r}$.

We now define the reduced incidence Hopf algebra of posets, which we denote $\mathscr{I}$. See [5, 10] for a more complete description of the subject. Also see [15] for the algebraic concepts of coalgebra, bialgebra, and Hopf algebra. Two posets have the same *type* if they are isomorphic, that is, there is an order preserving bijection between them. We denote the type of a poset P by $\bar{P}$. Let 1 be the type of a one element poset. The Cartesian product is compatible with isomorphism; this implies that the product of two types is a well-defined type. Let $\mathscr{I}$ be the linear space, over a field $\mathbf{k}$, spanned by all types. One can extend by linearity the product of types to a product on the space $\mathscr{I}$, thus making $\mathscr{I}$ into an algebra where 1 is the unit.

The algebra $\mathscr{I}$ can be made into a bialgebra by introducing the coproduct $\Delta: \mathscr{I} \to \mathscr{I} \otimes \mathscr{I}$ by

$$\Delta(\bar{P}) = \sum_{x \in P} \overline{[\hat{0}, x]} \otimes \overline{[x, \hat{1}]},$$

and the augmentation (or counit) $\varepsilon\colon \mathscr{I} \to \mathbf{k}$ by

$$\varepsilon(\bar{P}) = \begin{cases} 1, & \text{if } \quad \bar{P} = 1, \\ 0, & \text{if } \quad \bar{P} \neq 1, \end{cases}$$

and extend both definitions by linearity. The coproduct is sometimes written in the Sweedler notation, that is

$$\Delta(\bar{P}) = \sum_{\bar{P}} \bar{P}_{(1)} \otimes \bar{P}_{(2)}.$$

For our two examples we have that

$$\Delta(B_m) = \sum_{k+r=m} \binom{m}{k} \cdot B_k \otimes B_r,$$

$$\Delta(C_m) = \sum_{k+r=m} C_k \otimes C_r.$$

For any Hopf algebra H, with product $\mu\colon H \otimes H \to H$ and coproduct $\Delta\colon H \to H \otimes H$, one defines $\mu^n\colon H^{\otimes n} \to H$ and $\Delta^n\colon H \to H^{\otimes n}$ by the recursions

$$\mu^n = \begin{cases} \mu \circ (\mu^{n-1} \otimes 1), & \text{if } \quad n > 1, \\ 1, & \text{if } \quad n = 1; \end{cases} \qquad \Delta^n = \begin{cases} (\Delta^{n-1} \otimes 1) \circ \Delta, & \text{if } \quad n > 1, \\ 1, & \text{if } \quad n = 1. \end{cases}$$

Observe that $\mu^2 = \mu$ and $\Delta^2 = \Delta$. As an example, Δ^n applied to a type $\bar{P}$ in the Hopf algebra $\mathscr{I}$ is given by

$$\Delta^n(\bar{P}) = \sum_{\hat{0} = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n = \hat{1}} \overline{[x_0, x_1]} \otimes \overline{[x_1, x_2]} \otimes \cdots \otimes \overline{[x_{n-1}, x_n]}. \tag{1}$$

The *antipode* is the linear map $S\colon \mathscr{I} \to \mathscr{I}$ satisfying the relation

$$\varepsilon(\bar{P}) \cdot 1 = \sum_{\bar{P}} S(\bar{P}_{(1)}) \cdot \bar{P}_{(2)}.$$

One can view this relation as a recursive definition of the antipode; hence we observe that the antipode is well-defined. By an easy induction argument, one may show that the antipode of the Boolean algebra B_m satisfies $S(B_m) = (-1)^m \cdot B_m$. In general, the antipode of a type is a linear combination of types. Schmitt gives a closed form formula of the antipode in [10].

The antipode can be thought of as a generalization of the Möbius function in the following manner. Define the linear function $\phi\colon \mathscr{I} \to \mathbf{k}$ by $\phi(\bar{P}) = 1$ for each type $\bar{P}$. Then one has that $\mu(P) = \phi(S(\bar{P}))$, where $\mu(P)$ is the Möbius function of P.

We may formalize the idea that the antipode in a graded bialgebra satisfies a recursion.

LEMMA 2.1. *Let B be a graded bialgebra. That is, $B=\bigoplus_{n\geqslant 0} B_n$, $B_0=\mathbf{k}$, $B_i\cdot B_j\subseteq B_{i+j}$, $\Delta(B_n)\subseteq\bigoplus_{i+j=n} B_i\otimes B_j$. Then B is a Hopf algebra and the antipode may be recursively defined by $S(1)=1$, and for $x\in B_n$, $n\geqslant 1$,*

$$S(x)=-\sum_{i=1}^{m} S(y_i)\cdot z_i,$$

where

$$\Delta(x)=x\otimes 1+\sum_{i=1}^{m} y_i\otimes z_i$$

and $\deg(y_i)<n$.

3. THE HOPF ALGEBRA OF QUASI-SYMMETRIC FUNCTIONS

We will now introduce the quasi-symmetric functions and their algebraic structures.

DEFINITION 3.1. A formal power series in the variables w_1, w_2,... over a field $\mathbf{k}$ is a formal expression of the form

$$\sum_I a_I\cdot\prod_{i\in I} w_i,$$

where I ranges over all finite multisets of positive integers, and the coefficients a_I are in the field $\mathbf{k}$. The degree of a monomial is the cardinality of the corresponding multiset. Moreover, if the monomials in a power series f have bounded degree, then we call f a *formal polynomial.*

DEFINITION 3.2. A quasi-symmetric function $f(w_1, w_2, ...)$ over a field $\mathbf{k}$ is a formal polynomial in the variables w_1, w_2, ..., such that for all sequences $1\leqslant i_1<i_2<\cdots$, we have that

$$f(y_1, y_2, ...)=f(w_1, w_2, ...),$$

where

$$y_j=\begin{cases} w_k, & \text{if } \ j=i_k \text{ for some } k,\\ 0, & \text{otherwise.}\end{cases}$$

We denote the linear space of quasi-symmetric functions by Ω.

A *composition of an nonnegative integer* m is a sequence $(a_1, a_2, ..., a_k)$ of positive integers such that $a_1 + a_2 + \cdots + a_k = m$. Observe that there is only one composition of 0, namely the empty sequence $\varnothing$. Let Σ_m be the set of all compositions of m. Observe that the cardinality of Σ_m is 2^{m-1} for $m \geqslant 1$. Let Σ be the set of all compositions, that is, $\Sigma = \bigcup_{m \geqslant 0} \Sigma_m$. Let the length of a composition $\mathbf{a} = (a_1, a_2, ..., a_k)$ be the number of parts, that is, k. We denote the length by $l(\mathbf{a})$. Define the reverse of a composition $\mathbf{a} = (a_1, a_2, ..., a_k)$ to be $\mathbf{a}^* = (a_k, a_{k-1}, ..., a_1)$.

