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Juggling and applications to q-analogues

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Abstract

We consider juggling patterns where the juggler can only catch and throw one ball at a time, and patterns where the juggler can handle many balls at the same time. Using a crossing statistic, we obtain explicit q-enumeration formulas. Our techniques give a natural combinatorial interpretation of the q-Stirling numbers of the second kind and a bijective proof of an identity of Carlitz. By generalizing these techniques, we give a bijective proof of a q-identity involving unitary compositions due to Haglund. Also, juggling patterns enable us to easily compute the Poincaré series of the affine Weyl group $\tilde{A}_{d-1}$.

Résumé

Nous considérons des configurations de jonglerie dans lesquelles le jongleur ne peut attraper ou lancer qu'une seule balle à la fois, ainsi que les configurations où le jongleur peut manipuler plusieurs balles à la fois. En considérant une statistique de croisements, nous obtenons des formules explicites de q-énumération. Nos techniques fournissent des interprétations combinatoires naturelles pour les q-nombres de Stirling de deuxième espèce ainsi qu'une preuve bijective d'une identité de Carlitz. Généralisant ces techniques, nous donnons une preuve bijective d'une q-identité des compositions unitaires due à Haglund. Les configurations de jonglerie nous permettent aussi de calculer la série de Poincaré du groupe du Weyl affine $\tilde{A}_{d-1}$.

1. Introduction

Consider the pattern in Fig. 1. We can think of this picture as the pattern that juggling balls describe as they are juggled. The horizontal axis is the time axis. At each integer time point one ball is caught and then thrown. At time points $0, 3, 6, \ldots$ each

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ball is thrown high enough so that it lands one time unit later. Similarly, at time points $1, 4, 7, \ldots$ each ball is thrown so that it lands two time units later, while at time points $2, 5, 8, \ldots$ each ball will land three time units later. Thus this pattern is periodic with period $d = 3$. In this pattern there are two balls since the arcs describe two infinite paths.

In this paper we will enumerate periodic patterns like the one just described. Fig. 1 shows a pattern where the juggler can only catch and throw one ball at a time. We will also consider patterns where the juggler has the ability to catch and throw many balls at a time. See Fig. 2 for an example of such a pattern. Among jugglers this is called *multiplex*. We say that a juggling pattern is *simple* if the juggler can only catch and throw one ball at a time.

We denote the pattern in Fig. 1 by the vectors $\boldsymbol{x} = (0, 1, 2)$ and $\boldsymbol{a} = (1, 2, 3)$. The fact that there is one 0 in the vector $\boldsymbol{x}$ means that at times $0 \bmod d$ the juggler catches and throws one ball. If there were three 1's appearing in the vector $\boldsymbol{x}$, this would mean that the juggler catches and throws three balls at times $1 \bmod d$. The entries of the vector $\boldsymbol{a}$ indicate how far each ball is thrown, that is, when it will return to the juggler's hand. Thus at time periods $x_i \bmod d$ the juggler throws a ball a_i time units. The pattern in Fig. 2 is represented by $d = 2$, $\boldsymbol{x} = (0, 0, 1)$, and $\boldsymbol{a} = (1, 4, 1)$.

Buhler et al. [2] proved that the number of simple juggling patterns of period d and at most n balls is equal to n^d. Their proof uses the fact that the number of permutations with k excedances is equal to the Eulerian number $A(n, k + 1)$ [17, Proposition 1.3.12]. Stanley bijectified their proof [18]. Using a completely different approach, we simultaneously generalize the n^d result in two ways. We include juggling patterns with multiplex and give q-analogues of these results.

Between time points 1 and 2 in Fig. 1, the paths of the two balls cross. We call this a *crossing*. Since the pattern is periodic, similar crossings appear between 4 and 5, 7 and 8, etc. There is one more crossing, namely between time points 2 and 3. Thus we

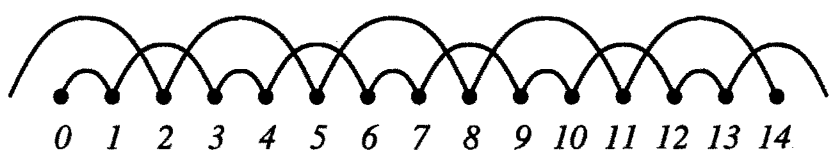

Fig. 1. A juggling pattern with $d = 3$, $\boldsymbol{x} = (0, 1, 2)$, and $\boldsymbol{a} = (1, 2, 3)$.

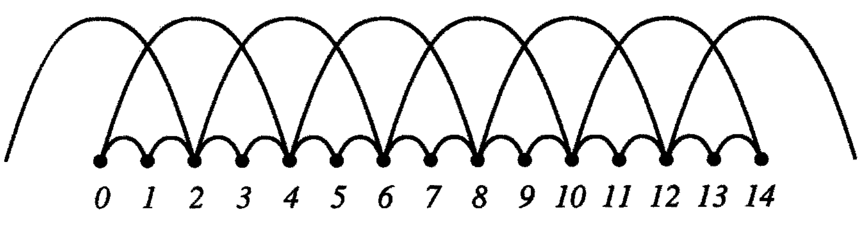

Fig. 2. A juggling pattern with $d = 2$, $\boldsymbol{x} = (0, 0, 1)$, and $\boldsymbol{a} = (1, 4, 1)$.

say that this pattern has two crossings. Define the *weight* of a juggling pattern to be q to the power of the number of crossings of the pattern. The q-analogue of the n^d result, which is proved bijectively, is the following theorem:

Theorem 1.1. *The sum of the weight of simple juggling sequences, with period d and at most n balls, is equal to*

$$(1 + q + \cdots + q^{n-1})^d.$$

As a corollary to this theorem, we are able to easily compute the Poincaré series of the affine Weyl group $\tilde{A}_{d-1}$.

The results for multiplex include a product of Gaussian coefficients, which is presented in Theorem 5.1. While studying the multiplex case, we came across a natural combinatorial interpretation of $S[n,k]$, the q-Stirling numbers of the second kind, using intertwining numbers of blocks. This method can easily be shown to be equivalent to Garsia and Remmel's [6] idea of obtaining the q-Stirling numbers from rook placements. We give a bijective proof of an identity of Carlitz [4] involving $S[n,k]$ by contracting simple juggling graphs.

The method of contracting simple juggling graphs may be extended to contracting multiplex juggling graphs. This enables us to prove a q-identity, due to Haglund [7]. Upon setting $q = 1$, this identity involves enumerating unitary compositions and is due to MacMahon [9].

Observe for a multiplex juggling pattern that at each time point the number of balls the juggler catches is equal to the number of balls he throws at that time point. In the last section of the paper, we enumerate patterns without this property (see Theorem 8.2). For this generalization of juggling patterns, we use two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ to describe such patterns. The vector $\boldsymbol{x}$ describes how many balls are thrown at each time point, while the vector $\boldsymbol{y}$ describes the number of balls caught.

