THE TORIC h-VECTORS OF PARTIALLY ORDERED SETS

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ABSTRACT. An explicit formula for the toric h-vector of an Eulerian poset in terms of the **cd**-index is developed using coalgebra techniques. The same techniques produce a formula in terms of the flag h-vector. For this, another proof based on Fine's algorithm and lattice-path counts is given. As a consequence, it is shown that the Kalai relation on dual posets, $g_{n/2}(P) = g_{n/2}(P^*)$, is the only equation relating the h-vectors of posets and their duals. A result on the h-vectors of oriented matroids is given. A simple formula for the **cd**-index in terms of the flag h-vector is derived.

1. INTRODUCTION

In his paper [12] on face numbers of simplicial polytopes, Sommerville found a transformation of the f-vector that puts the linear relations on f-vectors into a simple form. Fifty years later the transformed vector, now called the h-vector, proved crucial in the Upper Bound Theorem [11] and, finally, the characterization of f-vectors of simplicial polytopes [6, 13]. The h-vector can be interpreted in several ways, in particular, as the Betti numbers of the toric variety associated with a simplicial polytope. This can be generalized to define a "toric" h-vector for every rational polytope, the vector of middle perversity intersection homology ranks of the toric variety. The combinatorial formula for this toric h-vector makes sense for all Eulerian posets, and following Stanley [14, Section 3.14] we define and study it in this general context.

The formula for the toric h-vector is a recursion on the poset. The recursion can be used to show that the h-vector can be obtained by a linear transformation from the flag f-vector of the poset. An explicit formula for that linear transformation was lacking, however. A recursion for the linear transformation from **cd**-index to toric h-vector appears in [3]. In 1993 Fine gave a nonrecursive, combinatorial algorithm for computing the coefficients of the h-vector in terms of the flag f-vector; see [1].

In Sections 3 and 4 we give closed formulas for the toric h-vector in terms of the flag h-vector, and in terms of the **cd**-index. In Section 7 these formulas are proved using coalgebra techniques. A sketch of another proof using Fine's algorithm is given. We note that the formulas can be shown to satisfy the Bayer-Klapper recursion, which gives a third method of proof. Section 5 includes a proof that the

Kalai relation on dual posets, $g_{n/2}(P) = g_{n/2}(P^*)$, is the only equation relating the h-vectors of posets and their duals. A result on the h-vectors of oriented matroids is also given there. A simple formula for the **cd**-index in terms of the flag h-vector (or **ab**-index) is derived in Section 6.

2. Definitions

A partially ordered set (poset) P is ranked if there is a function $\rho : P \longrightarrow \mathbf{Z}$ such that for two elements $x \leq y$ the cardinality of every maximal chain $x = x_0 < x_1 < \cdots < x_k = y$ is given by $\rho(y) - \rho(x) + 1$. A poset P is graded if it has a minimal element $\hat{0}$, a maximal element $\hat{1}$, and it is ranked such that $\rho(\hat{0}) = 0$. The rank of a graded poset P is defined by $\rho(P) = \rho(\hat{1})$. All the posets in this paper are graded.

For two elements $x \leq y$ in a poset P define the *interval* $[x, y]$ to be the set $\{z : x \leq z \leq y\}$. Observe that all intervals of a graded poset are also graded posets. The rank of the interval $[x, y]$ is given by $\rho(y) - \rho(x)$.

Let P be a graded poset of rank $n + 1$. For S a subset of $\{1, 2, \ldots, n\}$, let f_S be the number of chains in the poset P such that the set of ranks of elements in the chain is exactly the set S. The collection of f_S where the set S ranges over all subsets of the set $\{1, 2, \ldots, n\}$ is called the flag f-vector. The flag h-vector is defined by the alternating sum

$$h_S = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_T.$$

One can recover the flag f-vector from the flag h-vector by the inverse relation

$$f_S = \sum_{T \subseteq S} h_T.$$

Hence the flag f-vector and the flag h-vector encode the same information of the poset.

It is convenient to write a generating function for the flag h-vector. The **ab**-index is a polynomial in the noncommuting variables **a** and **b**. For n a nonnegative integer and S a subset of $\{1, 2, \ldots, n\}$, define the **ab**-monomial $u_S = u_1 \cdots u_n$ by letting $u_i = \mathbf{a}$ if $i \notin S$, and $u_i = \mathbf{b}$ otherwise. The **ab**-index of a graded poset P of rank $n + 1$ is the polynomial

$$\Psi(P) = \sum_S h_S u_S,$$

where the sum ranges over all subsets S of $\{1, 2, \ldots, n\}$. Observe that the **ab**-index $\Psi(P)$ is a homogeneous polynomial of degree one less than the rank of the poset P.

A poset P is *Eulerian* if its Möbius function is given by $\mu(x, y) = (-1)^{\rho(y) - \rho(x)}$. An equivalent definition is that a poset P is Eulerian if every interval of the poset satisfies the Euler-Poincaré relation $f_0 - f_1 + \cdots + (-1)^k f_k = 0$, where k is the rank of the interval and f_i denotes the number of elements in the interval of rank i. Examples of Eulerian posets are face lattices of convex polytopes and the strong Bruhat order in Coxeter groups.

Fine [3] observed that when a poset P is Eulerian, then its **ab**-index $\Psi(P)$ may be written as a polynomial in $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. When $\Psi(P)$ is written in terms of $\mathbf{c}$ and $\mathbf{d}$, it is called the **cd**-index of the poset. The number of coefficients in the **cd**-index is the nth Fibonacci number, which is the dimension of the span of flag vectors of Eulerian posets. In comparison, the number of coefficients in the

ab-index (and the flag vector) is 2^n, which is much greater. For a short proof of the existence of the **cd**-index for Eulerian posets see Stanley [15]. That a poset P has a **cd**-index is equivalent to the fact that the flag f-vector of P satisfies the generalized Dehn-Sommerville relations [2].

The **ab**-index and **cd**-index are easy to use because they are coalgebra homomorphisms. We include a short explanation here; for more details, see [8]. On the algebra $\mathbf{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ define a coproduct $\Delta : \mathbf{Z}\langle \mathbf{a}, \mathbf{b} \rangle \longrightarrow \mathbf{Z}\langle \mathbf{a}, \mathbf{b} \rangle \otimes \mathbf{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ by

$$\Delta(v_1 \cdots v_n) = \sum_{i=1}^{n} v_1 \cdots v_{i-1} \otimes v_{i+1} \cdots v_n,$$

for a monomial $v_1 \cdots v_n$, and extend to $\mathbf{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ by linearity. We abbreviate this using the Sweedler notation for the coproduct $\Delta(v) = \sum_v v_{(1)} \otimes v_{(2)}$. There is no co-unit; hence $\mathbf{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ is not a coalgebra in the classical sense. This coproduct does not extend to a bialgebra with the ordinary multiplication. Instead it satisfies the Newtonian condition (see [9, 10]):

$$(2.1) \qquad \Delta(uv) = \sum_u u_{(1)} \otimes (u_{(2)}v) + \sum_v (uv_{(1)}) \otimes v_{(2)}.$$

Using the Newtonian condition it is straightforward to show that the coproduct is closed on the subalgebra $\mathbf{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ generated by $\mathbf{c}$ and $\mathbf{d}$.

The following proposition states that the **ab**-index (and hence the **cd**-index) is a coalgebra homomorphism. For more details on the corresponding coalgebra structure of posets see [8].

Proposition 2.1 (Ehrenborg and Readdy). *Let P be a graded poset of rank at least one. Then the coproduct of the **ab**-index of the poset is given by*

$$\Delta(\Psi(P)) = \sum_{\hat{0} < y < \hat{1}} \Psi([\hat{0}, y]) \otimes \Psi([y, \hat{1}]).$$

This result is useful for reducing calculations on posets to computations involving only their **ab**-indices.