The following linear basis of Ω is the quasi-symmetric analogue to the monomial basis for symmetric functions. This basis for Ω is indexed by compositions and is given by

$$M_{\mathbf{a}} = \sum_{i_1 < i_2 < \cdots < i_k} w_{i_1}^{a_1} \cdot w_{i_2}^{a_2} \cdots w_{i_k}^{a_k},$$

where $\mathbf{a} \in \Sigma$ and $\mathbf{a} = (a_1, ..., a_k)$. We have the convention that $M_{\varnothing} = 1$. In analogy with the theory of symmetric functions, we will call $M_{\mathbf{a}}$ a *monomial quasi-symmetric function*. We say that two compositions $\mathbf{a} = (a_1, ..., a_k)$ and $\mathbf{b} = (b_1, ..., b_k)$ of the same length are *similar* if there exists a permutation $\pi \in S_k$ such that $a_i = b_{\pi(i)}$. Then

$$m_\lambda = \sum_{\mathbf{a}} M_{\mathbf{a}},$$

where m_λ is the monomial symmetric function and the sum is over all composition $\mathbf{a} \in \Sigma$ similar to $\lambda = (\lambda_1, ..., \lambda_k)$.

It follows directly from the definition that quasi-symmetric functions are closed under multiplication. Thus they form an algebra. The linear map $M_{\mathbf{a}} \mapsto M_{\mathbf{a}^*}$ extends to an algebra automorphism, which we will denote by $f \mapsto f^*$. Moreover, quasi-symmetric functions also form a coalgebra, with the coproduct and the augmentation given by the following formulas on the basis elements:

$$\Delta(M_{\mathbf{a}}) = \sum_{i=0}^{k} M_{(a_1, ..., a_i)} \otimes M_{(a_{i+1}, ..., a_k)},$$

$$\varepsilon(M_{\mathbf{a}}) = \delta_{\mathbf{a}, \varnothing}.$$

To see that this is in fact a bialgebra, one may write the coproduct and the augmentation as

$$\Delta(f(w_1, w_2, ...)) = f(w_1 \otimes 1, w_2 \otimes 1, ..., 1 \otimes w_1, 1 \otimes w_2, ...),$$

$$\varepsilon(f(w_1, w_2, ...)) = f(0, 0, ...).$$

Thus the augmentation is the constant term of the quasi-symmetric function. Now it is easy to see that both the coproduct and the augmentation are algebra maps, and hence it is a bialgebra.

To see that the bialgebra of quasi-symmetric functions is a Hopf algebra, we must show the antipode exists. This bialgebra is graded, by letting the degree of $M_{\mathbf{a}}$ be m, where $\mathbf{a}$ is a composition of m. Hence by Lemma 2.1 the quasi-symmetric functions form a Hopf algebra.

We will now give a formula for the antipode of a monomial quasi-symmetric function. By Hopf algebra techniques one can easily show that the antipode is given by

$$S(M_{\mathbf{a}}) = \sum_{j \geqslant 0} (-1)^j \cdot \sum_{0 = i_0 < i_1 < \cdots < i_j = k} \prod_{l=1}^{j} M_{(a_{i_{l-1}+1}, \ldots, a_{i_l})},$$

where the term $k=0$ only occurs when $(a_1, \ldots, a_n) = \varnothing$ and the product is 1 in this case. This formula is quite similar to Schmitt's formula for the antipode in the reduced incidence Hopf algebra; see [10].

There is a more explicit formula for the antipode. To be able to present it, we need to write the product of two monomial quasi-symmetric functions as a linear combination of monomial quasi-symmetric functions. We will be using strictly order preserving maps τ that are surjective, from the disjoint union of two chains $C' = \{x_1 < x_2 < \cdots < x_k\}$ and $C'' = \{y_1 < y_2 < \cdots < y_l\}$ to the chain $C = \{z_1 < z_2 < \cdots < z_r\}$. That is, τ is a map from $C' \cup C''$ to C.

Lemma 3.3. *Let* $\mathbf{a} = (a_1, \ldots, a_k)$ and $\mathbf{b} = (b_1, \ldots, b_l)$ *be two compositions. Then*

$$M_{\mathbf{a}} \cdot M_{\mathbf{b}} = \sum_{r=\max(k,l)}^{k+l} \sum_{\tau} M_{(\hat{\tau}(1), \hat{\tau}(2), \ldots, \hat{\tau}(r))},$$

where τ ranges over all strictly order-preserving surjective maps $C' \cup C'' \to C$, and $\hat{\tau}$ is defined by

$$\hat{\tau}(p) = \begin{cases} a_i + b_j, & \text{if there exist } i \text{ and } j \text{ such that } \tau(x_i) = p \text{ and } \tau(y_j) = p, \\ a_i, & \text{if there exist } i \text{ such that } \tau(x_i) = p \text{ but no } j \\ & \text{such that } \tau(y_j) = p, \\ b_j, & \text{if there exist } j \text{ such that } \tau(y_j) = p \text{ but no } i \\ & \text{such that } \tau(x_i) = p. \end{cases}$$

In the linear combination for $M_{\mathbf{a}} \cdot M_{\mathbf{b}}$ observe that the first entry in the composition of each monomial quasi-symmetric function is either $a_1 + b_1$, a_1, or b_1.

As an example of this lemma, we will write out the product of the two monomial quasi-symmetric functions $M_{(a,b)}$ and $M_{(c,d)}$:

$$\begin{aligned}
M_{(a,b)} \cdot M_{(c,d)} \\
&= M_{(a+c,b+d)} + M_{(a+c,b,d)} + M_{(a+c,d,b)} + M_{(c,a+d,b)} + M_{(a,b+c,d)} \\
&\quad + M_{(a,c,b+d)} + M_{(c,a,b+d)} + M_{(a,b,c,d)} + M_{(a,c,b,d)} \\
&\quad + M_{(a,c,d,b)} + M_{(c,a,b,d)} + M_{(c,a,d,b)} + M_{(c,d,a,b)}.
\end{aligned}$$

We may define a partial order on the compositions of the integer m by defining the covering relation to be

$$(a_1, ..., a_i + a_{i+1}, ..., a_k) \prec (a_1, ..., a_i, a_{i+1}, ..., a_k).$$

This makes Σ_m into a partially ordered set. In this order the composition $(1, 1, ..., 1)$ is the maximum element, and (m) the minimum element. Observe that the partial order Σ_m is isomorphic to the Boolean algebra B_{m-1}. This is easily seen by considering the expression $(1 * 1 * \cdots * 1)$, and replacing each star by either a plus sign or a comma.

PROPOSITION 3.4. *Let* $\mathbf{a} = (a_1, ..., a_k)$ *be a composition of m. Then the antipode is given by*

$$S(M_{\mathbf{a}}) = (-1)^{l(\mathbf{a})} \cdot \sum_{\mathbf{b} \leqslant \mathbf{a}} M_{\mathbf{b}^*},$$

where $\mathbf{b}$ *belongs to* Σ_m.