2. Definitions

We say that two vectors $\boldsymbol{u} = (u_1, u_2, \ldots, u_m)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_m)$ are *similar* if there exists a permutation $\pi \in S_m$ such that $u_i = v_{\pi(i)}$ for all $i = 1, 2, \ldots, m$. We write $\boldsymbol{u} \sim \boldsymbol{v}$ when $\boldsymbol{u}$ and $\boldsymbol{v}$ are similar.

Definition 2.1. A *juggling triple* $(d, \boldsymbol{x}, \boldsymbol{a})$ consists of a positive integer d, a vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_m)$ of integers and a vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_m)$ of positive integers, such that the following two conditions hold:

1. $0 \leqslant x_i \leqslant d - 1$ for all $i = 1, 2, \ldots, m$.
2. $(\boldsymbol{a} + \boldsymbol{x}) \bmod d \sim \boldsymbol{x}$, where the $\bmod d$ applies component-wise.

We call d the *period*, $\boldsymbol{x}$ the *base vector*, and $\boldsymbol{a}$ the *juggling sequence*.

To a juggling triple $(d,\boldsymbol{x},\boldsymbol{a})$ we associate the following directed multigraph G on the integers $\mathbb{Z}$. The vertex set of the graph is $\mathbb{Z}$ and the directed edge set is given by

$$E(G) = \{(x_i + k \cdot d, x_i + a_i + k \cdot d)\colon 1 \leqslant i \leqslant m,\ k \in \mathbb{Z}\}.$$

Observe that all the edges are directed increasingly with respect to time. Moreover, the condition $(\boldsymbol{a} + \boldsymbol{x}) \bmod d \sim \boldsymbol{x}$ implies that for every vertex its outdegree is equal to its indegree. Hence we can decompose the graph into a finite number of edge-disjoint paths that are increasing. This composition is not unique. However, the number of paths is always the same. We call the number of edge-disjoint paths the *number of balls* of the juggling triple $(d,\boldsymbol{x},\boldsymbol{a})$. We denote this number by $\mathrm{ball}(d,\boldsymbol{x},\boldsymbol{a})$.

Remark 2.2. It is easy to show that the number of balls of the juggling triple $(d,\boldsymbol{x},\boldsymbol{a})$ is given by $(1/d)\cdot(a_1 + a_2 + \cdots + a_m)$.

Let α_j be equal to the outdegree at vertex j in the associated graph. That is, for $0 \leqslant j \leqslant d-1$, α_j is the cardinality of the set $\{i\colon x_i = j\}$. We say that a juggling triple is a *simple* juggling triple if $m = d$ and the base vector $\boldsymbol{x}$ is given by $\boldsymbol{x} = (0, 1, \dots, d-1)$. This implies that $\alpha_0 = \alpha_1 = \cdots = \alpha_{d-1} = 1$. Every vertex in the associated graph of a simple juggling sequence has outdegree and indegree one. The more general case, where the out and indegrees may be greater than one, is called *multiplex*.

In the directed graph G we define a *crossing* to be a pair of two edges (x,y) and (u,v) such that $x < u < y < v$. We say that two crossings (x_1,y_1) and (u_1,v_1), and (x_2,y_2) and (u_2,v_2) are *equivalent* if there exists an integer k such that

$$x_1 = x_2 + k \cdot d, \qquad y_1 = y_2 + k \cdot d, \qquad u_1 = u_2 + k \cdot d, \quad \text{and} \quad v_1 = v_2 + k \cdot d.$$

Define the number of *external crossings* to be the number of classes of equivalent crossings of the graph.

Remark 2.3. The number of external crossings of the juggling triple $(d,\boldsymbol{x},\boldsymbol{a})$ is explicitly given by

$$\sum_i \left\lfloor \frac{a_i - 1}{d} \right\rfloor + \sum_{i \neq j,\, a_i \leqslant a_j} \left\lfloor \frac{x_i + a_i - x_j - 1}{d} \right\rfloor - \left\lfloor \frac{x_i - x_j}{d} \right\rfloor + \left\lfloor \frac{x_i + a_i - x_j - a_j - 1}{d} \right\rfloor - \left\lfloor \frac{x_i - x_j - a_j}{d} \right\rfloor.$$

An *internal crossing* of a juggling triple $(d,\boldsymbol{x},\boldsymbol{a})$ is a pair (i,j) such that $1 \leqslant i < j \leqslant m$, $x_i = x_j$, and $a_i > a_j$. For example, the juggling triple $(2,(0,0,1),(1,4,1))$ that appears in Fig. 2 has no internal crossings, whereas the juggling triple $(2,(0,0,1),(4,1,1))$ has one internal crossing. Observe that these two juggling triples have the same associated graph. No internal crossings occur for a simple juggling triple since all the entries of the base vector are different. The *number of crossings* of a juggling triple $(d,\boldsymbol{x},\boldsymbol{a})$ is the sum of the number of external and internal crossings. We denote the number of crossings of a juggling triple by $\mathrm{cross}(d,\boldsymbol{x},\boldsymbol{a})$. We define the *weight* of a juggling triple $(d,\boldsymbol{x},\boldsymbol{a})$ to be q to the power of the number of crossings, that is, $q^{\mathrm{cross}(d,\boldsymbol{x},\boldsymbol{a})}$.

Following the convention for q-analogues, we define $[n] = 1 + q + \cdots + q^{n-1}$ and $[n]! = [1]\cdot[2]\cdots[n]$. The *Gaussian coefficient*, or *q-binomial coefficient*, is given by

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]!\cdot[n-m]!}.$$

A combinatorial interpretation of the Gaussian coefficient, which will be used in the proofs of Theorems 5.1 and 8.2, is an identity due to Schützenberger [13]. Let x and y be noncommutative variables, with the relation $yx = q\cdot xy$. Then

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \cdot x^k y^{n-k}. \tag{1}$$

The coefficient of $x^k y^{n-k}$ is found by taking all monomials having k x's and $(n-k)$ y's and sorting them to the form $x^k y^{n-k}$ by replacing each occurrence of yx by $q\cdot xy$. Notice that the power of q for a given monomial is equal to the number of 'swaps' that are made.

3. Simple juggling

We will now consider simple juggling and present the proof of Theorem 1.1. In this section the base vector will be $(0,1,\ldots,d-1)$, which we denote δ_d. Also, let $\mathbf{1}_d$ be the vector $\underbrace{(1,1,\ldots,1)}_{d}$.