3. THE TORIC h-VECTOR

One important invariant of a graded poset is the toric h-vector. We follow Stanley [14, Section 3.14] in defining it. We begin by defining three linear maps $U_{\leq m}$, $U_{\geq m}$, and $U_{=m}$ on the polynomial ring $\mathbf{Z}[x]$ by $U_{\leq m}[p(x)] = \sum_{i=0}^{m} a_i x^i$, $U_{\geq m}[p(x)] = \sum_{i \geq m} a_i x^i$, and $U_{=m}[p(x)] = a_m x^m$, where $p(x) = \sum_{i \geq 0} a_i x^i$.

Now for each poset we define two polynomials $f(P, x) = f(P)$ and $g(P, x) = g(P)$, which are called the f-polynomial and the g-polynomial of the poset P. Their definitions are two intertwined recursions.

Definition.

- For P a poset of rank $n + 1$,
$$f(P, x) = (x - 1)^n + \sum_{\hat{0} < y < \hat{1}} g([\hat{0}, y])(x - 1)^{\rho([y, \hat{1}]) - 1}.$$

- For P a poset of rank $n + 1$ and $m = \lfloor n/2 \rfloor$,
$$g(P, x) = U_{\leq m}\left[(1 - x)f(P, x)\right].$$

The relations imply that for the unique graded poset of rank 1, both the f- and g-polynomial are equal to 1.

The *toric h-vector* of a poset P is defined as the vector of coefficients of the polynomial $f(P)$, that is,

$$f(P, x) = \sum_{i=0}^{n} h_i(P)x^i.$$

Fine described a combinatorial way to compute the f-polynomial of P in terms of the flag f-vector of P; see [1]. (A similar formula was found by Brenti [7].) Billera suggested that Fine's formula can be simplified by converting from the flag f-vector to the flag h-vector. Indeed it then becomes possible to write the coefficients explicitly, in reasonably simple form.

Let S be a subset of the set $\{1, 2, \dots, n\}$. Consider $\{1, 2, \dots, n\}$ partitioned into maximal consecutive strings of $\overline{S}$ and S, with blocks of the partition ordered by least element. Write the sequence of sizes of the blocks, but add 1 to the leftmost term if $1 \notin S$. Call the resulting sequence $\sigma(S) = \sigma_1, \sigma_2, \dots, \sigma_r$.

A set $S \subseteq \{1, 2, \dots, n\}$ is *odd* if and only if in the corresponding sequence $\sigma(S) = \sigma_1, \sigma_2, \dots, \sigma_r$, all terms except σ_r are odd, and σ_r has the same parity as n if $n \in S$, and the opposite parity from n if $n \notin S$.

The following theorem gives the toric h-vector in terms of the flag h-vector. The proof is postponed until Section 7. We use the following notation: $p(n, k) = \binom{n}{k} - \binom{n}{k-1}$. Note that for n even, $p(n, n/2)$ is the Catalan number $C_{n/2}$.

Theorem 3.1. *Let P be a graded poset of rank $n + 1$.*

1. *For $0 \le i < n/2$,*

$$h_i(P) = \sum (-1)^{|S|+n-i} p(\sigma_r - 1, (\sigma_r - n + 2i)/2) \prod_{t=1}^{r-1} p(\sigma_t - 1, (\sigma_t - 1)/2) h_S(P),$$

   where the sum is over odd sets $S \subseteq \{1, 2, \dots, n\}$ for which $n \in S$ and $\sigma_r \ge n - 2i$. (Here σ_r is the length of the final consecutive string in S.)
2. *For $n/2 \le i \le n$,*

$$h_i(P) = \sum (-1)^{|S|+n-i} p(\sigma_r - 1, (\sigma_r + n - 1 - 2i)/2)$$
$$\times \prod_{t=1}^{r-1} p(\sigma_t - 1, (\sigma_t - 1)/2) h_S(P),$$

   where the sum is over odd sets $S \subseteq \{1, 2, \dots, n\}$ for which $n \notin S$ and $\sigma_r \ge 2i + 1 - n$. (Here $n - \sigma_r$ is the maximum element of S if $S \ne \emptyset$, and $\sigma_r = n + 1$ if $S = \emptyset$.)

In terms of the sequence $\sigma(S) = \sigma_1, \sigma_2, \dots, \sigma_r$, the **ab**-monomial u_S is as follows:

1. If $1 \notin S$ and $n \notin S$, then r is odd and $u_S = \mathbf{a}^{\sigma_1-1}\mathbf{b}^{\sigma_2}\mathbf{a}^{\sigma_3} \cdots \mathbf{b}^{\sigma_{r-1}}\mathbf{a}^{\sigma_r}$.
2. If $1 \notin S$ and $n \in S$, then r is even and $u_S = \mathbf{a}^{\sigma_1-1}\mathbf{b}^{\sigma_2}\mathbf{a}^{\sigma_3} \cdots \mathbf{a}^{\sigma_{r-1}}\mathbf{b}^{\sigma_r}$.
3. If $1 \in S$ and $n \notin S$, then r is even and $u_S = \mathbf{b}^{\sigma_1}\mathbf{a}^{\sigma_2}\mathbf{b}^{\sigma_3} \cdots \mathbf{b}^{\sigma_{r-1}}\mathbf{a}^{\sigma_r}$.
4. If $1 \in S$ and $n \in S$, then r is odd and $u_S = \mathbf{b}^{\sigma_1}\mathbf{a}^{\sigma_2}\mathbf{b}^{\sigma_3} \cdots \mathbf{a}^{\sigma_{r-1}}\mathbf{b}^{\sigma_r}$.

For $n = 3$, here is a listing of the subsets of the set $\{1, 2, 3\}$, and the corresponding **ab**-words u_S and sequences $\sigma(S)$:

S	u_S	$\sigma(S)$	S	u_S	$\sigma(S)$
$\emptyset$	**aaa**	4	$\{3\}$	**aab**	$3, 1$
$\{1\}$	**baa**	$1, 2$	$\{1, 3\}$	**bab**	$1, 1, 1$
$\{2\}$	**aba**	$2, 1, 1$	$\{2, 3\}$	**abb**	$2, 2$
$\{1, 2\}$	**bba**	$2, 1$	$\{1, 2, 3\}$	**bbb**	3

4. THE FORMULA IN TERMS OF THE **ab**-INDEX AND **cd**-INDEX

In this section the formula of Theorem 3.1 is presented in another guise. We specify the portion of the toric h-vector associated with each **ab**-monomial and with each **cd**-monomial; this extends to the computation of the toric h-vector of P from the **ab**-index and **cd**-index of P.

It is convenient to define three sequences of polynomials. For nonnegative integers n, let $Q_n(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k p(n-1, k) x^k$, and let $R_n(x) = x^{n-1} Q_n(x^{-1})$. Thus $Q_n(x)$ is of degree $\lfloor (n-1)/2 \rfloor$ and has constant term 1, while $R_n(x)$ is of degree $n-1$, has no terms of degree less than $(n-1)/2$, and has leading coefficient 1. Finally, for n odd we define a monomial,

$$T_n(x) = (-1)^{(n-1)/2} p(n-1, (n-1)/2) x^{(n-1)/2}.$$

(For even n, let $T_n(x) = 0$.)