This formula for the antipode has been obtained independently by Malvenuto and Reutenauer in [8]. As an example, we present the antipode of $M_{(a,b,c)}$.

$$S(M_{(a,b,c)}) = -M_{(c,b,a)} - M_{(c+b,a)} - M_{(c,b+a)} - M_{(c+b+a)}.$$

Proof of Proposition 3.4. We proceed by induction on the length k. When k is 0 or 1, the result follows directly from the definition. Consider the defining relation of the antipode, as a recursive formula:

$$\begin{aligned}
S(M_{\mathbf{a}}) &= -\sum_{i=0}^{k-1} S(M_{(a_1,...,a_i)}) \cdot M_{(a_{i+1},...,a_k)} \\
&= -\sum_{i=0}^{k-1} (-1)^i \cdot \sum_{\mathbf{b} \leqslant (a_i,...,a_1)} M_{\mathbf{b}} \cdot M_{(a_{i+1},...,a_k)}. \qquad (2)
\end{aligned}$$

The first entry of $\mathbf{b}$ is of the form $a_i + \cdots + a_j$, where $j \leqslant i$. When we expand $M_{\mathbf{b}} \cdot M_{(a_{i+1}, \ldots, a_k)}$, every monomial quasi-symmetric function in the linear combination will have its first entry as in one of the following three cases:

1. $a_i + \cdots + a_j$, where $j \leqslant i$.
2. $a_i + \cdots + a_j + a_{i+1}$, where $j \leqslant i$.
3. a_{i+1}.

In the first case, we say that the monomial quasi-symmetric function has type i. In the two last cases, we say that the type is $i+1$. Hence the type is the largest index which occurs in the first entry of its composition.

Consider a monomial quasi-symmetric funcion $M_{\mathbf{c}}$ that occurs in the above expansion of $S(M_{\mathbf{a}})$. If it has type i, where $1 \leqslant i \leqslant k-1$, then it will occur in both the ith term and the $(i-1)$st term. Observe that these two occurrences will have opposite signs and hence will cancel each other. Thus the only monomials left are those of type k. They will all appear in the $(k-1)$st term and have the coefficient $(-1)^k$.

Finally, notice that a monomial quasi-symmetric function $M_{\mathbf{c}}$ of type k in (2) will satisfy $\mathbf{c} \leqslant \mathbf{a}^*$. This completes the proof. ∎

By definition of a monomial quasi-symmetric function, we can write the expression of the antipode in Proposition 3.4 as

$$S(M_{\mathbf{a}}) = (-1)^k \cdot \sum_{i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k} w_{i_1}^{a_k} \cdot w_{i_2}^{a_{k-1}} \cdots w_{i_k}^{a_1}.$$

4. The F-Quasi-Symmetric Function of a Poset

Definition 4.1. Let P be a graded poset. Define the F-quasi-symmetric function $F(P)$ by

$$F(P) = \sum_{\hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}} M_{(\rho(x_0, x_1), \rho(x_1, x_2), \ldots, \rho(x_{n-1}, x_n))},$$

where the sum ranges over all chains from $\hat{0}$ and $\hat{1}$.

Since $F(P)$ depends only on the type of the poset P, we can extend the definition of F by linearity to the reduced incidence Hopf algebra $\mathscr{I}$. By abuse of notation, we use the same symbol F for the F-quasi-symmetric function of a poset, and the F-quasi-symmetric function of a linear combination of types. That is,

$$F\left(\sum c_i \overline{P_i}\right) = \sum c_i F(P_i).$$

Observe that $F(P)$ is homogeneous of degree $\rho(P)$.

For a graded poset P of rank m and $S \subseteq \{1, 2, ..., m-1\}$, Stanley [12] defines the *rank-selected poset* $P(S)$ to be the poset $P(S) = \{x \in P : \rho(x) \in S\} \cup \{\hat{0}, \hat{1}\}$, and $\alpha(S)$ to be the number of maximal chains in $P(S)$. The vector $(\cdots, \alpha(S), \cdots)_{S \subseteq \{1, ..., m-1\}}$ is called the *flag f-vector* of P. The coefficients of the F-quasi-symmetric function encode the same information as the flag f-vector.

LEMMA 4.2. *Let P be a graded poset of rank m. Then*

$$F(P) = \sum_{S \subseteq \{1, ..., m-1\}} \alpha(S) \cdot M_{(s_1 - s_0, s_2 - s_1, ..., s_k - s_{k-1})},$$

where $S = \{s_1 < s_2 < \cdots < s_{k-1}\}$, $s_0 = 0$, and $s_k = m$.

The F-quasi-symmetric function of the Boolean algebra and the chain are given by

$$F(B_m) = h_1^m, \qquad F(C_m) = h_m,$$

where h_m is the mth complete symmetric function. Also, the F-quasi-symmetric function of the dual poset of P is determined by $F(P^*) = F(P)^*$.

Define $F_n(P; w_1, ..., w_n) = F(P; w_1, ..., w_n, 0, 0, ...)$. Observe that $F_n(P)$ is a polynomial in the variables $w_1, ..., w_n$.

LEMMA 4.3. *The following equality holds:*

$$F_n(P; w_1, ..., w_n) = \sum_{\hat{0} = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n = \hat{1}} w_1^{\rho(x_0, x_1)} \cdot w_2^{\rho(x_1, x_2)} \cdots w_n^{\rho(x_{n-1}, x_n)}.$$

Observe that $F_2(P; w_1, w_2)$ is the *homogeneous rank-generating function* of the poset P.

In perfect analogy with [6, Section I.2] we may observe that the ring Ω is isomorphic to the graded inverse limit of the rings Ω_n of quasi-symmetric functions in n variables. Under this isomorphism, every $f \in \Omega$ is identified with the limit $\lim_{n \to \infty} f(w_1, ..., w_n, 0, 0, ...)$. (Observe that $f(w_1, ..., w_n, 0, 0, ...)$ may be considered as an element of Ω_n.) In particular, we have

$$F(P) = \lim_{n \to \infty} F_n(P). \tag{3}$$

PROPOSITION 4.4. *The linear map F from the reduced incidence Hopf algebra $\mathscr{I}$ to the Hopf algebra of quasi-symmetric functions, Ω, is a Hopf algebra homomorphism.*

Proof. It is easy to see that $F(1)=1$ and that F commutes with the augmentation. To see that F is an algebra map, consider the polynomials F_n in the variables $w_1, ..., w_n$. Let us define $\kappa_i\colon \mathscr{I} \to \mathbf{k}[w_i]$ by $\kappa_i(\bar{P}) = w_i^{\rho(P)}$ and extend this definition by linearity. Since the rank of the Cartesian product of two posets is the sum of their ranks, we conclude that κ_i is an algebra map. By the identity (1) and Lemma 4.3 we can write F_n as

$$F_n = \mu^n \circ (\kappa_1 \otimes \cdots \otimes \kappa_n) \circ \Delta^n.$$

Since every operator on the right-hand side of this expression is an algebra map, it follows that $F_n(P \times Q) = F_n(P) \cdot F_n(Q)$. By the limit in Eq. (3) it follows that $F(P \times Q) = F(P) \cdot F(Q)$.