Proof of Theorem 1.1. Consider the three cards in Fig. 3. Each card has three paths, and one of these paths goes through the time point (alias the juggler's hand). By taking d such cards, placing them side-by-side, and repeating the pattern periodically, we obtain a picture of three balls going from the left to the right. Observe that if we do not

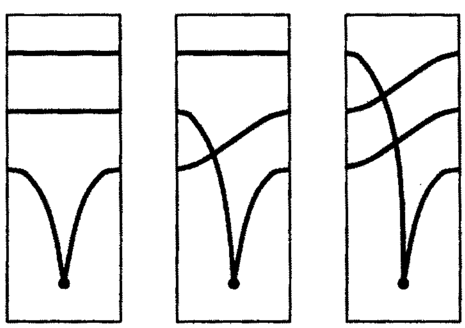

Fig. 3. The three juggling cards C_0, C_1, and C_2 for simple juggling with at most three balls.

use the last card C_2, we will obtain a picture having (at least) one continuous horizontal line. For example, if we were to only use the first card C_0, we would have a picture with two horizontal lines and below there is one ball being caught and thrown at every time point. Each of the horizontal lines corresponds to a ball 'hanging in the air'. Hence, these are the cards to use to describe the number of juggling patterns with at most three balls.

To be able to make this argument for at most n balls, we will have n different cards, say $C_0, C_1, \ldots, C_{n-1}$. On the card C_i the $(i+1)$st path, counted from below, goes down and touches the time point and then continues as the lowest path. The weight of a card will be q to the number of crossings that occurs on the card. Thus card C_i will have weight q^i.

There are n^d different patterns we can make with d cards. The sum of the weights of all the different patterns are

$$(1 + q + q^2 + \cdots + q^{n-1})^d = [n]^d,$$

since the sum of the weights of the n types of cards is $1 + q + q^2 + \cdots + q^{n-1}$. It remains to show that these $[n]^d$ patterns that we can construct with the cards are in a one-to-one correspondence with simple juggling triples.

Clearly a pattern of d cards will be a simple juggling sequence of period d. Say that an edge (x, y) of a simple juggling pattern is crossed from the inside by an edge (u, v) if $x < u < y < v$. Let (x, y) be an edge in a juggling pattern constructed from cards. Observe that all the crossings of (x, y) from the inside will occur on the card that covers the time point y.

Define the map Φ from simple juggling triples to $\mathbb{N}^d$ by $\Phi(d, \delta_d, \boldsymbol{a}) = (\phi_1, \phi_2, \ldots, \phi_d)$, where

$$\phi_i = |\{(u, v) \in E(G) \colon i - 1 < u < i - 1 + a_i < v\}|.$$

That is, ϕ_i counts the number of directed edges of the associated graph that crosses the edge $(i-1, i-1+a_i)$ from the 'inside'. For $0 \leqslant j \leqslant d-1$, let $\mu_j = \phi_i$ where $i - 1 + a_i \equiv j \bmod d$. Hence μ_j counts the number of crossings from the 'inside' at the edge that goes to time point j. Thus to construct the simple juggling pattern corresponding to the triple $(d, \boldsymbol{x}, \boldsymbol{a})$, it suffices to use the cards $C_{\mu_0}, C_{\mu_1}, \ldots, C_{\mu_{d-1}}$. □

It follows from this proof that Φ is a bijection between simple juggling triples of period d having at most n balls and the set $\{0, 1, \ldots, n-1\}^d$. The following identity also holds

$$\mathrm{ball}(d, \delta_d, \boldsymbol{a}) = \max(\phi_1, \phi_2, \ldots, \phi_d) + 1,$$

since the cards $C_{\max+1}, C_{\max+2}, \ldots, C_{n-1}$ are not used in the pattern. The $n - \max - 1$ paths on the top of each card in the pattern represents $n - \max - 1$ balls hanging in the air. Thus the juggling pattern involves $\max + 1$ balls.

4. The affine Weyl group $\tilde{A}_{d-1}$

We will now consider the affine Weyl group $\tilde{A}_{d-1}$. For more detailed accounts, see [8, 14].

Definition 4.1. Let $\tilde{A}_{d-1}$ be the group of bijections $\sigma\colon \mathbb{Z} \to \mathbb{Z}$ under composition, where the bijections satisfy the following two conditions:

1. $\sigma(i + d) = \sigma(i) + d$ for all i,
2. $\sum_{i=1}^{d}(\sigma(i) - i) = 0$.

This combinatorial description of $\tilde{A}_{d-1}$ is due to Lusztig. $\tilde{A}_{d-1}$ is a Coxeter group, and when $d \geqslant 2$ it is generated by the simple reflections $s_0, s_1, \ldots, s_{d-1}$, where

$$s_i(k) = \begin{cases} k+1 & \text{if } k \equiv i \bmod d \\ k-1 & \text{if } k \equiv i+1 \bmod d \\ k & \text{if } k \not\equiv i, i+1 \bmod d. \end{cases}$$

An element $\sigma \in \tilde{A}_{d-1}$ may be written as a product of simple reflections. Define the *length* $l(\sigma)$ of the element σ as the smallest integer r such that one can write σ as a product of r simple reflections. Observe that $\tilde{A}_0$ is the one element group.

Theorem 4.2. *Let σ be an element in $\tilde{A}_{d-1}$ and n a positive integer such that $n > i - \sigma(i)$ for all $i = 1, 2, \ldots, d$. Form the sequence $\boldsymbol{a} = (a_1, a_2, \ldots, a_d)$, where $a_i = \sigma(i) - i + n$. Then $(d, \delta_d, \boldsymbol{a})$ is a juggling triple with $\mathrm{ball}(d, \delta_d, \boldsymbol{a}) = n$ and $\mathrm{cross}(d, \delta_d, \boldsymbol{a}) = (n-1)\cdot d - l(\sigma)$.*

Proof. Since $n > i - \sigma(i)$ for all $i = 1, \ldots, d$, we have $a_i = \sigma(i) - i + n > 0$ and hence the a_i are positive integers. Because $\sigma(i + d) = \sigma(i) + d$ and σ is a bijection, we know that σ permutes the congruence classes modulo d. Hence, we know that the two vectors $\boldsymbol{a} + \delta_d$ and δ_d are similar modulo d. Thus $(d, \delta_d, \boldsymbol{a})$ is a simple juggling triple. Directly we obtain

$$\mathrm{ball}(d, \delta_d, \boldsymbol{a}) = \frac{1}{d} \cdot \sum_{i=1}^{d} (\sigma(i) - i + n) = n.$$

By induction on $l(\sigma)$, we prove the identity for the crossing number. If σ is the identity element, then $\boldsymbol{a} = (n, n, \ldots, n)$, and $\mathrm{cross}(d, \delta_d, \boldsymbol{a}) = (n-1)\cdot d$. Assume now that $\sigma = \tau \circ s_i$, where $l(\sigma) > l(\tau)$. By [14, Corollary 4.2.3] this is equivalent to $\tau(i) < \tau(i+1)$. Thus we have $\sigma(i) > \sigma(i+1)$. Let $\boldsymbol{a}$ be the juggling sequence derived from σ, and let G be the associated graph. Similarly, let $\boldsymbol{b}$ be the juggling sequence and H be the graph derived from τ. Observe that these two juggling sequences only differ in the entries i and $i + 1$. In other words, we have $b_i = a_{i+1} + 1$, $b_{i+1} = a_i - 1$, and $b_j = a_j$ for $j \neq i, i + 1$.