Here is a table of these polynomials for small n:

n	$Q_n(x)$	$R_n(x)$	$T_n(x)$
1	1	1	1
2	1	x	0
3	$-x + 1$	$x^2 - x$	$-x$
4	$-2x + 1$	$x^3 - 2x^2$	0
5	$2x^2 - 3x + 1$	$x^4 - 3x^3 + 2x^2$	$2x^2$
6	$5x^2 - 4x + 1$	$x^5 - 4x^4 + 5x^3$	0
7	$-5x^3 + 9x^2 - 5x + 1$	$x^6 - 5x^5 + 9x^4 - 5x^3$	$-5x^3$

Theorem 4.1. *For P a poset of rank $n + 1$ with **ab**-index $\Psi(P) = \sum_S h_S u_S$, the toric h-vector is given by $f(P, x) = \sum_S h_S f(u_S, x)$, where $f(u_S, x)$ equals*

1. $x^{\lfloor (r-1)/2 \rfloor} Q_{\sigma_r}(x) \prod_{t=1}^{r-1} T_{\sigma_t}(x)$ *if $n \in S$,*

2. $x^{\lfloor r/2 \rfloor} R_{\sigma_r}(x) \prod_{t=1}^{r-1} T_{\sigma_t}(x)$ *if $n \notin S$.*

Proof of the equivalence of Theorem 3.1 and Theorem 4.1. We show that the two statements give the same coefficient of $h_S(P)$ in the h-vector polynomial $f(P, x)$. Below are the details for the case where $n \in S$; minor modifications give the case where $n \notin S$. Let $\alpha(n, S, i) = |S| + n - i - (n - \sigma_r)/2 + \lfloor (r-1)/2 \rfloor$. Theorem 3.1

gives the coefficient of $h_S(P)$ as

$$\sum_{i=(n-\sigma_r)/2}^{\lfloor (n-1)/2 \rfloor} (-1)^{|S|+n-i} p(\sigma_r - 1, (\sigma_r - n + 2i)/2) \prod_{t=1}^{r-1} p(\sigma_t - 1, (\sigma_t - 1)/2) x^i$$

$$= \left[\prod_{t=1}^{r-1} (-1)^{(\sigma_t-1)/2} p(\sigma_t - 1, (\sigma_t - 1)/2) x^{(\sigma_t-1)/2} \right] x^{\lfloor (r-1)/2 \rfloor}$$

$$\times \sum_{i=(n-\sigma_r)/2}^{\lfloor (n-1)/2 \rfloor} (-1)^{\alpha(n,S,i)} p(\sigma_r - 1, (\sigma_r - n + 2i)/2) x^{i-(n-\sigma_r)/2}$$

$$= \left[\prod_{t=1}^{r-1} (-1)^{(\sigma_t-1)/2} p(\sigma_t - 1, (\sigma_t - 1)/2) x^{(\sigma_t-1)/2} \right] x^{\lfloor (r-1)/2 \rfloor}$$

$$\times \sum_{k=0}^{\lfloor (\sigma_r-1)/2 \rfloor} (-1)^{|S|+n+\lfloor (r-1)/2 \rfloor + k} p(\sigma_r - 1, k) x^k.$$

Now $|S| + n + \lfloor (r-1)/2 \rfloor \equiv 0 \pmod 2$, so this equals

$$\left[\prod_{t=1}^{r-1} (-1)^{(\sigma_t-1)/2} p(\sigma_t - 1, (\sigma_t - 1)/2) x^{(\sigma_t-1)/2} \right] x^{\lfloor (r-1)/2 \rfloor}$$

$$\times \sum_{k=0}^{\lfloor (\sigma_r-1)/2 \rfloor} (-1)^k p(\sigma_r - 1, k) x^k,$$

which is part (1) of Theorem 4.1. $\qquad \square$

For P an Eulerian poset, the **ab**-index gives rise to the more compact **cd**-index. The toric h-vector can be computed from the **cd**-index in a way similar to that given by Theorem 4.1.

Theorem 4.2. *Let P be an Eulerian poset of rank $n + 1$ with **cd**-index $\Psi(P) = \sum_w \xi_w w$. The toric h-vector of P is given by $f(P, x) = \sum_w \xi_w f(w, x)$, where*

$$f(\mathbf{c}^{k_1} \mathbf{dc}^{k_2} \mathbf{d} \cdots \mathbf{dc}^{k_r} \mathbf{dc}^k, x) = x^r (x R_k(x) + Q_k(x)) \prod_{j=1}^{r} T_{k_j+1}(x).$$

Similarly, writing the g-polynomial as $g(P, x) = \sum_w \xi_w g(w, x)$,

$$g(\mathbf{c}^{k_1} \mathbf{dc}^{k_2} \mathbf{d} \cdots \mathbf{dc}^{k_r} \mathbf{dc}^k, x) = x^r Q_{k+1}(x) \prod_{j=1}^{r} T_{k_j+1}(x).$$

(Recall that $T_n(x) = 0$ if n is even. Also set $x R_0(x) + Q_0(x) = 1$.)

Observe that $g(\mathbf{c}^{k_1} \mathbf{dc}^{k_2} \mathbf{d} \cdots \mathbf{dc}^{k_r} \mathbf{dc}^k, x)$ is zero if any of $k_1, \ldots, k_r$ is odd. There are only $2^{\lfloor n/2 \rfloor}$ **cd**-monomials such that all the k_i's are even. Hence the g-polynomial $g(P, x)$ depends on only $2^{\lfloor n/2 \rfloor}$ of the coefficients of $\Psi(P)$. A similar observation holds for the f-polynomial.

Section 6 gives a formula for the **cd**-index in terms of the **ab**-index, which connects Theorems 4.1 and 4.2. The proofs of the formulas for the toric h-vector in terms of the **ab**-index and the **cd**-index are found in Section 7. Another way to prove the f-polynomial formula of Theorem 4.2 is to show (with tedious calculations) that it solves the recurrence given in [3].

5. CONSEQUENCES FOR THE g-VECTOR

From the formula of Theorem 4.2 it is easy to prove the following identity conjectured by Kalai. This was already done using the recurrence formula in [3]. Recall that $g_i = h_i - h_{i-1}$.

Theorem 5.1 (Kalai, Bayer-Klapper). *If P is an Eulerian poset of odd rank $n+1$, and P^* is the dual poset, then $g_{n/2}(P) = g_{n/2}(P^*)$.*

We can now prove that other than the trivial relation, $g_0(P) = g_0(P^*)$, this is the only linear relation between the g-vectors of Eulerian posets and their duals.

Let $\mathcal{P}_n$ be the set of rank $n+1$ Eulerian posets. Let $g_i : \mathcal{P}_n \to \mathbf{Z}$ be the function giving the coefficient of x^i in $g(P, x)$. Let $g_i^* : \mathcal{P}_n \to \mathbf{Z}$ be the function whose value at P is the coefficient of x^i in the g-polynomial of the dual polytope P^*.

Theorem 5.2. *For fixed n the only linear relations among the functions g_0, g_1, $\ldots$, $g_{\lfloor n/2 \rfloor}$, g_0^*, g_1^*, $\ldots$, $g_{\lfloor n/2 \rfloor}^*$ are $g_0 = g_0^*$ and, for n even, $g_{n/2} = g_{n/2}^*$. The set of g's and g^*'s thus has dimension n.*

Proof. We write a matrix M for the functions g_i and g_i^* in terms of the **cd**-index. Order the n functions g_0, g_1, g_1^*, g_2, g_2^*, $\ldots$, $g_{\lfloor n/2 \rfloor - 1}^*$, $g_{\lfloor n/2 \rfloor}$, ending in $g_{\lfloor n/2 \rfloor}^*$ if n is odd. A **cd**-word is called a *border* word if it is of the form $\mathbf{c}^{n-2j}\mathbf{d}^j$ or $\mathbf{d}^j\mathbf{c}^{n-2j}$, $0 \le j \le n/2$. There are n border words. Order all **cd**-words with the border words first, in increasing degree of $\mathbf{d}$, and with $\mathbf{d}^j\mathbf{c}^{n-2j}$ preceding $\mathbf{c}^{n-2j}\mathbf{d}^j$. The nonborder words follow the border words in any order. Now write the matrix M with n rows, indexed by the functions in the order described above, and a Fibonacci number of columns, indexed by the **cd**-words in the order described above.