To see that F is a coalgebra map, consider the identities

$$\begin{aligned}
((F \otimes F) \circ \Delta)(P) &= \sum_{\hat{0} \leqslant x \leqslant \hat{1}} F([\hat{0}, x]) \otimes F([x, \hat{1}]) \\
&= \sum_{\hat{0} \leqslant x \leqslant \hat{1}} \left(\sum_{\hat{0} = y_0 < y_1 < \cdots < y_k = x} M_{(\rho(y_0, y_1), ..., \rho(y_{k-1}, y_k))} \right) \\
&\quad \otimes \left(\sum_{x = z_0 < z_1 < \cdots < z_m = \hat{1}} M_{(\rho(z_0, z_1), ..., \rho(z_{m-1}, z_m))} \right) \\
&= \sum_{\hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}} \sum_{k=0}^{n} M_{(\rho(x_0, x_1), ..., \rho(x_{k-1}, x_k))} \\
&\quad \otimes M_{(\rho(x_k, x_{k+1}), ..., \rho(x_{n-1}, x_n))} \\
&= \sum_{\hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}} \Delta(M_{(\rho(x_0, x_1), ..., \rho(x_{n-1}, x_n))}) \\
&= \Delta \left(\sum_{\hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}} M_{(\rho(x_0, x_1), ..., \rho(x_{n-1}, x_n))} \right) \\
&= \Delta(F(P)).
\end{aligned}$$

Hence, F is a bialgebra map. But any bialgebra map between two Hopf algebras is a Hopf algebra homomorphism, which concludes the proof. ∎

DEFINITION 4.5. A poset P is *triangular* if the number of maximal chains in an interval $[x, y]$ only depends on the rank of x and the rank of y. Let $B(n, m)$ denote the number of maximal chains in the interval $[x, y]$, where $n = \rho(x)$ and $m = \rho(y)$.

PROPOSITION 4.6. *Let P be a graded triangular poset of rank m, with function $B(\cdot, \cdot)$. Then the F-quasi-symmetric function of P is given by*

$$F(P) = \sum \frac{B(0, m)}{B(b_0, b_1) \cdot B(b_1, b_2) \cdots B(b_{k-1}, b_k)} \cdot M_{\mathbf{a}},$$

where the sum ranges over all compositions $\mathbf{a} = (a_1, ..., a_k) \in \Sigma_m$, and $b_i = a_1 + \cdots + a_i$ for all i.

Proof. In order to prove the identity, we need only count the number of chains $\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}$ such that $\rho(x_{i-1}, x_i) = a_i = b_i - b_{i-1}$ for all $i = 1, 2, ..., k$. We can choose the maximal chain between x_{i-1} and x_i in $B(b_{i-1}, b_i)$ possible ways. Hence we can extend the chain $\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}$ in exactly $B(b_0, b_1) \cdot B(b_1, b_2) \cdots B(b_{k-1}, b_k)$ ways to a maximal chain. But there are $B(0, m)$ maximal chains. Hence the number of such chains is $B(0, m)/[B(b_0, b_1)\, B(b_1, b_2) \cdots B(b_{k-1}, b_k)]$. ∎

Examples of triangular posets are the face lattice of the m-dimensional cube, and $I_m(\mathbb{F}_q)$, the lattice of isotropic subspaces of a vector space of dimension $2n$ over the finite field $\mathbb{F}_q$. For more examples, see [1, 2].

A triangular poset P is binomial if the function $B(n, m)$ only depends on the difference $m - n$. In this case we write $B(n, m) = B(m - n)$, and the function $B(\cdot)$ is called the factorial function. See [12, Section 3.15] for more about binomial posets.

COROLLARY 4.7. *Let P be a graded binomial poset of rank m, with factorial function $B(\cdot)$. Then the F-quasi-symmetric function of P is given by*

$$F(P) = \sum_{\mathbf{a} \in \Sigma_m} \frac{B(m)}{B(a_1) \cdot B(a_2) \cdots B(a_k)} \cdot M_{\mathbf{a}}.$$

The class of binomial posets includes the chain, the Boolean algebra, and the lattice of subspaces of a finite dimensional vector space over a finite field. Note that the coefficients of the monomial quasi-symmetric functions in Corollary 4.7 are equal for compositions that are similar. Thus we conclude that the F-quasi-symmetric function of a binomial poset is a symmetric function.

One may ask which graded posets have a symmetric F-quasi-symmetric function. As we just observed, binomial posets have this property. The Cartesian product of two posets each with a symmetric F-quasi-symmetric function is a poset with a symmetric F-quasi-symmetric function. But

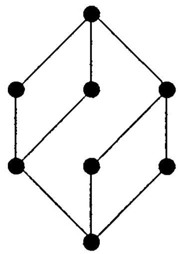

FIG. 1. A poset P such that $F(P)$ is a symmetric function.

there are also other posets with this property. For example, see the poset in Fig. 1. Its F-quasi-symmetric function is equal to $M_{(3)} + 3M_{(2,1)} + 3M_{(1,2)} + 4M_{(1,1,1)}$. For more on this question, see [13].

5. AN INVOLUTION ON THE QUASI-SYMMETRIC FUNCTIONS

Define the linear map $\omega\colon \Omega \to \Omega$ by

$$\omega(M_{\mathbf{a}}) = (-1)^{m-k} \cdot \sum_{\mathbf{b} \leqslant \mathbf{a}} M_{\mathbf{b}},$$

where $\mathbf{a}$ is a composition of m into k parts. It has been shown by Malvenuto and Reutenauer [7, 8] that this map is a Hopf algebra homomorphism. The map ω is also an involution, that is, $\omega^2 = 1$. Moreover, when ω is restricted to symmetric functions, we obtain the classical involution ω on symmetric functions.

Observe that the antipode of a monomial quasi-symmetric function may now be described by

$$S(M_{\mathbf{a}}) = (-1)^{|\mathbf{a}|} \cdot \omega(M_{\mathbf{a}^*}).$$

By linearity the antipode of a homogeneous quasi-symmetric function f of degree m is given by $S(f) = (-1)^m \cdot \omega(f^*)$.

For a graded poset P we define the $\tilde{F}$-quasi-symmetric function $\tilde{F}(P)$ by

$$\tilde{F}(P) = (-1)^{\rho(P)} \cdot \sum_{\hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}} \mu(x_0, x_1) \cdots \mu(x_{n-1}, x_n) \cdot M_{(\rho(x_0, x_1), \ldots, \rho(x_{n-1}, x_n))},$$

where the sum ranges over all chains from $\hat{0}$ and $\hat{1}$.