We will now compare the crossings of the two graphs G and H. Since $\sigma(i) > \sigma(i+1)$, we have that $(i-1, i-1+a_i)$ and $(i, i+a_{i+1})$ is not a crossing in G. But $(i-1, i-1+b_i)$ and $(i, i+b_{i+1})$ is a crossing in H, since $\tau(i) < \tau(i+1)$.

If (x, y) and (u, v) is a crossing in G and $x, u \not\equiv i-1, i \bmod d$, then it is also a crossing in H. If $(i-1, i-1+a_i)$ and (x, y) is a crossing in G, then $(i, i+b_{i+1})$ and (x, y) is a crossing in H. Similarly, if $(i, i+a_{i+1})$ and (x, y) is a crossing in G, then $(i-1, i-1+b_i)$ and (x, y) is a crossing in H. Thus the juggling graph H will have one more equivalence class of crossings than the graph G. Thus we conclude $\mathrm{cross}(d, \delta_d, \boldsymbol{a}) = \mathrm{cross}(d, \delta_d, \boldsymbol{b}) - 1$, and the induction is complete. $\square$

By Theorems 1.1 and 4.2 we obtain:

Corollary 4.3. *The Poincaré series of the group $\tilde{A}_{d-1}$ is given by:*

$$\sum_{\sigma \in \tilde{A}_{d-1}} q^{l(\sigma)} = \frac{1-q^d}{(1-q)^d}.$$

Proof. Let P_n be the subset of $\tilde{A}_{d-1}$ defined by $P_n = \{\sigma \in \tilde{A}_{d-1}\colon n > \max(i - \sigma(i))\}$. Then

$$\sum_{\sigma \in P_n} q^{(n-1)\cdot d - l(\sigma)} = (1 + q + \cdots + q^{n-1})^d - (1 + q + \cdots + q^{n-2})^d,$$

which implies

$$\begin{aligned}\sum_{\sigma \in P_n} q^{l(\sigma)} &= (1 + q + \cdots + q^{n-1})^d - (q + \cdots + q^{n-1})^d \\ &= \left(\frac{1-q^n}{1-q}\right)^d - q^d \cdot \left(\frac{1-q^{n-1}}{1-q}\right)^d.\end{aligned}$$

Since $\bigcup_{n \geq 1} P_n = \tilde{A}_{d-1}$, we obtain the result by letting $n \to \infty$. $\square$

5. Multiplex juggling

In this section we enumerate multiplex juggling patterns. Recall that α_j is the outdegree (and indegree) at time point j in the associated graph of a multiplex juggling triple. Observe that when $\alpha_0 = \alpha_1 = \cdots = \alpha_{d-1} = 1$ we obtain Theorem 1.1, the simple juggling enumeration theorem.

Theorem 5.1. *The sum of the weight of juggling triples, with period d, base vector $\boldsymbol{x}$, and at most n balls, is equal to*

$$\begin{bmatrix} n \\ \alpha_0 \end{bmatrix} \cdot \begin{bmatrix} n \\ \alpha_1 \end{bmatrix} \cdots \begin{bmatrix} n \\ \alpha_{d-1} \end{bmatrix},$$

where $\alpha_j = \mathrm{Card}\{i\colon x_i = j\}$ for $j = 0, \ldots, d-1$.

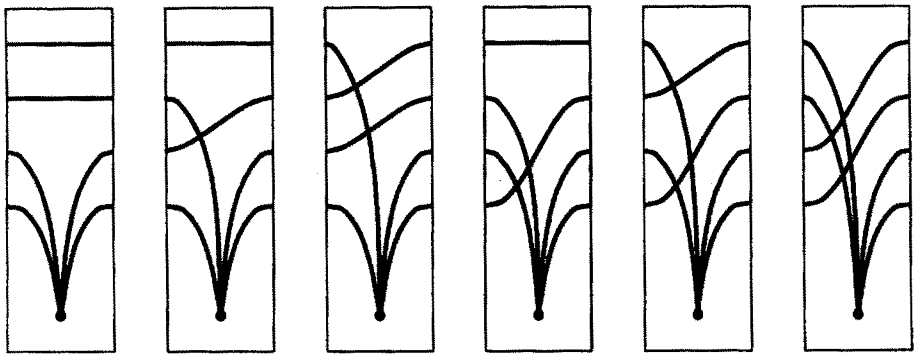

Fig. 4. The $6 = \binom{4}{2}$ juggling cards for multiplex juggling with at most four balls at a time where the juggler catches two balls. Observe that the sum of the weights is $\left[{4 \atop 2}\right] = 1 + q + 2\cdot q^2 + q^3 + q^4$.

Proof. The proof is based upon the same cards idea as in the proof of Theorem 1.1 except that we will be using different decks of cards. The kth deck D_k will consist of $\binom{n}{k}$ cards. For $n = 4$ the six cards in deck D_2 are pictured in Fig. 4. The cards in deck D_k will show all possible ways to choose k balls out of the n balls in the air, have the juggler catch these k balls, then throw these k balls up so that they are placed in the lowest orbits.

We will index the cards in the deck D_k by multisets M of cardinality k, where the entries of the multisets are integers between 0 and $n - k$. The card C_M, where $M = \{m_1 \leqslant m_2 \leqslant \cdots \leqslant m_k\}$, is the card where the ith lowest ball that the juggler catches has m_i crossings. For example, in Fig. 4 we have the cards, reading from left to right, $C_{\{0,0\}}$, $C_{\{0,1\}}$, $C_{\{0,2\}}$, $C_{\{1,1\}}$, $C_{\{1,2\}}$, and $C_{\{2,2\}}$. The weight of a card is defined to be q to the power of the number of crossings that appear on that card. Hence the weight of card C_M is $q^{\sum_{m \in M} m}$. By the combinatorial interpretation of the Gaussian coefficient given in Eq. (1), the sum of the weights in deck D_k is $\left[{n \atop k}\right]$.

At time point i we will use deck D_{α_i}, which consists of $\binom{n}{\alpha_i}$ cards. Thus we are able to construct $\binom{n}{\alpha_0} \cdots \binom{n}{\alpha_{d-1}}$ patterns having period d. The sum of the weights of all these patterns is

$$\begin{bmatrix} n \\ \alpha_0 \end{bmatrix} \cdot \begin{bmatrix} n \\ \alpha_1 \end{bmatrix} \cdots \begin{bmatrix} n \\ \alpha_{d-1} \end{bmatrix}.$$

We claim that there is one-to-one correspondence between these patterns and multiplex juggling triples. Moreover, this correspondence is weight-preserving.

Given a juggling triple (d, x, a), define $\Psi(d, x, a) = (\psi_1, \psi_2, \ldots, \psi_m)$, where

$$\psi_i = |\{(u, v) \in E(G)\colon x_i < u < x_i + a_i < v\}| + |\{j\colon 1 \leqslant j < i,\ x_i = x_j,\ a_j > a_i\}|.$$

Thus ψ_i is equal to the number of crossings from the inside of the edge $(x_i, x_i + a_i)$ plus the number internal crossings this edge has with other edges with a smaller index. Define the multiset M_j, where $0 \leqslant j \leqslant d - 1$, by $M_j = \{\psi_i\colon\ x_i + a_i \equiv j \bmod d\}$.

