Hamiltonian cycles on Archimedean solids are twisting free*

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Abstract

We prove that a Hamiltonian cycle on the faces of an Archimedean solid is *twisting free*, that is, when returning to the first facet of the cycle, it has the same orientation as in the beginning. We also explore a continuous analogue on the unit sphere.

Consider the following two nets of the cube in Figure 1.

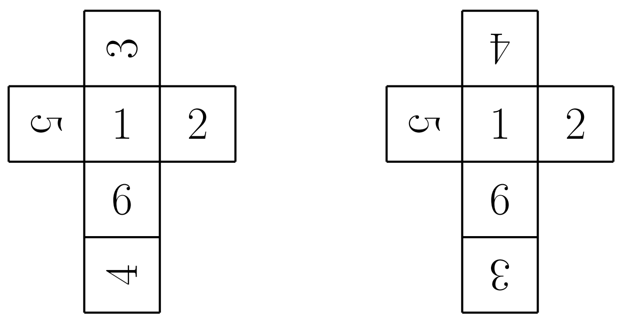

Figure 1

Cut out both nets and fold them into cubes. Place one of the cubes in front of you so that the number 1 is on the top face and it is oriented so that you can read it. Turn the cube so that the number 2 is facing upwards, and continue to turn the cube such that you have seen the numbers 3 through 6 in that order. Now make the last turn such that the number 1 is on top again. You will note that it has the right orientation.

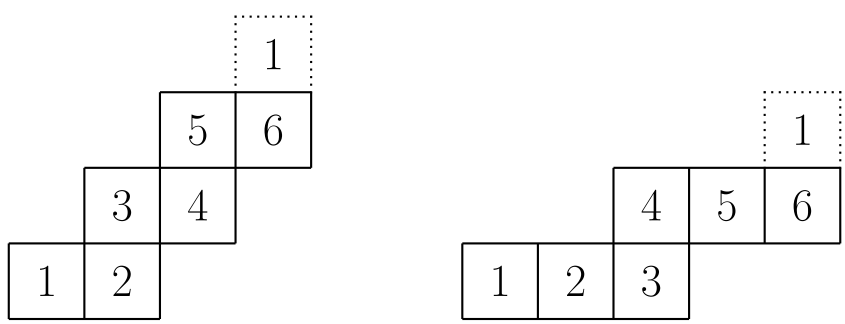

Figure 2

*The American Mathematical Monthly, Vol. 121, No. 2 (February 2014), pp. 158–161.

Observe that up to symmetry there are two Hamiltonian cycles on the faces of a cube. Instead of having the classical cross as the net of the cube, we could instead have the nets follow the corresponding Hamiltonian cycle as illustrated in Figure 2.

Recall that an *Archimedean solid* is a vertex transitive three-dimensional polyhedron whose faces are regular polygons. The vertex transitivity implies that each vertex looks the same. In our discussion, we include the five Platonic solids, the prisms and the antiprisms among the Archimedean solids. The pseudo rhombicubeoctahedron, which has a smaller symmetry group than the classical rhombicubeoctahedron, is also included.

Recall that a Hamiltonian cycle in a graph is a cycle on the vertices in which each vertex is visited exactly once. A planar graph has a dual graph. We call a Hamiltonian cycle of the dual graph a Hamiltonian cycle on the faces of the original graph.

Theorem 1. *A Hamiltonian cycle on the faces of an Archimedean solid is twisting free.*

As an example, Figure 3 is a net of the truncated tetrahedron showing a Hamiltonian cycle on the faces. By folding it together, one observes that the two faces labeled 1 have the same orientation.

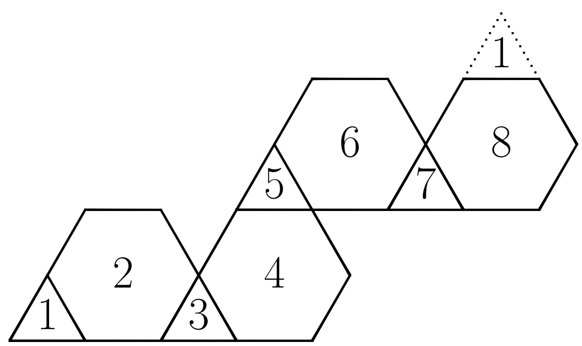

Figure 3

We note that there are six Archimedean solids that do not have any Hamiltonian cycles on their faces. Coxeter was the first to observe this for the cuboctahedron [1, 4]. For each of these six solids one can observe this fact by finding an independent set of faces (that is, no two faces are adjacent) which is greater in cardinality than the remaining set of faces. See Weisstein [6] for the complete classification.

We begin to consider the dual polyhedron. That is, consider a polyhedron where each face is a q-gon, but not necessarily a regular q-gon. The cycles we consider are on the vertices of the polyhedron.