Proposition 5.1. *For a graded poset P, we have that*

$$\tilde{F}(P) = \omega(F(P)).$$

Proof. Let $\hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}$ be a chain in the poset P. By Philip Hall's formula for the Möbius function we have

$$\mu(x_{i-1}, x_i) = \sum_{x_{i-1} = y_0 < y_1 < \cdots < y_h = x_i} (-1)^h.$$

Taking the product of this equation for $i = 1, 2, ..., n$, we obtain

$$\mu(x_0, x_1) \cdot \mu(x_1, x_2) \cdots \mu(x_{n-1}, x_n) = \sum_{\hat{0} = z_0 < z_1 < \cdots < z_k = \hat{1}} (-1)^k,$$

where the chain $\hat{0} = z_0 < z_1 < \cdots < z_k = \hat{1}$ contains our original chain $\hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}$. By multiplying by $M_{(\rho(x_0, x_1), ..., \rho(x_{n-1}, x_n))}$ and summing over all chains $\hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}$ in the poset P, we deduce that

$$\begin{aligned}
&(-1)^{\rho(P)} \cdot \tilde{F}(P) \\
&\quad = \sum_{\hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}} M_{(\rho(x_0, x_1), ..., \rho(x_{n-1}, x_n))} \cdot \sum_{\hat{0} = z_0 < z_1 < \cdots < z_k = \hat{1}} (-1)^k \\
&\quad = \sum_{\hat{0} = z_0 < z_1 < \cdots < z_k = \hat{1}} (-1)^k \cdot \sum_{\hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}} M_{(\rho(x_0, x_1), ..., \rho(x_{n-1}, x_n))} \\
&\quad = \sum_{\hat{0} = z_0 < z_1 < \cdots < z_k = \hat{1}} (-1)^k \cdot \sum_{\mathbf{b} \leqslant (\rho(z_0, z_1), \rho(z_1, z_2), ..., \rho(z_{k-1}, z_k))} M_{\mathbf{b}} \\
&\quad = \sum_{\hat{0} = z_0 < z_1 < \cdots < z_k = \hat{1}} (-1)^{\rho(P)} \cdot \omega(M_{(\rho(z_0, z_1), \rho(z_1, z_2), ..., \rho(z_{k-1}, z_k))}) \\
&\quad = (-1)^{\rho(P)} \cdot \omega(F(P)). \quad \blacksquare
\end{aligned}$$

The $\tilde{F}$-quasi-symmetric function of a chain is given by $\tilde{F}(C_m) = \omega(F(C_m)) = \omega(h_m) = e_m$, where e_m is the mth elementary symmetric function. This can also be observed directly from the definition of the $\tilde{F}$-quasi-symmetric function.

Corollary 5.2. *The linear map $\tilde{F}$ from the reduced incidence Hopf algebra $\mathscr{I}$ to the Hopf algebra Ω of quasi-symmetric functions is a Hopf algebra homomorphism.*

6. Zeta and Characteristic Polynomials

The polynomial algebra $\mathbf{k}[x]$ may be considered as a Hopf algebra, by defining the coproduct, the augmentation, and the antipode by

$$\begin{aligned} \Delta(p(x)) &= p(x \otimes 1 + 1 \otimes x), \\ \varepsilon(p(x)) &= p(0), \\ S(p(x)) &= p(-x). \end{aligned}$$

It is easy to verify that this is a Hopf algebra.

For a nonnegative integer x, define the infinite vector

$$\mathbf{1}_x = (\underbrace{1, 1, ..., 1}_{x}, 0, 0, ...).$$

Observe that

$$M_{\mathbf{a}}(\mathbf{1}_x) = \binom{x}{k},$$

where the composition $\mathbf{a}$ has length k, is a polynomial in x. Thus for a quasi-symmetric function f, which is a linear combination of $M_{\mathbf{a}}$'s we have that $f(\mathbf{1}_x)$ is a polynomial in x. Hence we can define the function $\Psi: \mathscr{I} \to \mathbf{k}[x]$ by

$$\Psi(f) = f(\mathbf{1}_x),$$

and extend it to all values of x.

Proposition 6.1. *The function Ψ is a Hopf algebra homomorphism from the Hopf algebra Ω of quasi-symmetric functions to the Hopf algebra $\mathbf{k}[x]$.*

Proof. It follows from the definition of Ψ that it is an algebra homomorphism. Recall the Vandermonde identity

$$\binom{x+y}{k} = \sum_{i=0}^{k} \binom{x}{i} \cdot \binom{y}{k-i}.$$

This implies that

$$\begin{aligned} \Delta(M_{\mathbf{a}}(\mathbf{1}_x)) &= \Delta\left(\binom{x}{k}\right) \\ &= \binom{x \otimes 1 + 1 \otimes x}{k} \end{aligned}$$

$$= \sum_{i=0}^{k} \binom{x}{i} \otimes \binom{x}{k-i}$$

$$= \sum_{i=0}^{k} M_{(a_1,\ldots,a_i)}(\mathbf{1}_x) \otimes M_{(a_{i+1},\ldots,a_k)}(\mathbf{1}_x).$$

By linearity this gives $\Delta \circ \Psi = (\Psi \otimes \Psi) \circ \Delta$. Also,

$$\begin{aligned}\varepsilon(M_{\mathbf{a}}(\mathbf{1}_x)) &= M_{\mathbf{a}}(\mathbf{1}_0) \\ &= M_{\mathbf{a}}(0, 0, \ldots) \\ &= \varepsilon(M_{\mathbf{a}}).\end{aligned}$$

Hence $\varepsilon \circ \Psi = \varepsilon$, and we conclude that Ψ is a bialgebra homomorphism. Since the antipode is uniquely determined by its defining relation, it is easy to see that Ψ commutes with the antipode. ∎

COROLLARY 6.2. *Let $Z\colon \mathcal{I} \to \mathbf{k}[x]$ be the homomorphism defined by $Z = \Psi \circ F$. Then Z is a Hopf algebra homomorphism, and for a poset P the polynomial $Z(P)$ is the zeta polynomial of P.*

Proof. Directly from Propositions 4.4 and 6.1, it follows that Z is a Hopf algebra homomorphism. By definition, the zeta polynomial of a poset P evaluated at a nonnegative integer n is equal to the number of multichains of length n from $\hat{0}$ to $\hat{1}$ in the poset P. But this is true for $Z(P) = (\Psi \circ F)(P)$, since

$$\begin{aligned}Z(P; n) &= F(P; \mathbf{1}_n) \\ &= F_n(P; 1, 1, \ldots, 1) \\ &= \sum_{\hat{0} = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n = \hat{1}} 1^{\rho(x_0, x_1)} 1^{\rho(x_1, x_2)} \cdots 1^{\rho(x_{n-1}, x_n)} \\ &= \text{number of multichains from } \hat{0} \text{ to } \hat{1} \text{ of length } n \text{ in the poset } P,\end{aligned}$$

where the third equality is by Lemma 4.3. ∎

Since for a composition $\mathbf{a}$ of length k, $M_{\mathbf{a}}(\mathbf{1}_x) = \binom{x}{k} = M_{\mathbf{a}^*}(\mathbf{1}_x)$ holds, we have $\Psi(f) = \Psi(f^*)$ for all quasi-symmetric functions f. Notably, for a poset P we have $Z(P) = Z(P^*)$.

The *homogeneous characteristic polynomial* of a graded poset P of rank m is defined as

$$\chi(P; q_1, q_2) = \sum_{x \in P} \mu(\hat{0}, x) \cdot q_1^{\rho(x)} \cdot q_2^{m - \rho(x)}.$$

By setting $q_1 = 1$, we recover the characteristic polynomial. We may compute the characteristic polynomial of a poset P by knowing the F-quasi-symmetric function of this poset. In order to do this, let us define $\psi_q\colon \Omega \to \mathbf{k}[q]$ by $\psi_q(f) = f(q, 0, 0, \ldots)$.

PROPOSITION 6.3. *Let P be a graded poset. Then*

$$\chi(P; q_1, q_2) = \mu \circ (\psi_{q_1} \otimes \psi_{q_2}) \circ (S \otimes 1) \circ \Delta \circ F(P),$$

where μ is the multiplication map $\mu\colon \mathbf{k}[q_1] \otimes \mathbf{k}[q_2] \to \mathbf{k}[q_1, q_2]$.