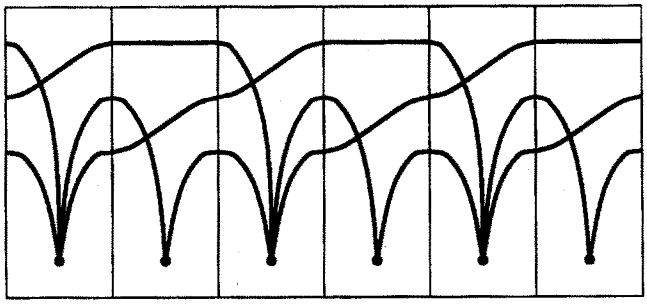

Fig. 5. The juggling triple $d = 2$, $x = (0, 0, 1)$, and $a = (4, 1, 1)$ constructed by cards. Observe the internal crossing that occurs.

Observe that the cardinality of the multiset M_j is α_j. To construct the associated graph of the juggling triple (d, x, a), use the cards $C_{M_0}, C_{M_1}, \ldots, C_{M_{d-1}}$.

By this construction, the associated graph of a juggling triple is drawn so that each crossing will appear as far right as possible. That is, all edges that cross an edge (x, y) from the inside will do this just before the time point y. Moreover, this rule also applies to internal crossings. Hence the internal crossing must be drawn outside the time point where they occur.

Now from a pattern constructed with juggling cards, we can obtain the associated juggling graph by reading off the edges, that is, where a ball gets caught and thrown. Also by keeping track of where the internal crossings appear, we will obtain the juggling triple. □

Fig. 5 illustrates one such situation when an internal crossing appears outside. Observe that this juggling triple has the same associated graph as the juggling triple in Fig. 2.

6. q-Stirling numbers of the second kind

The usual definition for the q-Stirling numbers of the second kind follows:

Definition 6.1. The *q-Stirling numbers of the second kind*, $S[n, k]$, are defined by the recursion

$$S[n, k] = q^{k-1} \cdot S[n-1, k-1] + [k] \cdot S[n-1, k],$$

where $n, k \geqslant 1$. When $n = 0$ or $k = 0$, define $S[n, k] = \delta_{n,k}$.

The q-Stirling numbers of the second kind have been well-studied in the literature. See for example [5, 6, 10–12, 19]. Recall that the Stirling numbers of the second kind, $S(n, k)$, are defined by the number of partitions of the set $\{1, \ldots, n\}$ into k blocks. In what follows, we give a combinatorial interpretation which extends the partition model of the Stirling numbers of the second kind in a very natural way to its q-analogue.

Let $\Pi_k[n]$ denote the set of all partitions of $\{1, 2, \ldots, n\}$ into k blocks. For two integers i and j define the interval $\mathrm{int}(i, j)$ to be the set

$$\mathrm{int}(i,j) = \{n \in \mathbb{Z}: \min(i,j) < n < \max(i,j)\}.$$

Observe that the interval is symmetric in i and j, that is, $\mathrm{int}(i,j) = \mathrm{int}(j,i)$.

Definition 6.2. For two disjoint nonempty subsets B, C of $\{1, 2, \ldots, n\}$, define the *intertwining number* $\imath(B,C)$ to be the cardinality of the set $\{(b,c) \in B \times C: \mathrm{int}(b,c) \cap (B \cup C) = \emptyset\}$. The intertwining number is independent of order, that is, $\imath(B,C) = \imath(C,B)$. For a partition $\pi = \{B_1, B_2, \ldots, B_k\}$ of the set $\{1, 2, \ldots, n\}$ define the intertwining number $\imath(\pi)$ to be

$$\imath(\pi) = \sum_{1 \leqslant i < j \leqslant k} \imath(B_i, B_j).$$

Since the intertwining number of two blocks is independent of their order, the intertwining number of a partition does not depend upon how the blocks are ordered.

As an example, consider the partition $\pi = \{\{1,3,6\}, \{2,4\}, \{5\}\}$ in Fig. 6. The intertwining number of the two blocks $\{1,3,6\}$ and $\{2,4\}$ is 4, which is equal to the number of crossings between the solid line and the dashed line. Also the intertwining number of π is equal to 7, which is the total number of crossings in Fig. 6.

Proposition 6.3. *The q-Stirling numbers of the second kind, $S[n,k]$, satisfy*

$$S[n,k] = \sum_{\pi} q^{\imath(\pi)} \quad (n \geqslant 1 \text{ and } k \geqslant 1),$$

where the sum ranges over all partitions π into k blocks, that is, $\Pi_k[n]$.

By conditioning on the block in which the element n appears, we can easily derive the recurrence in Definition 6.1.

Observe that the intertwining number of two disjoint blocks is greater than or equal to 1. Hence the intertwining number of a partition π is greater than or equal to $\binom{k}{2}$,

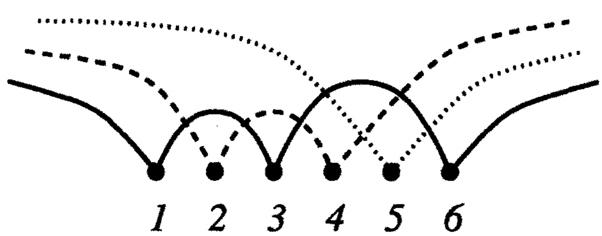

Fig. 6. Computation of the intertwining number of the partition $\pi = \{\{1,3,6\}, \{2,4\}, \{5\}\}$. $\imath(\pi) = \imath(\{1,3,6\}, \{2,4\}) + \imath(\{1,3,6\}, \{5\}) + \imath(\{2,4\}, \{5\}) = 4 + 2 + 1 = 7$.

where k is the number of blocks of π. This implies the Stirling number $S[n,k]$ is divisible by $q^{\binom{k}{2}}$.

There is a natural bijection between partitions of $\{1,2,\dots,n\}$ into k blocks, and rook placements of $n-k$ rooks on the triangular board of shape $(0,1,\dots,n-1)$. (This bijection is given after Corollary 2.4.2 in [17].) It is easy to see that the intertwining number of a partition is equal to the Garsia–Remmel statistic 'inv' of the corresponding rook placement under this bijection [6]. However, the intertwining number is different from the q-Stirling distributed set partitions statistics of Milne [11], Sagan [12], and Wachs and White [19].

The following identity is due to Carlitz [4]. It is a q-analogue of a well-known identity for Stirling numbers of the second kind. (See for example [17].) Milne proved this q-identity by using finite operator techniques on restricted growth functions [10]. See also de Médicis and Leroux [5] for a combinatorial proof.