We show that the first n columns of M form a lower triangular submatrix. Theorem 4.2 gives, for $1 \le j \le n/2$,

$$g(\mathbf{c}^{n-2j}\mathbf{d}^j, x) = \begin{cases} (-1)^{(n-2j)/2} p(n - 2j, (n-2j)/2) x^{n/2} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

So the column of M indexed by the **cd**-word $\mathbf{c}^{n-2j}\mathbf{d}^j$ ($j \ge 1$) has nonzero entries in row g_i only if $i = n/2$. Duality says that the column of M indexed by the **cd**-word $\mathbf{d}^j\mathbf{c}^{n-2j}$ has nonzero entries in row g_i^* only if $i = n/2$. For these latter border words, $g(\mathbf{d}^j\mathbf{c}^{n-2j}, x) = x^j Q_{n-2j}(x)$, which includes only terms of degree at least j. So the column of M indexed by the **cd**-word $\mathbf{d}^j\mathbf{c}^{n-2j}$ has nonzero entries in row g_i only if $i \ge j$ (and has entry 1 in row g_j). Similarly, the column of M indexed by the **cd**-word $\mathbf{c}^{n-2j}\mathbf{d}^j$ has nonzero entries in row g_i^* only if $i \ge j$ (and has entry 1 in row g_j^*).

In summary, the first nonzero entry in the column indexed by $\mathbf{d}^j\mathbf{c}^{n-2j}$ is in row g_j, and the first nonzero entry in the column indexed by $\mathbf{c}^{n-2j}\mathbf{d}^j$ is in row g_j^*. Thus the first n columns of the matrix M form a lower triangular submatrix. Since the coefficients of the **cd**-words, as functions on polytopes, are linearly independent, so are the functions in the set $\{g_0, g_1, \ldots, g_{\lfloor n/2 \rfloor}, g_1^*, \ldots, g_{\lfloor (n-1)/2 \rfloor}^*\}$. $\square$

We close this section with an observation on g-vectors of zonotopes, or, more generally, oriented matroids.

Proposition 5.3. *Let P be the lattice of regions of an oriented matroid. Let $n+1$ be the rank of P. Then*

$$g(P, x) \equiv U_{\le \lfloor n/2 \rfloor}\left[(1 + x)^{n+1}\right] \pmod 2.$$

Proof. Let P be the lattice of regions of an oriented matroid of rank $n + 1$. In [5] it is shown that in the **cd**-index of P, 2^j divides the coefficient ξ_w of a **cd**-word w, if w contains j **d**'s. In particular, the only coefficient that can be odd is the coefficient of $w = \mathbf{c}^n$, which is always 1. So $g(P, x) \equiv \xi_{\mathbf{c}^n} g(\mathbf{c}^n, x) \equiv Q_{n+1}(x)$ (mod 2). Thus for $0 \le i \le n/2$, $g_i(P) \equiv p(n, i) \equiv \binom{n}{i} + \binom{n}{i-1} \equiv \binom{n+1}{i}$ (mod 2). So $g(P, x) \equiv U_{\le \lfloor n/2 \rfloor} \left[(1 + x)^{n+1} \right]$ (mod 2). $\qquad\square$

6. Consequences for the **cd**-index

A comparison of the formulas for the toric h-vector in terms of the **ab**-index and **cd**-index suggests a transformation from **cd**-index to **ab**-index. For Eulerian posets of rank $n + 1$, the **cd**-index consists of a Fibonacci number of coefficients encoding the **ab**-index, which has 2^n coefficients. Each **ab**-coefficient can be written in a unique way as a linear combination of **cd**-coefficients, but not vice versa. We present here a simple algorithm for computing the **cd**-index from the **ab**-index.

Let $v = \mathbf{c}^{k_1} \mathbf{dc}^{k_2} \mathbf{d} \cdots \mathbf{dc}^{k_r} \mathbf{dc}^{k}$, with the degree of v equal to n. One of the **ab**-words in the expansion of v is obtained by replacing each **d** by **ba**, replacing the last k **c**'s by k **a**'s, and replacing all other **c**'s by **b**'s. We distinguish this **ab**-word as $s(v) = \mathbf{b}^{k_1+1} \mathbf{ab}^{k_2+1} \mathbf{a} \cdots \mathbf{ab}^{k_r+1} \mathbf{a}^{k+1}$. Write T_v for the set of locations of the **b**'s in $s(v)$. Partition T_v into its maximal intervals, $T_v = T_1 \cup T_2 \cup \cdots \cup T_r$, where $|T_i| = k_i + 1$. In the following write $[r]$ for $\{1, 2, \ldots, r\}$. For $I \subseteq [r]$, let $\langle I \rangle = \bigcup_{i \in I} T_i \subseteq T_v$. Thus $u_{\langle I \rangle}$ is the **ab**-word obtained from $s(v)$ by replacing the ith substring of $k_i + 1$ **b**'s by $k_i + 1$ **a**'s, for all $i \notin I$. In particular, $u_{\langle \emptyset \rangle} = u_\emptyset = \mathbf{a}^n$, and $u_{\langle [r] \rangle} = u_{T_v} = s(v)$. Recall that $h_{\langle I \rangle}$ is the coefficient of $u_{\langle I \rangle}$ in the **ab**-index $\Psi(P)$ of a poset P and, for a **cd**-word w, ξ_w is the coefficient of w when $\Psi(P)$ is written in terms of **cd**-words.

Theorem 6.1. $\xi_v = \displaystyle\sum_{I \subseteq [r]} (-1)^{r-|I|} h_{\langle I \rangle}.$

For example, $s(\mathbf{dccd}) = \mathbf{babbba}$ and $\xi_{\mathbf{dccd}} = h_{1345} - h_{345} - h_1 + h_\emptyset$.

Proof. For w a **cd**-word and u an **ab**-word, w *covers* u if and only if u occurs as a monomial in the expansion of w by $\mathbf{c} = \mathbf{a} + \mathbf{b}$, $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. Equivalently, if we match up the symbols of w with those of u by position, each **c** is matched to a single **a** or **b**, and each **d** covers either **ab** or **ba** (not **aa** or **bb**). The symbol **d** represents a place where the **ab**-word *must* have a change from **a** to **b** or vice versa. (Changes are also *permitted* where w has **c**'s.)

For each set $S \subseteq [n]$, h_S is the sum of ξ_w, over those **cd**-words w for which w covers u_S. Thus,

$$\sum_{I \subseteq [r]} (-1)^{r-|I|} h_{\langle I \rangle} = \sum_{I \subseteq [r]} (-1)^{r-|I|} \sum_{\substack{w \\ w \text{ covers } u_{\langle I \rangle}}} \xi_w.$$

Note that, by construction, if **ab** or **ba** occurs in positions j and $j+1$ of $u_{\langle I \rangle}$, then the same pair occurs in positions j and $j+1$ of $s(v)$. Thus, if a **cd**-word w covers $u_{\langle I \rangle}$ for some I, then w covers $s(v)$.