Proof. First observe that for a poset P, we have $\psi_q(F(P)) = q^{\rho(P)}$. In particular, $\psi_1(F(P)) = 1 = \phi(P)$, where $\phi(P)$ is as defined at the end of Section 2. Hence, the Möbius function $\mu(P)$ is given by $\mu(P) = \phi(S(P)) = \psi_1(F(S(P))) = \psi_1(S(F(P)))$. It follows that $\psi_q(S(F(P))) = \mu(P) \cdot q^{\rho(P)}$. Hence we have

$$\begin{aligned}
&\mu \circ (\psi_{q_1} \otimes \psi_{q_2}) \circ (S \otimes 1) \circ \Delta \circ F(P) \\
&\quad = \mu \circ (\psi_{q_1} \otimes \psi_{q_2}) \circ (S \otimes 1) \circ (F \otimes F) \circ \Delta(P) \\
&\quad = \sum_{\hat{0} \leqslant x \leqslant \hat{1}} \psi_{q_1}(S(F([\hat{0}, x]))) \cdot \psi_{q_2}(F([x, \hat{1}])) \\
&\quad = \sum_{\hat{0} \leqslant x \leqslant \hat{1}} \mu(P) \cdot q_1^{\rho(x)} \cdot q_2^{\rho(x, \hat{1})} \\
&\quad = \sum_{\hat{0} \leqslant x \leqslant \hat{1}} \mu(P) \cdot q_1^{\rho(x)} \cdot q_2^{m - \rho(x)} \\
&\quad = \chi(P; q_1, q_2),
\end{aligned}$$

where m is the rank of the poset P. ∎

7. EULERIAN POSETS

PROPOSITION 7.1. *Let P be an Eulerian poset. Then $\tilde{F}(P) = F(P)$.*

Proof. This follows directly from the definitions of F and $\tilde{F}$, and the identity

$$\begin{aligned}
&\mu(x_0, x_1) \cdot \mu(x_1, x_2) \cdots \mu(x_{n-1}, x_n) \\
&\quad = (-1)^{\rho(x_0, x_1)} \cdot (-1)^{\rho(x_1, x_2)} \cdots (-1)^{\rho(x_{n-1}, x_n)} \\
&\quad = (-1)^{\rho(x_0, x_1) + \rho(x_1, x_2) + \cdots + \rho(x_{n-1}, x_n)} = (-1)^{\rho(P)},
\end{aligned}$$

where $\hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}$ is chain in the poset P. ∎

For a graded poset P, Stanley [12] also defined *the rank-selected Möbius invariant* $\beta(S)$ by

$$\beta(S) = \sum_{T \subseteq S} (-1)^{|S| - |T|} \cdot \alpha(T).$$

This identity is equivalent to $\beta(S) = (-1)^{|S|-1} \cdot \mu(P_S)$ is the Möbius function of the rank-selected poset P_S. A well-known fact about Eulerian posets is that $\beta(S) = \beta(\bar{S})$ for all $S \subseteq \{1, 2, ..., m-1\}$, where $m = \rho(P)$ and $\bar{S}$ is the complement of S. That is, $\bar{S} = \{1, 2, ..., m-1\} - S$. Proposition 7.1 is equivalent to this fact.

PROPOSITION 7.2. *For a graded poset P, the following three conditions are equivalent:*

(1) $F(P) = \tilde{F}(P)$,

(2) $F(S(P)) = (-1)^{\rho(P)} \cdot F(P^*)$,

(3) *for all subsets S of* $\{1, 2, ..., \rho(P) - 1\}$ *we have* $\beta(S) = \beta(\bar{S})$.

Proof. As before, let m be the rank of the poset P. To see that conditions (1) and (2) are equivalent, consider the two identities

$$\begin{aligned} F(S(P)) &= S(F(P)) \\ &= (-1)^m \cdot \omega(F(P)^*) \\ &= (-1)^m \cdot \omega(F(P))^* \\ &= (-1)^m \cdot \tilde{F}(P)^* \end{aligned}$$

and

$$(-1)^m \cdot F(P)^* = (-1)^m \cdot F(P^*).$$

If condition (1) holds, then we may concatenate these two identities and obtain condition (2). Similarly, if condition (2) holds, then we may concatenate the reverse of the two identities above and obtain condition (1).

To see that condition (1) implies condition (3), write

$$F(P) = \sum_{\mathbf{a} \in \Sigma_m} c_{\mathbf{a}} \cdot M_{\mathbf{a}}.$$

Then we have that

$$\begin{aligned} \tilde{F}(P) &= \omega(F(P)) \\ &= \sum_{\mathbf{b} \in \Sigma_m} c_{\mathbf{b}} \cdot \omega(M_{\mathbf{b}}) \end{aligned}$$

$$= (-1)^m \cdot \sum_{\mathbf{b} \in \Sigma_m} c_{\mathbf{b}} \cdot (-1)^{l(\mathbf{b})} \cdot \sum_{\mathbf{a} \leqslant \mathbf{b}} M_{\mathbf{a}}$$

$$= (-1)^m \cdot \sum_{\mathbf{a} \in \Sigma_m} \left(\sum_{\mathbf{b} \geqslant \mathbf{a}} (-1)^{l(\mathbf{b})} \cdot c_{\mathbf{b}} \right) \cdot M_{\mathbf{a}}.$$

Comparing the coefficients of $M_{\mathbf{a}}$ we obtain

$$c_{\mathbf{a}} = (-1)^m \cdot \sum_{\mathbf{b} \geqslant \mathbf{a}} (-1)^{l(\mathbf{b})} \cdot c_{\mathbf{b}}.$$

By Lemma 4.2 this is equivalent to

$$\alpha(T) = (-1)^m \cdot \sum_{S \supseteq T} (-1)^{|S|+1} \cdot \alpha(S).$$

Substituting this last equation into the definition of β, we get

$$\begin{aligned} \beta(U) &= \sum_{T \subseteq U} (-1)^{|U|-|T|} \, \alpha(T) \\ &= (-1)^m \cdot \sum_{T \subseteq U} (-1)^{|U|-|T|} \cdot \sum_{S \supseteq T} (-1)^{|S|+1} \cdot \alpha(S) \\ &= (-1)^m \cdot \sum_{S} \left(\sum_{T \subseteq S \cap U} (-1)^{|T|} \right) \cdot (-1)^{|U|+|S|+1} \cdot \alpha(S). \end{aligned}$$

The inner sum is equal to 0, unless $S \cap U = \varnothing$, which is equivalent to $S \subseteq \bar{U}$. Thus

$$\begin{aligned} \beta(U) &= (-1)^m \cdot \sum_{S \subseteq \bar{U}} (-1)^{|U|+|S|+1} \cdot \alpha(S) \\ &= \sum_{S \subseteq \bar{U}} (-1)^{|\bar{U}|-|S|} \cdot \alpha(S) \\ &= \beta(\bar{U}). \end{aligned}$$

This shows the implication. A similar calculation shows that condition (3) implies condition (1). ▮

Corollary 7.3. *Let P be an Eulerian poset. Then*

$$Z(P; -x) = (-1)^{\rho(P)} \cdot Z(P; x).$$

Proof. This is a straightforward consequence of Propositions 7.1 and 7.2 and the fact that the antipode in $\mathbf{k}[x]$ is given by $S(p(x)) = p(-x)$. ▮

8. Hierarchical Simplicial Complexes

We will generalize the previous theory concerning graded posets to completely balanced simplicial complexes. In order to this let us only recall the most essential definitions.