Theorem 6.4 (Carlitz [4]).

$$[n]^d = \sum_{m=0}^{d} S[d,m]\cdot[m]!\cdot\begin{bmatrix} n \\ m \end{bmatrix}.$$

Proof. The idea of the proof is to study simple juggling graphs of period d. We contract d consecutive vertices of the graph to form a multiplex juggling graph of period 1. Carlitz's identity will follow by keeping track of what happens to the crossings in the graph under contraction.

Let $(d,\delta_d,\boldsymbol{a})$ be a simple juggling triple. Observe that $(d,\delta_d,\boldsymbol{a})$ does not have any internal crossings. By Theorem 1.1, we know that the sum of the weight of such juggling triples with at most n balls is $[n]^d$. Contract the vertices $k\cdot d, k\cdot d+1,\dots,(k+1)\cdot d-1$ of the associated graph G into a new vertex k. We then obtain a graph associated with a juggling triple $(1,\mathbf{0}_m,\boldsymbol{c})$. Observe that $1\leqslant m\leqslant d$, since some arcs will be contracted to arcs of length 0. Thus we remove them. Formally, this contraction is described by letting $b_i=\lfloor (a_i+i-1)/d\rfloor$, and removing the zero entries from the sequence $(b_1,b_2,\dots,b_d)$ to produce the juggling sequence $\boldsymbol{c}=(c_1,c_2,\dots,c_m)$. Note that $\mathrm{ball}(d,\delta_d,\boldsymbol{a})=\mathrm{ball}(1,\mathbf{0}_m,\boldsymbol{c})$.

Observe that m edge-disjoint paths partition the vertex set $\{0,1,\dots,d-1\}$ into m disjoint blocks. Thus this is a partition π with m blocks. Moreover, the intertwining number of π is the number of crossings that occur between time points 0 and $d-1$.

Now we see what happens to a crossing $(x,y),(u,v)$ when the graph G is contracted. Four cases occur. First, if the vertices y and u are contracted together, then the crossing is counted by the q-Stirling number $S[d,m]$. In the three remaining cases, we may assume that y and u are not contracted together. If none of the vertices x,y,u, and v are contracted together, then the crossing remains an external crossing of $(1,\mathbf{0}_m,\boldsymbol{c})$ and thus is counted by $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]$. If x and u are contracted together, but not y and v, then the crossing becomes an internal crossing of $(1,\mathbf{0}_m,\boldsymbol{c})$, and thus is counted by $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]$. Finally, if y and v are contracted together, then we may view this as an inversion of a permutation of m elements. The weight of all such inversions is counted by the factor $[m]!$. $\square$

7. Unitary compositions

In this section we use the techniques developed in Section 6 to prove another identity involving Gaussian coefficients. This identity will involve a generalization of the concept of a partition of a set, namely, unitary compositions of a vector. The number of unitary compositions, $g_k(\alpha)$, was first studied by MacMahon [9, Article 134]. A q-analogue of $g_k(\alpha)$, $g_k[\alpha]$, and its connection with rook placements was recently studied by Haglund [7]. We will define $g_k[\alpha]$ from a juggling perspective and obtain a bijective proof of the identity in Theorem 7.4.

Definition 7.1. A *composition* of a vector $\alpha \in \mathbb{N}^d$ into k parts is defined to be a list of k nonzero vectors $v_1, v_2, \ldots, v_k \in \mathbb{N}^d$ such that $v_1 + v_2 + \cdots + v_k = \alpha$. A composition is *unitary* if no part has an entry larger than one. Denote by $g_k(\alpha)$ the number of unitary compositions of α into k parts.

For example, there are two unitary compositions of $(2,1)$ into two parts, namely $(1,1)+(1,0)$ and $(1,0)+(1,1)$; thus $g_2((2,1)) = 2$. Similarly there are three unitary compositions of $(2,1)$ into three parts: $(1,0)+(1,0)+(0,1)$, $(1,0)+(0,1)+(1,0)$, and $(0,1)+(1,0)+(1,0)$; hence $g_3((2,1)) = 3$. Also, there is a connection to Stirling numbers of the second kind, namely $g_k((\underbrace{1,1,\ldots,1}_{n})) = k! \cdot S(n,k)$.

We next define the concept of unitary compositions in terms of juggling. For a vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}^d$ we denote $\|\alpha\| = \sum_{i=1}^{d} \alpha_i$ and $z(\alpha)$ by

$$z(\alpha) = (\underbrace{1,\ldots,1}_{\alpha_1}, \underbrace{2,\ldots,2}_{\alpha_2}, \ldots, \underbrace{d,\ldots,d}_{\alpha_d}).$$

Definition 7.2. A *unitary compositional triple* of a vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}^d$ into k parts is a triple of vectors (x, y, a), where each vector has length $m = \|\alpha\| + k$, satisfying

1. $x = (0_k, z(\alpha))$ and $y = (z(\alpha), (d+1)_k)$, where the operation is concatenation of two vectors.
2. The entries of the vector a are positive integers less than or equal to d.
3. $a + x \sim y$, that is, the two vectors $a + x$ and y have the same entries.

Observe that vectors x and y only depend on α and k.

As an example of this definition, consider the vector $\alpha = (2,1)$. There are two unitary compositional triples of α into two parts, each with $x = (0,0,1,1,2)$ and

$\boldsymbol{y} = (1, 1, 2, 3, 3)$. The two unitary compositional triples have $\boldsymbol{a}$-vectors equal to

$$(1, 1, 1, 2, 1) \quad \text{and} \quad (1, 1, 2, 1, 1).$$

Similarly, there are three unitary compositional triples of $\boldsymbol{\alpha}$ into three parts, which have $\boldsymbol{a}$-vectors equal to

$$(1, 1, 2, 2, 2, 1), \qquad (1, 2, 1, 2, 2, 1), \quad \text{and} \quad (2, 1, 1, 2, 2, 1).$$

They have $\boldsymbol{x} = (0, 0, 0, 1, 1, 2)$ and $\boldsymbol{y} = (1, 1, 2, 3, 3, 3)$.

To see that unitary compositions and unitary compositional triples are equivalent, we study the graph of a unitary compositional triple. To the triple $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$ we associate the following directed multigraph G. The vertex set of the graph is $\{0, 1, \ldots, d, d+1\}$ and the directed edge set is given by

$$E(G) = \{(x_j, x_j + a_j) \colon 1 \leqslant j \leqslant m\}.$$

Observe that all the edges are directed increasingly with respect to time. We represent the edge $(x_j, x_j + a_j)$ by the index j. Also the vertex i, where $1 \leqslant i \leqslant d$, has outdegree and indegree equal to α_i. However the vertex 0 has outdegree k and the vertex $d+1$ has indegree k. Hence we may view the graph as a finite juggling pattern, where we begin to throw k balls at time point 0, and we catch all k balls at time point $d+1$. Condition 2 in Definition 7.2 implies that there is no edge $(0, d+1)$ in the graph G. Thus all of the balls are caught at least once inside the pattern.