Consider a fixed **cd**-word w that covers $s(v)$. Define I_w to be the set of $i \in [r]$ such that some **d** in w covers some **b** in the ith substring of **b**'s in $s(v)$. Then for $I \subseteq [r]$, $u_{\langle I \rangle}$ is covered by w if and only if $I_w \subseteq I$. This is because we can change a substring of **b**'s in $s(v)$ to **a**'s and preserve the covering relation of w as long as we do not change an **ab** or **ba** covered by **d**.

Thus,

$$\sum_{I\subseteq[r]}(-1)^{r-|I|}h_{\langle I\rangle} = \sum_{\substack{w \\ w \text{ covers } s(v)}} \xi_w \sum_{\substack{I \\ w \text{ covers } u_{\langle I\rangle}}} (-1)^{r-|I|}$$

$$= \sum_{\substack{w \\ w \text{ covers } s(v)}} \xi_w \sum_{I_w\subseteq I\subseteq[r]} (-1)^{r-|I|}$$

$$= \sum_{\substack{w \\ w \text{ covers } s(v) \\ \text{and } I_w=[r]}} \xi_w.$$

It remains to show that if w covers $s(v)$ and $I_w = [r]$, then $w = v$. The hypothesis says that for all i, at least one $\mathbf{b}$ in the ith substring of $\mathbf{b}$'s in $s(v)$ is covered by some $\mathbf{d}$ in w. That $\mathbf{b}$ must, of course, be the leftmost or rightmost $\mathbf{b}$ in the substring. Since $s(v)$ starts with a substring of (at least one) $\mathbf{b}$'s, the first $\mathbf{d}$ of w must cover the $\mathbf{ba}$ that occurs at the right-hand end of substring 1. The next $\mathbf{d}$ in w cannot cover the same $\mathbf{a}$, so it must cover the next $\mathbf{ba}$, occurring at the right-hand end of substring 2. Continuing, we see that the $\mathbf{d}$'s in w all occur in the locations of the $\mathbf{ba}$'s in $s(v)$, and thus $w = v$. So $\sum_{I\subseteq[r]}(-1)^{r-|I|}h_{\langle I\rangle} = \xi_v$. $\square$

Billera, Ehrenborg, and Readdy [5, 4] proved that for w a $\mathbf{cd}$-word containing r $\mathbf{d}$'s, the coefficient ξ_w in the $\mathbf{cd}$-index of an oriented matroid (in particular, a zonotope) is divisible by 2^r. Theorem 6.1 can be used to give another direct proof of this result. The original proof in [5] used the "sparse k-vector." Those authors later found a formula for the $\mathbf{cd}$-index in terms of the sparse k-vector. This means that the $\mathbf{cd}$-index can be written in terms of the sparse flag numbers, that is, those f_S for which S excludes n and contains no consecutive pair. This is of interest because the entire flag vector can be linearly generated by these sparse flag numbers.

7. Proofs of the h-vector formulas

7.1. **Preliminaries.** Define an algebra map κ from $\mathbf{Z}\langle\mathbf{a},\mathbf{b}\rangle$ to $\mathbf{Z}[x]$ by $\kappa(\mathbf{a}) = x-1$ and $\kappa(\mathbf{b}) = 0$. The $\mathbf{ab}$-index of a graded poset P contains the term $\mathbf{a}^{\rho(P)-1}$ with coefficient 1, so $\kappa(\Psi(P)) = (x-1)^{\rho(P)-1}$.

We define now two linear maps on $\mathbf{ab}$-polynomials that correspond to the f- and g-polynomials of posets.

Definition. Define two linear maps f and g from $\mathbf{Z}\langle\mathbf{a},\mathbf{b}\rangle$ to $\mathbf{Z}[x]$ by the following two relations:

- For any monomial v,

$$f(v) = \kappa(v) + \sum_v g(v_{(1)})\kappa(v_{(2)}).$$

- For v a monomial of degree n and $m = \lfloor n/2\rfloor$,

$$g(v) = U_{\leq m}\left[(1-x)f(v)\right].$$

The recursion for $f(v)$ without the Sweedler notation is

$$f(v) = \kappa(v) + \sum_{i=1}^{n} g(v_1\cdots v_{i-1})\kappa(v_{i+1}\cdots v_n),$$

where $v = v_1 \cdots v_n$. Observe that this definition gives $f(1) = 1$ and hence $g(1) = 1$.

Proposition 7.1. *For all graded posets P, $f(P) = f(\Psi(P))$ and $g(P) = g(\Psi(P))$.*

Proof. The proof is by induction on the rank of the poset P. If $\rho(P) = 1$, then $f(P) = g(P) = 1$, $\Psi(P) = 1$, and $f(1) = g(1) = 1$.

Assume now that P is a graded poset of rank $n + 1$. Let W be $\Psi(P)$. We begin by verifying the first identity,

$$
\begin{aligned}
f(P) &= (x - 1)^n + \sum_{\hat{0} < y < \hat{1}} g([\hat{0}, y])(x - 1)^{\rho([y, \hat{1}]) - 1} \\
&= \kappa(\Psi(P)) + \sum_{\hat{0} < y < \hat{1}} g(\Psi([\hat{0}, y]))\kappa(\Psi([y, \hat{1}])) \\
&= \kappa(W) + \sum_{W} g(W_{(1)})\kappa(W_{(2)}) \\
&= f(W).
\end{aligned}
$$

In the third step we used Proposition 2.1, that is, the fact that the **ab**-index is a co-algebra homomorphism. We finish by observing that $g(P) = U_{\leq m}[(1 - x)f(P)] = U_{\leq m}[(1 - x)f(W)] = g(W)$. $\qquad\qquad\square$

We need some facts about the polynomials $Q_n(x)$, $R_n(x)$, and $T_n(x)$.

Proposition 7.2. *For n a positive integer,*

$$
\begin{aligned}
Q_{n+1}(x) &= (1 - x)Q_n(x) + xT_n(x), \\
R_{n+1}(x) &= (x - 1)R_n(x) + T_n(x).
\end{aligned}
$$

An inductive proof of the first identity uses the following two facts about $p(n, k)$: $p(n + 1, k) = p(n, k) + p(n, k - 1)$ for all n and k, and for n odd $p(n, \lfloor n/2 \rfloor) = p(n + 1, \lfloor n/2 \rfloor + 1)$. The second identity follows by substituting $1/x$ into the first identity, multiplying by x^{n+1} and using $x^n T_n(1/x) = T_n(x)$.

Corollary 7.3. *The polynomial sequence $Q_n(x)$ satisfies the recursion*

$$
Q_{n+1}(x) = U_{\leq m}[(1 - x)Q_n(x)],
$$

where $m = \lfloor n/2 \rfloor$.

Proof. Observe first that $U_{\leq m}[xT_n(x)] = 0$, since if n is even $T_n(x)$ is equal to zero, and if n is odd the degree of $xT_n(x)$ is greater than m. Since $Q_{n+1}(x)$ has degree m,

$$
\begin{aligned}
Q_{n+1}(x) &= U_{\leq m}[Q_{n+1}(x)] \\
&= U_{\leq m}[(1 - x)Q_n(x) + xT_n(x)] \\
&= U_{\leq m}[(1 - x)Q_n(x)].
\end{aligned}
$$

$\qquad\qquad\square$

7.2. Proof of the ab-index formula.

Lemma 7.4. *For every* **ab***-monomial v,*

$$
\begin{aligned}
f(v\mathbf{a}) &= (x-1)f(v) + g(v), \\
f(v\mathbf{b}) &= g(v).
\end{aligned}
$$

Proof. Observe that $\kappa(v\mathbf{a}) = \kappa(v)(x-1)$. By the Newtonian condition (equation 2.1) $\Delta(v\mathbf{a}) = \sum_v v_{(1)} \otimes v_{(2)}\mathbf{a} + v \otimes 1$. Thus

$$
\begin{aligned}
f(v\mathbf{a}) &= \kappa(v\mathbf{a}) + \sum_v g(v_{(1)})\kappa(v_{(2)}\mathbf{a}) + g(v)\kappa(1) \\
&= \kappa(v)(x-1) + \sum_v g(v_{(1)})\kappa(v_{(2)})(x-1) + g(v) \\
&= f(v)(x-1) + g(v).
\end{aligned}
$$

The second identity is proved along the same lines, using $\kappa(v\mathbf{b}) = 0$. □

A direct application of this lemma gives the following.