Definition 8.1. An (abstract) *simplicial complex* Δ is a family of sets (called faces) on a vertex set V such that

(i) $\{v\} \in \Delta$ for every $v \in V$, and

(ii) if $F \in \Delta$ and $G \subseteq F$ then $G \in \Delta$.

For a face $\sigma \in \Delta$ we call $|\sigma| - 1$ the *dimension* of σ. A one-dimensional face is called an edge. Thus an edge consists of two vertices. The maximal faces are called *facets*. A simplical complex is *pure* if all of its facets have the same dimension. The dimension of a pure simplicial complex is the dimension of its facets.

Given a poset P, not necessarily graded, define the *order complex* $\Delta(P)$ to have vertex set P and let its faces be the chains in the poset P.

For a simplicial complex Δ, the *reduced Euler characteristic* $\tilde{\chi}(\Delta)$ is defined by the alternating sum

$$\tilde{\chi}(\Delta) = -f_{-1} + f_0 - f_1 + f_2 - \cdots,$$

where f_i is the number of i-dimensional faces of Δ. The above definition is related to Philip Hall's formula for the Möbius function of a poset. That is, if P is a poset with $\hat{0}$ and $\hat{1}$, then the Möbius funcion of P, $\mu(P)$, is equal to the reduced Euler characteristic of $\Delta(P - \{\hat{0}, \hat{1}\})$.

Definition 8.2. A *hierarchical simplicial complex* is a simplicial complex Δ such that each face of Δ is enriched with a linear order such that the orders are compatible. That is, if $x, y \in F, G$, where F and G are faces of Δ, then $x \leqslant_F y$ is equivalent to $x \leqslant_G y$.

We will write $F = \{x_0 < x_1 < \cdots < x_d\}$ for a face in a hierarchical simplicial complex when the face F is the set $F = \{x_0, x_1, ..., x_d\}$ and the linear order is given by $x_0 < x_1 < \cdots < x_d$.

Examples. 1. An example of a hierarchical simplicial complex is the order complex, $\Delta(P)$, of a poset P. The linear order in each face is just the order relation of the poset restricted to that face.

2. For an example of a hierarchical simplicial complex that is not an order complex, consider the simplicial complex on the set $V = \{a, b, c, d, e\}$

with facets $\{a<c<e\}$ and $\{b<c<d\}$. Observe that there is no transitivity between different faces.

3. Given a simplicial complex Δ on a vertex set V and a linear order on V, we obtain a hierarchical simplicial complex by letting each face inherit the given linear order.

4. For an example of a hierarchical simplicial complex that is not of the form in Example 3, consider the vertex set $V=\{a, b, c\}$. Let the facets be the following linearly ordered sets: $\{a<b\}$, $\{b<c\}$, and $\{c<a\}$.

We define the product $\Delta_1 \cdot \Delta_2$ of two hierarchical simplicial complexes Δ_1 and Δ_2 as follows. The vertex set of Δ is the Cartesian product of the vertex sets of Δ_1 and Δ_2. We will now describe the set of faces of Δ. Let $F=\{x_0<x_1<\cdots<x_{d_1}\}$ be a d_1-dimensional face of Δ_1, and $G=\{y_0<y_1<\cdots<y_{d_2}\}$ be a d_2-dimensional face of Δ_2. Let (i_0, j_0), (i_1, j_1), ..., (i_d, j_d) be a path from $(0, 0)$ to (d_1, d_2) only taking unit steps, where $d=d_1+d_2$. That is, $(i_{k+1}, j_{k+1})-(i_k, j_k)$ is equal to $(1, 0)$ or $(0, 1)$. Then we let the following linearly ordered set be a face of Δ:

$$\{(x_{i_0}, y_{j_0})<(x_{i_1}, y_{j_1})<\cdots<(x_{i_d}, y_{j_d})\}.$$

For each d_1-dimensional face of Δ_1 and d_2-dimensional face of Δ_2, we will obtain $\binom{d}{d_1}$ d-dimensional faces of Δ, since there are $\binom{d}{d_1}$ paths from $(0, 0)$ to (d_1, d_2).

It is easy to see that this product of hierarchical simplicial complexes is commutative and associative. It is also straightforward to see that the simplicial complex consisting only of one vertex, which is a face, is the unit under this product. Hence we will denote this one-element simplicial complex by 1.

By using a result in [3, Chapter II, Section 8], it is easy to show that if $|\Delta_i|$ is the geometric realization of the hierarchical simplicial complex Δ_i, then $|\Delta_1| \times |\Delta_2|$ is the geometric realization of hierarchical simplicial complex $\Delta_1 \cdot \Delta_2$. This triangulation of $|\Delta_1| \times |\Delta_2|$ is called the *staircase triangulation* in [14].

We say that a hierarchical simplicial complex Δ is *ranked* if there is a function $\rho\colon V \to \mathbb{Z}$ such that if x and y belongs to a facet F in Δ and x is covered by y then $\rho(x)+1=\rho(y)$. Moreover, a hierarchical simplicial complex Δ has a zero if there exists an element $\hat{0}$ which is the smallest element in each facet of Δ. Similarly, Δ has a one if there exists an element $\hat{1}$ which is the largest element in each facet of Δ. A hierarchical simplicial complex Δ with $\hat{0}$ and $\hat{1}$ is *graded* if there is a rank function such that $\rho(\hat{0})=0$. The dimension of a graded hierarchical simplicial complex Δ is equal to $\rho(\hat{1})$. The one-element simplicial complex 1 is considered to be a graded complex of dimension 0, with $\hat{0}_1=\hat{1}_1$.

Given a hierarchical simplicial complex Δ and an edge $\{x<z\}$ in Δ, define the *interval* $[x, z]$ to be the hierarchical subcomplex obtained by restricting the ordered face structure of Δ to the vertex set

$$\{y \in V : \exists F \in \Delta \text{ such that } x \leqslant_F y \leqslant_F z\}.$$

Observe that $[x, z]$ is a hierarchical simplicial complex where $\hat{0}_{[x,z]} = x$ and $\hat{1}_{[x,z]} = z$.

The class of graded hierarchical simplicial complexes is closed under taking intervals. Also, it is closed under taking products. Note that $\hat{0}_{\Delta_1 \cdot \Delta_2} = (\hat{0}_{\Delta_1}, \hat{0}_{\Delta_2})$, $\hat{1}_{\Delta_1 \cdot \Delta_2} = (\hat{1}_{\Delta_1}, \hat{1}_{\Delta_2})$, and $\rho_{\Delta_1 \cdot \Delta_2}((x, y)) = \rho_{\Delta_1}(x) + \rho_{\Delta_2}(y)$.