We will give a linear order to the α_i edges that come in at vertex i. We will also give a linear order to the α_i edges that leave the vertex i. By matching these two linear orders with each other, we will be able to connect the edges into paths. Hence we may view the graph G as being composed of k edge-disjoint paths.

The two linear orders are defined as follows: The order of the edges leaving vertex i is given by the underlying order of indices of the edges. That is, when $i = x_j = x_k$ we order two edges by $(x_j, x_j + a_j) < (x_k, x_k + a_k)$ if $j < k$. The order of the edges entering vertex i is given by the following rule. Assume that $i = x_j + a_j = x_k + a_k$. We order $(x_j, x_j + a_j) < (x_k, x_k + a_k)$ if $x_j > x_k$, or $x_j = x_k$ and $j < k$.

By pairing together at every vertex the entering edges with the outgoing edges (with respect to the two linear orders), we obtain in a canonical way a decomposition of the graph into k paths, $P_1, \ldots, P_k$. For a path P_i, consider the characteristic vector χ_{P_i} of the set of vertices the path P_i goes through, not including the vertices 0 and $d+1$. Clearly the sum of these k characteristic vectors $\chi_{P_1}, \ldots, \chi_{P_k}$ will be the vector $\boldsymbol{\alpha}$. Since the edges leaving the vertex 0 also receive a linear ordering, we obtain a unitary composition $\boldsymbol{\alpha}$. For example, the unitary compositional triple $((0, 0, 1, 1, 2), (1, 1, 2, 3, 3), (1, 1, 1, 2, 1))$ of the vector $(2, 1)$ has the two paths $0 \to 1 \to 2 \to 3$ and $0 \to 1 \to 3$. These paths correspond to the unitary composition $(1, 1) + (1, 0)$. Whereas, the other unitary compositional triple of the vector $(2, 1)$ into two parts corresponds to the unitary composition $(1, 0) + (1, 1)$.

We may also deduce that there is a unique unitary compositional triple for every unitary composition. Thus unitary composition and unitary compositional triple are equivalent concepts.

Similar to a juggling triple we can define external and internal crossings of a unitary compositional triple. We denote the number of crossings of the unitary composition $(\mathbf{x},\mathbf{y},\mathbf{a})$ by $\mathrm{cross}(\mathbf{x},\mathbf{y},\mathbf{a})$ and define the weight of a unitary compositional triple to be $q^{\mathrm{cross}(\mathbf{x},\mathbf{y},\mathbf{a})}$. This definition of weight agrees with Haglund's definition [7, Definition 2.1.6 and Theorem 4.2.2]. Hence we may define the q-analogue of $g_k(\boldsymbol{\alpha})$ as follows:

Definition 7.3. Define $g_k[\boldsymbol{\alpha}]$ to be

$$g_k[\boldsymbol{\alpha}] = \sum_{(\mathbf{x},\mathbf{y},\mathbf{a})} q^{\mathrm{cross}(\mathbf{x},\mathbf{y},\mathbf{a})},$$

where the sum ranges over all unitary compositional triples $(\mathbf{x},\mathbf{y},\mathbf{a})$ of $\boldsymbol{\alpha}$ into k parts.

Again let us consider the case when $\boldsymbol{\alpha} = (2,1)$. Since we already listed all the unitary compositions of $(2,1)$, we easily determine $g_2[(2,1)] = q + 1$ and $g_3[(2,1)] = q^4 + q^3 + q^2$.

Recall that $k!\cdot S(n,k)$ enumerates the number of compositions of n elements into k nonempty parts. Thus one may view $g_k[\boldsymbol{\alpha}]$ as a generalization of the quantity $[k]!\cdot S[n,k]$, since the following identity holds:

$$g_k\Bigl[\underbrace{(1,1,\ldots,1)}_{n}\Bigr] = [k]!\cdot S[n,k].$$

To prove this identity, use the interpretation of Stirling numbers of second kind given in Section 6. Observe that the factor $[k]!$ counts the weight of the internal crossings at the vertex 0.

The following identity is due to Haglund [7, Section 4.2, Eq. (8)]. The case when $q = 1$ of this identity was probably known to MacMahon, who studied the function $g_k(\boldsymbol{\alpha})$.

Theorem 7.4 (Haglund [7]).

$$\sum_{k=1}^{\|\boldsymbol{\alpha}\|} g_k[\boldsymbol{\alpha}]\cdot\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ \alpha_1 \end{bmatrix}\cdot\begin{bmatrix} n \\ \alpha_2 \end{bmatrix}\cdots\begin{bmatrix} n \\ \alpha_d \end{bmatrix}.$$

Sketch of proof. The proof is quite similar to the proof of Theorem 6.4. Consider all juggling triples of period d where the degrees of the vertices are respectively $\alpha_1, \alpha_2, \ldots, \alpha_d$. By Theorem 5.1 the sum of the weights of these triples is $[{n \atop \alpha_1}]\cdot[{n \atop \alpha_2}]\cdots[{n \atop \alpha_d}]$. Contract now every set of d vertices. We then obtain a juggling pattern of period 1, say with k balls. Such patterns are enumerated by $[{n \atop k}]$. The crossings that disappear when we contract the graph are counted by the factor $g_k[\boldsymbol{\alpha}]$. Thus the identity follows. $\square$

Corollary 7.5. *The value of $g_k[\alpha]$ does not depend on the order of the entries of α.*

8. A generalization of juggling

We now consider patterns where the number of balls caught at a particular time point does not necessarily equal the number of balls thrown at that same time point. Similar to a juggling triple we define a juggling quadruple.

Definition 8.1. A *juggling quadruple* $(d, \mathbf{x}, \mathbf{y}, \mathbf{a})$ consists of a positive integer d, two vectors $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ of integers, and a vector $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ of positive integers, such that the following two conditions hold:
1. $0 \leqslant x_i, y_i \leqslant d - 1$ for all $i = 1, 2, \ldots, m$.
2. $(\mathbf{a} + \mathbf{x}) \bmod d \sim \mathbf{y}$, where the $\bmod d$ applies component-wise.

We call d the *period*, $\mathbf{x}$ the *throw vector*, $\mathbf{y}$ the *catch vector*, and $\mathbf{a}$ the *juggling sequence.*

Observe that when $\mathbf{x} = \mathbf{y}$, this is equivalent to a juggling triple.

The number of balls of a juggling quadruple is not well-defined. Instead we will consider the sum $\sum_{i=1}^{m} a_i$. Observe that $\sum_{i=1}^{m} a_i \equiv \sum_{i=1}^{m} (y_i - x_i) \bmod d$.

As before, to a juggling quadruple $(d, \mathbf{x}, \mathbf{y}, \mathbf{a})$ we associate the following directed multigraph G on the integers $\mathbb{Z}$. The vertex set of the graph is $\mathbb{Z}$ and the directed edge set is given by

$$E(G) = \{(x_i + k \cdot d, x_i + a_i + k \cdot d) \colon 1 \leqslant i \leqslant m, k \in \mathbb{Z}\}.$$

Let α_j be equal to the outdegree at vertex j in the associated graph, and β_j equal to the indegree at vertex j. That is, for $0 \leqslant j \leqslant d - 1$, α_j is the cardinality of the set $\{i \colon x_i = j\}$, and similarly $\beta_j = |\{i \colon y_i = j\}|$. As for juggling triples, we define external and internal crossings in the same manner. We say that the weight of a juggling quadruple is equal to q to the number of crossings.