Lemma 7.5. *For v any* **ab***-monomial of degree n and $m = \lfloor (n+1)/2 \rfloor$,*

$$
g(v\mathbf{b}) = U_{\leq m}\left[(1-x)g(v)\right].
$$

If v is an **ab**-monomial of odd degree, then $g(v\mathbf{b}) = (1-x)g(v)$.

Lemma 7.6. *For v any* **ab***-monomial of degree n and $m = \lfloor n/2 \rfloor$,*

$$
g(v\mathbf{a}) = \begin{cases} U_{=m+1}\left[(x-1)f(v)\right] & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}
$$

Proof. By Lemma 7.4

$$
\begin{aligned}
f(v\mathbf{a}) &= (x-1)f(v) + g(v) \\
&= (x-1)f(v) + U_{\leq m}\left[(1-x)f(v)\right] \\
&= U_{>m}\left[(x-1)f(v)\right].
\end{aligned}
$$

Hence

$$
\begin{aligned}
g(v\mathbf{a}) &= U_{\leq \lfloor (n+1)/2 \rfloor}\left[(1-x)f(v\mathbf{a})\right] \\
&= U_{\leq \lfloor (n+1)/2 \rfloor}\left[(1-x)U_{>m}\left[(x-1)f(v)\right]\right].
\end{aligned}
$$

If n is even, then $U_{\leq m}\left[(1-x)U_{>m}\left[p(x)\right]\right] = 0$; hence $g(v\mathbf{a}) = 0$. If n is odd, then $\lfloor (n+1)/2 \rfloor = m+1$ so $g(v\mathbf{a}) = U_{=m+1}\left[(x-1)f(v)\right]$. □

Proposition 7.7. *For any two* **ab***-monomials u and v,*

$$
\begin{aligned}
f(u\mathbf{ab}v) &= g(u\mathbf{a})f(\mathbf{b}v), \\
g(u\mathbf{ab}v) &= g(u\mathbf{a})g(\mathbf{b}v).
\end{aligned}
$$

Proof. The proof is by induction on the degree $|v|$ of v. The induction basis is when $v = 1$. Let $|u| = n$ and let $m = \lfloor n/2 \rfloor$. In the first case, when n is even, $g(u\mathbf{ab}) = U_{\leq m+1} [(1 - x)g(u\mathbf{a})] = 0 = g(u\mathbf{a})$. When n is odd,

$$
\begin{aligned}
g(u\mathbf{ab}) &= U_{\leq m+1} [(1 - x)g(u\mathbf{a})] \\
&= U_{\leq m+1} [(1 - x)U_{=m+1} [(x - 1)f(u)]] \\
&= U_{=m+1} [(x - 1)f(u)] \\
&= g(u\mathbf{a}).
\end{aligned}
$$

Since $g(\mathbf{b}) = 1$ the second identity holds for $v = 1$. The first identity follows from $f(u\mathbf{ab}) = g(u\mathbf{a}) = g(u\mathbf{a})f(\mathbf{b})$, since $f(\mathbf{b}) = 1$.

Now consider the induction step. Assume that the two statements hold for words of degree less than $|v|$. In particular, it holds for the words $v_{(1)}$ in the coproduct. Then

$$
\Delta(u\mathbf{ab}v) = \sum_u u_{(1)} \otimes u_{(2)}\mathbf{ab}v + u \otimes \mathbf{b}v + u\mathbf{a} \otimes v + \sum_v u\mathbf{ab}v_{(1)} \otimes v_{(2)}.
$$

Hence, using $\kappa(\mathbf{b}) = 0$,

$$
\begin{aligned}
f(u\mathbf{ab}v) &= \kappa(u\mathbf{ab}v) + \sum_u g(u_{(1)})\kappa(u_{(2)}\mathbf{ab}v) + g(u)\kappa(\mathbf{b}v) \\
&\quad + g(u\mathbf{a})\kappa(v) + \sum_v g(u\mathbf{ab}v_{(1)})\kappa(v_{(2)}) \\
&= g(u\mathbf{a}) \left(\kappa(v) + \sum_v g(\mathbf{b}v_{(1)})\kappa(v_{(2)}) \right) \\
&= g(u\mathbf{a})f(\mathbf{b}v).
\end{aligned}
$$

This verifies the first identity. Assume that $|u| = n$ and $|v| = k$. Then when n is even,

$$
\begin{aligned}
g(u\mathbf{ab}v) &= U_{\leq \lfloor (n+k+2)/2 \rfloor} [(1 - x)f(u\mathbf{ab}v)] \\
&= U_{\leq \lfloor (n+k+2)/2 \rfloor} [(1 - x)g(u\mathbf{a})f(\mathbf{b}v)] = 0 \\
&= g(u\mathbf{a})g(\mathbf{b}v).
\end{aligned}
$$

Now let n be odd and let $m = \lfloor n/2 \rfloor$; thus $m + 1 = (n + 1)/2$. Then

$$
\begin{aligned}
g(u\mathbf{ab}v) &= U_{\leq \lfloor (n+k+2)/2 \rfloor} [(1 - x)g(u\mathbf{a})f(\mathbf{b}v)] \\
&= U_{\leq \lfloor (n+k+2)/2 \rfloor} \left[U_{=(n+1)/2} [(x - 1)f(v)] (1 - x)f(\mathbf{b}v) \right] \\
&= U_{=(n+1)/2} [(x - 1)f(v)] \, U_{\leq \lfloor (k+1)/2 \rfloor} [(1 - x)f(\mathbf{b}v)] \\
&= g(u\mathbf{a})g(\mathbf{b}v).
\end{aligned}
$$

This completes the induction. $\square$

In order to evaluate $f(v)$ and $g(v)$ for any $\mathbf{ab}$-monomial v, we need to know the f- and g-polynomials evaluated on the monomials $\mathbf{b}^n$, $\mathbf{a}^n$, and $\mathbf{b}^n\mathbf{a}^k$. The following proposition has a straightforward proof by induction.

Proposition 7.8. *For any positive integer n,*

$$
\begin{aligned}
f(\mathbf{b}^n) &= Q_n(x), & g(\mathbf{b}^n) &= Q_{n+1}(x), \\
f(\mathbf{a}^n) &= R_{n+1}(x), & g(\mathbf{a}^n) &= T_{n+1}(x).
\end{aligned}
$$

Moreover, for any positive integers n and k,

$$\begin{aligned} f(\mathbf{b}^n\mathbf{a}^k) &= xT_n(x)R_k(x), \\ g(\mathbf{b}^n\mathbf{a}^k) &= xT_n(x)T_k(x). \end{aligned}$$

Theorem 4.1 follows now by applying Proposition 7.1 to the poset, and evaluating $f(\Psi(P))$ with Propositions 7.7 and 7.8.

7.3. **Proof of the cd-index formula.** The algebra map κ from $\mathbf{Z}\langle\mathbf{a},\mathbf{b}\rangle$ to $\mathbf{Z}[x]$ restricts to an algebra map from $\mathbf{Z}\langle\mathbf{c},\mathbf{d}\rangle$ to $\mathbf{Z}[x]$. Observe that $\kappa(\mathbf{c}) = x - 1$ and $\kappa(\mathbf{d}) = 0$. The next lemma follows easily from Lemma 7.4.