Stanley [11] defines a pure simplicial complex Δ of dimension d to be *completely balanced* when there exists a partition $\{V_0, V_1, ..., V_d\}$ of the vertex set V such that for each facet F and each block V_i in the partition we have $|F \cap V_i| = 1$. It is clear from this definition that graded hierarchical simplicial complexes are completely balanced.

It is easy to see that the reduced Euler characteristic of a hierarchical simplicial complex, where $\hat{0} \neq \hat{1}$, is equal to 0. Instead, we are interested in the statistic $E(\Delta) = \tilde{\chi}(\Delta - \{\hat{0}, \hat{1}\})$. We can express this as

$$E(\Delta) = \sum_{\{\hat{0}, \hat{1}\} \subseteq F \in \Delta} (-1)^{\dim(F)}.$$

Observe that for a graded poset P with a minimal and a maximal element, we have $\mu(P) = E(\Delta(P))$, where $\Delta(P)$ denotes the order complex of P.

9. The Hopf Algebra of Graded Hierarchical Simplicial Complexes

We will now develop the Hopf algebra of graded hierarchical simplicial complexes. We say that two hierarchical simplicial complexes have the same *type* if there is a bijection between them that carries faces to faces and preserves the ordering of each face. We denote the type of hierarchical simplicial complex Δ by $\bar{\Delta}$. Observe that the product of types is well-defined, that is, $\overline{\Delta_1} \cdot \overline{\Delta_2} = \overline{\Delta_1 \cdot \Delta_2}$. We write $\bar{1} = 1$ for the type of the one element simplicial complex.

We will now define the Hopf algebra spanned by types of hierarchical simplicial complexes. Let Υ be the linear span over the field $\mathbf{k}$ of types of graded hierarchical simplicial complexes. By the previous paragraph we may define an algebra structure on this space. Hence Υ is an algebra with identity 1.

We enrich the structure of the space Υ by defining a coproduct Δ: $\Upsilon \to \Upsilon \otimes \Upsilon$ and an augmentation ε: $\Upsilon \to \mathbf{k}$. The definition of the coproduct is

$$\Delta(\bar{\Delta}) = \sum_{x \in V} \overline{[\hat{0}, x]} \otimes \overline{[x, \hat{1}]},$$

where V is the vertex set of the complex $\varDelta$. The definition of the augmentation is

$$\varepsilon(\overline{\varDelta}) = \begin{cases} 1, & \text{if } \overline{\varDelta} = 1, \\ 0, & \text{if } \overline{\varDelta} \neq 1. \end{cases}$$

Both definitions are extended to Υ by linearity. It is easy to see that Υ, together with the coproduct $\varDelta$ and the augmentation ε, form a coalgebra.

It remains to show that Υ is a bialgebra. We will do this with the help of the following lemma.

LEMMA 9.1. *Let $\varDelta_1$ and $\varDelta_2$ be two hierarchical simplicial complexes and $\{y_i < z_i\}$ be an edge in $\varDelta_i$. Then*

$$[y_1, z_1] \cdot [y_2, z_2] = [(y_1, y_2), (z_1, z_2)],$$

where $[y_i, z_i]$ is an interval of the complex $\varDelta_i$ and $[(y_1, y_2), (z_1, z_2)]$ is an interval of the product complex $\varDelta_1 \cdot \varDelta_2$.

Let V_i be the vertex set of $\varDelta_i$. Then we have

$$\begin{aligned}
\varDelta(\varDelta_1 \cdot \varDelta_2) &= \sum_{(x_1, x_2) \in V_1 \times V_2} \overline{[(\hat{0}_{\varDelta_1}, \hat{0}_{\varDelta_2}), (x_1, x_2)]} \otimes \overline{[(x_1, x_2), (\hat{1}_{\varDelta_1}, \hat{1}_{\varDelta_2})]} \\
&= \sum_{x_1 \in V_1} \sum_{x_2 \in V_2} (\overline{[\hat{0}_{\varDelta_1}, x_1]} \cdot \overline{[\hat{0}_{\varDelta_2}, x_2]}) \otimes (\overline{[x_1, \hat{1}_{\varDelta_1}]} \cdot \overline{[x_2, \hat{1}_{\varDelta_2}]}) \\
&= \sum_{x_1 \in V_1} \sum_{x_2 \in V_2} (\overline{[\hat{0}_{\varDelta_1}, x_1]} \otimes \overline{[x_1, \hat{1}_{\varDelta_1}]}) \cdot (\overline{[\hat{0}_{\varDelta_2}, x_2]} \otimes \overline{[x_2, \hat{1}_{\varDelta_2}]}) \\
&= \varDelta(\varDelta_1) \cdot \varDelta(\varDelta_2),
\end{aligned}$$

and hence Υ is a bialgebra.

The bialgebra Υ is graded: the grading is given by the dimensions of the hierarchical simplicial complexes. Hence by Lemma 2.1, we conclude that Υ forms a Hopf algebra.

It is easy to verify that the map $\overline{P} \mapsto \overline{\varDelta(P)}$, where P is a graded poset and $\varDelta(P)$ is its order complex, is a natural Hopf algebra homomorphism from $\mathscr{I}$ to Υ.

10. HIERARCHICAL COMPLEXES AND THEIR QUASI-SYMMETRIC FUNCTIONS

For an edge $\{x < y\}$ in a graded hierarchical simplicial complex $\varDelta$, we denote $\rho(x, y) = \rho(y) - \rho(x)$. Observe that $\rho(x, y) \geqslant 1$ and that $\rho(x, y)$ is equal to the dimension of the interval $[x, y]$.

DEFINITION 10.1. Let Δ be a graded hierarchical simplicial complex. Define the *F-quasi-symmetric* function $F(\Delta)$ by

$$F(\Delta) = \sum_{F = \{x_0 < x_1 < \cdots < x_n\}} M_{(\rho(x_0, x_1), \rho(x_1, x_2), ..., \rho(x_{n-1}, x_n))},$$

where the sum ranges over all faces $F = \{x_0 < x_1 < \cdots < x_n\}$ of Δ such that $x_0 = \hat{0}$ and $x_n = \hat{1}$. Similarly define

$$\tilde{F}(\Delta) = \sum_{F = \{x_0 < x_1 < \cdots < x_n\}} E([x_0, x_1]) \cdots E([x_{n-1}, x_n]) \cdot M_{(\rho(x_0, x_1), ..., \rho(x_{n-1}, x_n))}.$$

PROPOSITION 10.2. *For a graded hierarchical simplicial complex Δ, we have*

$$\tilde{F}(\Delta) = \omega(F(\Delta)).$$

That is, $\tilde{F} = \omega \circ F$.

The proof is quite similar to the proof of Proposition 5.1.

PROPOSITION 10.3. *The two linear maps F and $\tilde{F}$ from the reduced incidence Hopf algebra Υ to the Hopf algebra of quasi-symmetric functions, Ω, are Hopf algebra homomorphisms.*

The proof that $F: \Upsilon \to \Omega$ is a Hopf algebra homomorphism is similar to Proposition 4.4. By the identity $\tilde{F} = \omega \circ F$, it follows that $\tilde{F}$ is a Hopf algebra homomorphism.

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