Theorem 8.2. *The sum of the weight of juggling quadruples $(d, \mathbf{x}, \mathbf{y}, \mathbf{a})$ having period d, throw vector $\mathbf{x}$, catch vector $\mathbf{y}$, and $\sum_{i=1}^{d} a_i \leqslant N$, where $N \equiv \sum_{i=1}^{m} (y_i - x_i) \bmod d$, is*

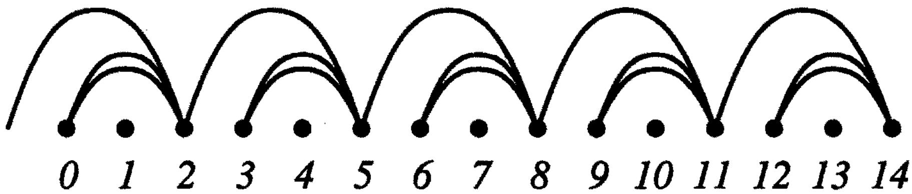

Fig. 7. A juggling pattern with $d = 3$, $\mathbf{x} = (0, 0, 2)$, $\mathbf{y} = (2, 2, 2)$, and $\mathbf{a} = (2, 2, 3)$.

equal to

$$\begin{bmatrix} n_0 \\ \beta_0 \end{bmatrix} \cdot \begin{bmatrix} n_1 \\ \beta_1 \end{bmatrix} \cdots \begin{bmatrix} n_{d-1} \\ \beta_{d-1} \end{bmatrix},$$

where $n_0, n_1, \ldots, n_{d-1}$ is the unique solution to the system of equations

$$N = n_0 + n_1 + \cdots + n_{d-1},$$

$$n_{i+1} = n_i - \beta_i + \alpha_i, \quad i = 0, 1, \ldots, d-1,$$

and the indices are considered modulo d.

In the case when $x = y$, Theorem 8.2 implies Theorem 5.1.

Sketch of proof of Theorem 8.2. The proof is similar to the proof of Theorem 5.1. Assume that there are n_i balls in the air just before time point i. It is easy to see that $n_0, n_1, \ldots, n_{d-1}$ will satisfy the system of equations given in the theorem, and that it is the unique solution. Thus cards in the deck used at time i will show how to catch β_i balls out of n_i balls, and throw α_i balls in the α_i lowest orbits. Thus the sum of the weights in the ith deck is $\left[\begin{smallmatrix} n_i \\ \beta_i \end{smallmatrix}\right]$, and the result follows. □

We can extend the notion of unitary compositional triples which were studied in the previous section.

Definition 8.3. A *generalized unitary compositional triple* of two vectors $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^d$ into k parts, such that $\|\boldsymbol{\alpha}\| = \|\boldsymbol{\beta}\|$, is a triple of vectors $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$, where each vector has length $m = \|\boldsymbol{\alpha}\| + k$, satisfying

1. $\boldsymbol{x} = (\boldsymbol{0}_k, z(\boldsymbol{\alpha}))$ and $\boldsymbol{y} = (z(\boldsymbol{\beta}), (d+1)_k)$, where the operation is concatenation of two vectors.
2. The entries of the vector $\boldsymbol{a}$ are positive integers less than or equal to d.
3. $\boldsymbol{a} + \boldsymbol{x} \sim \boldsymbol{y}$, that is, the two vectors $\boldsymbol{a} + \boldsymbol{x}$ and $\boldsymbol{y}$ have the same entries.

As before, we can define the graph and the number of crossings of a generalized unitary compositional triple.

Definition 8.4. Define $c_k[\boldsymbol{\alpha}, \boldsymbol{\beta}]$ to be

$$c_k[\boldsymbol{\alpha}, \boldsymbol{\beta}] = \sum_{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})} q^{\operatorname{cross}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})},$$

where the sum ranges over all generalized unitary compositional triples $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$ of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ into k parts.

Notice that $c_k[\boldsymbol{\alpha}, \boldsymbol{\alpha}] = g_k[\boldsymbol{\alpha}]$. By applying the techniques of Sections 6 and 7, we can obtain a generalization of Haglund's identity (see Theorem 7.4).

Theorem 8.5.

$$\sum_{k=1}^{\|\alpha\|} c_k[\alpha,\beta]\cdot\begin{bmatrix} n_0 \\ k \end{bmatrix} = \begin{bmatrix} n_0 \\ \beta_0 \end{bmatrix}\cdot\begin{bmatrix} n_1 \\ \beta_1 \end{bmatrix}\cdots\begin{bmatrix} n_{d-1} \\ \beta_{d-1} \end{bmatrix}.$$

9. Concluding remarks

A number of open questions remain to be answered. For instance, what happens if we consider colored balls? Is there a juggling interpretation of $s[n,k]$, the q-Stirling numbers of the first kind?

Fomin has posed the problem of studying how juggling balls in a simple juggling pattern are permuted. That is, we would like to compute

$$(1 + q\cdot s_1 + q^2\cdot s_2 s_1 + \cdots + q^{n-1}\cdot s_{n-1}s_{n-2}\cdots s_1)^d,$$

where $s_i = (i, i+1)$ is the ith elementary reflection in the symmetric group on n elements and the computation takes place in the group algebra of the symmetric group.

The computation of the Poincaré series for $\tilde{A}_n$ leads us to ask if a version of juggling could be used to compute the Poincaré series of other affine Weyl groups. Recall that there are combinatorial descriptions of $\tilde{B}_n$, $\tilde{C}_n$, and $\tilde{D}_n$ using permutations of the integers (see [1, 15]). We are now considering $\tilde{C}_n$, which appears to be the most promising candidate.

MacMahon also studied $f_k(\alpha)$, the number of compositions of a vector α into k parts. Haglund has defined a q-analogue of $f_k(\alpha)$, namely $f_k[\alpha]$, and proved the following identity [7, Section 4.1, Eq. (4)]

$$\sum_{k=1}^{\|\alpha\|} f_k[\alpha]\cdot\begin{bmatrix} n \\ k \end{bmatrix} = \prod_{i=1}^{d}\begin{bmatrix} n+\alpha_i-1 \\ \alpha_i \end{bmatrix}.$$

MacMahon did give, in the case when $q = 1$, a proof of an equivalent identity [9, Article 17]. Haglund has asked if there is a combinatorial model, such as juggling, which can prove this identity bijectively.

Juggling patterns may be generalized in the following manner. Given a group G and a normal subgroup H of G, consider all the permutations π of G such that $\pi(g\cdot h) = \pi(g)\cdot h$ for all $h \in H$. Does this approach give more generalized enumerative results?

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