Lemma 7.9. *For any* **cd**-*monomial v,*

$$\begin{aligned} f(v\mathbf{c}) &= (x-1)f(v) + 2g(v), \\ f(v\mathbf{d}) &= (x-1)g(v) + g(v\mathbf{c}). \end{aligned}$$

The reciprocal theorem [14, Theorem 3.14.9] says that if P is an Eulerian poset of rank $n + 1$, then $f(P, x) = x^n f(P, 1/x)$. Since the **cd**-indices of Eulerian posets span the linear space of all **cd**-polynomials, the reciprocal theorem also holds for a **cd**-monomial v. That is, $f(v, x) = x^n f(v, 1/x)$. Hence if we know the g-polynomial of a monomial, we are able to compute the f-polynomial of the monomial.

Proposition 7.10. *For v any* **cd**-*monomial of degree n and $m = \lfloor(n+1)/2\rfloor$,*

$$g(v\mathbf{c}) = U_{\leq m}\left[(1-x)g(v)\right].$$

Proof. Suppose n is odd, $m = (n + 1)/2$. Expand the **cd**-monomial v as a sum of **ab**-monomials. Applying Lemma 7.6 to each summand and recombining gives $g(v\mathbf{a}) = U_{=m}\left[(x-1)f(v)\right]$. Write $f(v) = \sum_{i=0}^{n} h_i x^i$. By the reciprocal theorem [14, Theorem 3.14.9], $h_i = h_{n-i}$. In particular, for n odd $h_m = h_{m-1}$, so $g(v\mathbf{a}) = U_{=m}\left[(x-1)f(v)\right] = 0$. Lemma 7.6 also gives $g(v\mathbf{a}) = 0$ for n even. So $g(v\mathbf{c}) = g(v\mathbf{a}) + g(v\mathbf{b}) = g(v\mathbf{b}) = U_{\leq m}\left[(1-x)g(v)\right]$, by applying Lemma 7.5 to each **ab**-summand of v. □

If v is a **cd**-monomial of odd degree, then $g(v\mathbf{c}) = (1-x)g(v)$.

Proposition 7.11. *Let v be a* **cd**-*monomial of degree n. Let $m = \lfloor n/2 \rfloor$. Then*

$$g(v\mathbf{d}) = f(v\mathbf{d}) = \begin{cases} xU_{=m}\left[g(v)\right] & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. By Lemma 7.9 and Proposition 7.10,

$$\begin{aligned} f(v\mathbf{d}) &= (x-1)g(v) + g(v\mathbf{c}) \\ &= (x-1)g(v) + U_{\leq\lfloor(n+1)/2\rfloor}\left[(1-x)g(v)\right]. \end{aligned}$$

The degree of $(x-1)g(v)$ is $\lfloor n/2 \rfloor + 1$. So for n odd all terms cancel, and $g(v\mathbf{d}) = f(v\mathbf{d}) = 0$. For n even, $m = n/2$, and $f(v\mathbf{d}) = U_{=m+1}\left[(x-1)g(v)\right] = xU_{=m}\left[g(v)\right]$.

Hence

$$
\begin{aligned}
g(v\mathbf{d}) &= U_{\leq m+1}\left[(1-x)f(v\mathbf{d})\right] \\
&= U_{\leq m+1}\left[(1-x)xU_{=m}\left[g(v)\right]\right] \\
&= xU_{=m}\left[g(v)\right].
\end{aligned}
$$

$\square$

By Propositions 7.10 and 7.11 we are now able to evaluate the g-polynomial of any $\mathbf{cd}$-word. For instance, $g(\mathbf{d}^n) = x^n$. The following result facilitates these computations.

Proposition 7.12. *For two $\mathbf{cd}$-monomials u and v,*

$$
\begin{aligned}
g(u\mathbf{d}v) &= g(u\mathbf{d})g(v), \\
f(u\mathbf{d}v) &= g(u\mathbf{d})f(v).
\end{aligned}
$$

Proof. We begin by proving the first identity. Consider the two cases when $|u|$ is odd and when $|u|$ is even. First, when $|u|$ is odd, the right hand side is equal to 0. The left hand side may be computed by the recursions of the last two propositions from knowing $g(u\mathbf{d})$. Hence the left hand side is also equal to zero.

Hence it remains to consider the case when $|u|$ is even. Assume that $|u\mathbf{d}| = 2k$. The proof is by induction on the degree of v. The base case is $v = 1$, where there is nothing to prove. Consider first the case when $v = v'\mathbf{c}$, where $n = |v|$. So $|u\mathbf{d}v'\mathbf{c}| = 2k + n$. Then

$$
\begin{aligned}
g(u\mathbf{d}v'\mathbf{c}) &= U_{\leq \lfloor (2k+n)/2 \rfloor}\left[(1-x)g(u\mathbf{d}v')\right] \\
&= U_{\leq k+\lfloor n/2 \rfloor}\left[(1-x)g(u\mathbf{d})g(v')\right] \\
&= g(u\mathbf{d})U_{\leq \lfloor n/2 \rfloor}\left[(1-x)g(v')\right] \\
&= g(u\mathbf{d})g(v'\mathbf{c}).
\end{aligned}
$$

The second case is when $v = v'\mathbf{d}$. Let $n = |v'|$. So $|u\mathbf{d}v'| = 2k + n$. If n is odd, then $g(u\mathbf{d}v'\mathbf{d}) = 0 = g(u\mathbf{d})g(v'\mathbf{d})$. If n is even, then

$$
\begin{aligned}
g(u\mathbf{d}v'\mathbf{d}) &= xU_{=\lfloor (2k+n)/2 \rfloor}\left[g(u\mathbf{d}v')\right] \\
&= xU_{=k+\lfloor n/2 \rfloor}\left[g(u\mathbf{d})g(v')\right] \\
&= xg(u\mathbf{d})U_{=\lfloor n/2 \rfloor}\left[g(v')\right] \\
&= g(u\mathbf{d})g(v'\mathbf{d}).
\end{aligned}
$$

This completes the induction, and hence proves the first identity. A proof for the second identity is as follows. By the definition of f and by recalling that $\kappa(\mathbf{d}) = 0$, we obtain the expansion

$$
\begin{aligned}
f(u\mathbf{d}v) &= g(u)\kappa(\mathbf{c}v) + g(u\mathbf{c})\kappa(v) + \sum_v g(u\mathbf{d}v_{(1)})\kappa(v_{(2)}) \\
&= (x-1)g(u)\kappa(v) + g(u\mathbf{c})\kappa(v) + g(u\mathbf{d})\sum_v g(v_{(1)})\kappa(v_{(2)}) \\
&= (x-1)g(u)\kappa(v) + g(u\mathbf{c})\kappa(v) + g(u\mathbf{d})f(v) - g(u\mathbf{d})\kappa(v) \\
&= g(u\mathbf{d})f(v) + \left((x-1)g(u) + g(u\mathbf{c}) - f(u\mathbf{d})\right)\kappa(v) \\
&= g(u\mathbf{d})f(v).
\end{aligned}
$$

The last two lines come from Lemma 7.9 and Proposition 7.11. $\square$

In the proof we observe that $g(u\mathbf{d}v) = 0$ if $|u|$ is odd. This gives the fact, noted after the statement of Theorem 4.2, that $g(\mathbf{c}^{k_1}\mathbf{dc}^{k_2}\mathbf{d}\cdots\mathbf{dc}^{k_r}\mathbf{dc}^k, x) = 0$ if any of the k_i are odd.