Proposition 2. *Let P be a polyhedron where each face is a q-gon. Let R be a collection of faces of P such that R is homeomorphic to a disc, that is, the boundary of R is a cycle. Let B the number of vertices on the boundary of the collection R and I be the number of vertices in the interior of R. Then the number of faces of R is given by*

$$|R| = \frac{B + 2 \cdot I - 2}{q - 2}.$$

Proof. Let $f_i = f_i(R)$ denote the number of i-dimensional faces of the region R. Note that the number of vertices, f_0, is given by $B+I$. Since R is homeomorphic to a disc, we have by the Euler characteristic that

$$B + I - f_1 + f_2 = 1. \tag{1}$$

Next, observe that each edge on the boundary lies in one face of R, whereas each edge in the interior lies in two. Hence twice the number of edges is given by

$$2 \cdot f_1 = q \cdot f_2 + B, \tag{2}$$

since each face has q edges. Viewing equations (1) and (2) as a linear system in f_1 and f_2, and solving for $f_2 = |R|$ yields the result. □

Proposition 3. *A Hamiltonian cycle on the vertices of a polyhedron where all the faces are q-gons partitions the faces into two sets of equal size.*

Proof. The Hamiltonian cycle bounds two regions, each of which has no interior vertices. By Proposition 2, both regions have $(B-2)/(q-2)$ faces, that is, the same number. □

This proposition is reminiscent of the following classical problem. *Let m and n be two non-negative integers, not both odd. Consider a path on the grid $\{0, 1, \ldots, m\} \times \{0, 1, \ldots, n\}$ starting at $(0, 0)$ and ending at (m, n) going through all the $(m+1) \cdot (n+1)$ vertices. Then this path cuts the $m \cdot n$ squares into two sets of equal size.* We leave this problem for the reader, but the solution is not hard to pick by thinking outside the box.

By taking the dual, we obtain a result that applies to every Archimedean solid.

Corollary 4. *Let P be a polyhedron where q faces meet at every vertex. A Hamiltonian cycle on the faces of P partitions the vertices into two sets of equal size.*

Recall for a vertex v of a polyhedron P that the *angle defect* $\alpha(v)$ is 2π minus the sum of the angles at v of the faces meeting at v.

Proposition 5. *Consider a cycle C of faces in positive orientation on a polyhedron. Let W be the set of vertices that is surrounded by the cycle C. Then the twist when we travel around the cycle C is given by the negative of the sum of the defects in W; that is,*

$$\operatorname{twist}(C) = -\sum_{v \in W} \alpha(v).$$

Proof. It is enough to observe this when the cycle C goes around exactly one vertex v. Then by summing the cycles around each vertex in W, the result follows. □

Proof of Theorem 1. Descartes' theorem states that the sum of the angle defects of a polyhedron is 4π; see for instance [2, Section 2.5] or [3, Chapter 5]. By symmetry, an Archimedean solid has all the angle defects to be the same. Thus, the angle defect of every vertex is given by $4\pi/f_0$, where f_0 is the number of vertices. However, the Hamiltonian cycle goes around exactly half of the vertices. Hence, the sum of the angle defects of these vertices is $f_0/2 \cdot 4\pi/f_0 = 2\pi$. That is, there is no twist. □

Observe that in the proof of Theorem 1 we only used two properties of the Archimedean solids. Hence we have the following more general result.

Theorem 6. *Let P be a three-dimensional polyhedron such that each vertex is incident to the same number of edges and that the angle defect of every vertex is also the same. Then a Hamiltonian cycle on the faces of the polyhedron P is twisting free.*

Examples of such polyhedra are vertex transitive polyhedra.

We end by discussing a *continuous analogue* of Proposition 5. Consider the two-dimensional unit sphere S and choose a region M homeomorphic to a disc on the sphere S. Furthermore, assume its boundary ∂M is piecewise smooth. Pick a starting point x on the boundary ∂M. Place the sphere on a plane such that the plane is touching the sphere at the point x. Now roll the sphere on the plane such that the sphere only touches the plane at the boundary ∂M. After traversing the whole boundary ∂M and returning to the starting point x, the sphere has twisted by the angle given by the negative of the area of M.

The Gauss–Bonnet theorem (see for instance [5, Chapter 6]) states that

$$\int_M K\, dA + \int_{\partial M} k_g\, ds + \sum_{i=1}^{n} \delta_i = 2\pi \cdot \chi(M),$$

where K is the Gaussian curvature of M, k_g is the signed geodesic curvature of ∂M, and $\delta_1, \ldots, \delta_n$ are the exterior angles at the corners of M. Since M is a part of the unit sphere, the Gaussian curvature is 1, and hence the first integral is given by $\mathrm{Area}(M)$. Furthermore, the Euler characteristic $\chi(M)$ is 1. Hence, the integral over the geodesic curvature plus the sum of the exterior angles is given by

$$\int_{\partial M} k_g\, ds + \sum_{i=1}^{n} \delta_i = 2\pi - \mathrm{Area}(M).$$

But the left-hand side quantity measures how much the boundary is twisting. We do the same by rolling the sphere on the plane along ∂M. Hence, the total twist is given by $2\pi - \mathrm{Area}(M) \equiv -\mathrm{Area}(M) \bmod 2\pi$.

The most interesting case is when we choose M to have area 2π; that is, half of the area of the unit sphere. Then, when rolling the sphere on the plane along ∂M, the sphere is only translated and does not twist.

Acknowledgments. The author is grateful to Margaret Readdy, Richard Stanley and the referee for their comments on an earlier version of this note. The author was partially supported by National Science Foundation grant DMS 0902063.

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