In order to evaluate $g(v)$ and $f(v)$ for any $\mathbf{cd}$-monomial v, we need to know $g(\mathbf{c}^n)$, $g(\mathbf{c}^n\mathbf{d})$ and $f(\mathbf{c}^n)$.

Proposition 7.13. *For any nonnegative integer n,*

$$\begin{aligned}
g(\mathbf{c}^n) &= Q_{n+1}(x), \\
g(\mathbf{c}^n\mathbf{d}) &= xT_{n+1}(x), \\
f(\mathbf{c}^n) &= xR_n(x) + Q_n(x).
\end{aligned}$$

(Recall that $T_n(x) = 0$ if n is even, and $xR_0(x) + Q_0(x) = 1$.)

The first identity is due to Stanley and is essentially Exercise 70c in Chapter 3 in [14]. The first and third identities follow by induction. The second is proved by Proposition 7.11.

The proof of Theorem 4.2 is now similar to the proof of Theorem 4.1. Apply Proposition 7.1 to the $\mathbf{cd}$-index and evaluate the f- and g-polynomials with Propositions 7.12 and 7.13.

7.4. **A combinatorial approach.** The formulas of Theorems 3.1 and 4.1 for the h-vector involve the Catalan numbers, $p(n, n/2)$. In this section we give a combinatorial explanation for this.

A lattice path is a sequence of steps in $\mathbf{Z}^2$, where each step moves one to the right and either one up (adding $(1, 1)$) or one down (adding $(1, -1)$). For n even, $p(n, n/2)$ is the number of lattice paths from $(0, 0)$ to $(n, 0)$ that never go below the x-axis; see for instance [16].

Fine gave the following formula for the h-vector of a polytope P in terms of the flag vector of P. It applies more generally to Eulerian posets. Let P have rank $n + 1$ and let $S \subseteq [n]$. A *valid S-diagram* is a sequence of n $+1$'s and -1's so that for each $s \in S$, the sum of the terms up to position s is positive. Let $j_{S,i}$ be the number of valid S-diagrams with i -1's and $n - i$ $+1$'s. If $n = 0$ we understand $j_{\emptyset,0} = 1$.

Theorem 7.14 (Fine). *For any rank $n + 1$ Eulerian poset P,*

$$f(P, x) = \sum_{i=0}^{n} x^i \sum_{S \subseteq [n]} (-1)^{|S|+n-i} j_{S,i} f_S(P).$$

This can be proved directly from the recursions for the h-vector and g-vector. The idea is that the f_S-term in $h_k(P)$ comes from f_T-terms in $g_i(y)$, where $T = S \setminus \{\max S\}$, $i \leq k$, and y ranges over the elements of P of rank $\max S$. In $g_i(y)$, f_T occurs with coefficient $(-1)^{|T|+\max S - i}(j_{T,i} + j_{T,i-1})$. This is then multiplied by $(x - 1)^{n-\max S}$. On the other hand, a valid S-diagram with k -1's arises from valid T-diagrams with i or $i - 1$ -1's as follows. Start with the T-diagram; extend it in position $\max S$ with a $+1$ (if T has i -1's) or with a -1 (if T has $i - 1$ -1's); fill the diagram out to length n with $k - i$ -1's and the rest $+1$'s. The number of choices is the binomial coefficient contributed by $(x - 1)^{n-\max S}$ in the recursion for the h-vector.

Now Fine's flag f-vector formula can be converted easily to a flag h-vector formula. For fixed sequence λ of n ±1's, there is a unique maximal set $S = S(\lambda) \subseteq [n]$

for which λ is a valid S-diagram. Furthermore, for all $T \subseteq S(\lambda)$, λ is a valid T-diagram. Let i_λ be the number of -1's in λ. Then the formula above becomes

$$\begin{aligned} f(P,x) &= \sum_{\lambda\in\{+1,-1\}^n} x^{i_\lambda} \sum_{T\subseteq S(\lambda)} (-1)^{|T|+n-i_\lambda} f_T(P) \\ &= \sum_{\lambda\in\{+1,-1\}^n} x^{i_\lambda} (-1)^{|S(\lambda)|+n-i_\lambda} h_{S(\lambda)}(P). \end{aligned}$$

Thus $f(P,x) = \sum_{S\subseteq[n]} \sum_{i=0}^n (-1)^{|S|+n-i} m_{S,i} x^i h_S(P)$, where $m_{S,i}$ is the number of valid S-diagrams λ for which $S(\lambda) = S$ and $i_\lambda = i$. In other words, the coefficient of $h_S(P)$ in h_i is (plus or minus) the number of valid ±1-sequences having i -1's for which the set S is maximal.

This coefficient is the coefficient of x^i in $f(u_S,x)$, where u_S is the **ab**-word associated with the set S. Theorems 3.1 and 4.1 say it is the product of various $p(k,k/2)$, where $k+1$ is the length of a consecutive (but not final) string of **a**'s or **b**'s in u_S, and some coefficient of either $Q_{k-1}(x)$ (if u_S ends in $\mathbf{b}^k$) or $R_{k-1}(x)$ (if u_S ends in $\mathbf{a}^k$). We show how these factors come from counting valid ±1-sequences associated to the set S.

Break u_S into its maximal consecutive strings of **a**'s and **b**'s. For S to be maximal for a sequence λ, when restricted to the positions of a string of **a**'s, λ must have nonpositive partial sums; when restricted to the positions of a string of **b**'s, it must have positive partial sums. At the boundary between **a**-strings and **b**-strings, the partial sums must change between 0 and $+1$.

Here is an example of part of an S-diagram, where u_S contains the substring $\mathbf{b}^3\mathbf{a}^5$. The locations of the **b**'s are marked with vertical bars to the left.

```
    λ          ··· **|+1|*| * -1****|+1 ···
partial sums   ··· *0 +1 * +1 0 ***0 +1 ···
```

How many different λs can fill out this diagram? Associate with each sequence λ a lattice path, with $+1$ representing a step up and -1 representing a step down. The partial sum of a sequence to some point represents the height of the lattice path at that point. The possible choices of λ_i for the locations of the three **b**'s correspond to lattice subpaths that go from $(0,1)$ to $(2,1)$ without going below the line $y=1$. More generally, if there are $k+1$ consecutive **b**'s, the lattice subpaths go from $(0,1)$ to $(k,1)$. Here k must be even and the number of such subpaths is $p(k,k/2)$. Similarly, the possible choices of λ_i for the locations of the five **a**'s correspond to lattice subpaths that go from $(0,0)$ to $(4,0)$ without going above the line $y=0$. More generally, if there are $k+1$ consecutive **a**'s, the lattice subpaths go from $(0,0)$ to $(k,0)$. Again k must be even and the number of such subpaths is $p(k,k/2)$.

For a final substring of **a**'s or **b**'s, the right-hand partial sum (the total sum of the sequence) is no longer fixed; it is merely restricted to be positive (if u_S ends in **b**) or nonpositive (if u_S ends in **a**). Sequences λ with different total sums count towards different h_i. The possible choices of λ_i for the locations of a final string of k **b**'s correspond to lattice subpaths that start at $(0,1)$ and do not go below the line $y=1$. The number of these containing $j \le (k-1)/2$ downward steps (-1's in λ) is $p(k-1,j)$. A similar argument counts the number of subsequences of λ that

can fill a final substring of $\mathbf{a}^k$. Here there must be more -1's than $+1$'s, and the number with j -1's is $p(k-1, k-1-j)$.

Details can be added to complete an alternative proof of the formula for the h-vector in terms of the $\mathbf{ab}$-index.

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