# Flags and shellings of Eulerian cubical posets*†

Richard EHRENBORG
School of Mathematics
Institute for Advanced Study
Olden Lane
Princeton, NJ 08540

Gábor HETYEI‡
Mathematics Department
405 Snow Hall
University of Kansas
Lawrence KS 66045-2142

**Abstract**

A cubical analogue of Stanley's theorem expressing the **cd**-index of an Eulerian simplicial poset in terms of its $h$-vector is presented. This result implies that the **cd**-index conjecture for Gorenstein* cubical posets follows from Ron Adin's conjecture on the non-negativity of his cubical $h$-vector for Cohen-Macaulay cubical posets. For cubical spheres the standard definition of shelling is shown to be equivalent to the spherical one. A cubical analogue of Stanley's conjecture about the connection between the **cd**-index of semisuspended simplicial shelling components and the reduced variation polynomials of certain subclasses of André permutations is established. The notion of signed André permutation used in this result is a common generalization of two earlier definitions of signed André permutations.

## Introduction

In recent years many efforts were made to describe the flag $f$-vector (the array giving the numbers of flags) of Eulerian partially ordered sets. Bayer and Billera characterized in [5] all linear inequalities which are satisfied by the flag $f$-vector of all Eulerian partially ordered sets. Fine used this information to introduce a non-commutative polynomial, the **cd**-index, to efficiently encode the flag $f$-vector of an Eulerian poset. He also conjectured that for the face lattice of a convex polytope, the **cd**-index has non-negative coefficients.

Fine's conjecture was proved by Stanley in [26] for spherically shellable Eulerian posets, a class properly containing the face lattices of all convex polytopes. Stanley also made the conjecture [26, Conjecture 2.1] that every Eulerian poset whose order complex has the Cohen-Macaulay property, i.e., every Gorenstein* poset, has a non-negative **cd**-index. He showed that this conjecture, if true, gives a full description of all linear inequalities holding for the **cd**-indices of all Gorenstein* posets.

*This research began when both authors were postdoctoral fellows at LACIM, Université du Québec à Montréal.
†Appeared in *Annals of Combinatorics* **4** (2000), 199–226.
‡On leave from the Mathematical Institute of the Hungarian Academy of Sciences. Partially supported by Hungarian National Foundation for Scientific Research grant no. F 023436

A strengthened version of this conjecture appears in [27, Conjecture 2.7], stating that the **cd**-index of a Gorenstein* poset is greater than or equal to the **cd**-index of the Boolean algebra of the same rank. This strengthening was proved for polytopes by Billera and Ehrenborg [7]. Both conjectures remain open for Gorenstein* posets in general.

Stanley verified [26, Conjecture 2.1] for every Eulerian simplicial poset, by expressing its **cd**-index in terms of its $h$-vector and non-negative **cd**-polynomials. He also conjectured a combinatorial description of the non-negative coefficients occurring in his formula. This conjecture was proved by Hetyei in [19].

In this paper we generalize Stanley's simplicial results to cubical posets. The most plausible analogue of the simplicial $h$-vector for Eulerian cubical posets turns out to be identical to the normalized version of a cubical $h$-vector suggested by Adin in [1, 2]. Thus we obtain that Adin's conjecture about the non-negativity of his cubical $h$-vector for Cohen-Macaulay cubical posets implies Stanley's conjecture [26, Conjecture 2.1] for Gorenstein* cubical posets.

Two other $h$-vectors defined for cubical complexes have been studied before. First, the toric $h$-vector defined by Stanley for Eulerian posets in general [24] and studied by Babson, Billera and Chan [3, 12]. Second, the $h$-vector of the Stanley ring of cubical complexes introduced in [17, 18]. Unfortunately, none of them have been useful when proving nontrivial inequalities about the $f$-vector of cubical complexes. Our main result indicates that Adin's cubical $h$-vector might be a good candidate for this purpose. Inspired by this finding, the second author proved later in [20] that among all reasonably definable cubical $h$-vectors, Adin's $h$-vector is the smallest. More precisely, if an invariant $I$ of $d$-dimensional cubical complexes depends linearly on the face numbers, is non-negative on the face complex of a $d$-cube, and changes by a non-negative number when a new cell is attached in a shelling, then this invariant is a non-negative linear combination of the entries in Adin's $h$-vector.

In Section 1 we recall the definition and fundamental properties of the **cd**-index of a graded Eulerian poset, with a special focus on $C$-shellable CW-spheres. We draw attention to a lemma by Stanley [26, Lemma 2.1] which allows us to greatly simplify the calculation of the change in the **cd**-index of a CW-sphere when we subdivide a facet into two facets.

In Section 2 we specialize the results of Section 1 to shellable cubical complexes. We show that for cubical spheres the usual notion of shelling (which we call $C$-shelling, following Stanley in [26]) and the notion of $S$-shelling or spherical shelling (also introduced in [26]) coincide. Using a consequence of [26, Lemma 2.1], we obtain a formula for the **cd**-index of a shellable cubical sphere.

In Section 3 we use [26, Lemma 2.1] to establish linear relations between the **cd**-indices of semi-suspended cubical shelling components. These relations allow us to express the **cd**-index of a shellable cubical sphere in terms of an $h$-vector which is an invertible linear function of the $f$-vector. This $h$-vector turns out to be the normalized version of the cubical $h$-vector suggested by Adin in [2]. The expression obtained is a cubical analogue of Stanley's theorem [26, Theorem 3.1] and also exhibits the behavior that each $h_i$ is multiplied by a **cd**-polynomial with non-negative coefficients. Adin has asked whether the cubical $h$-vector of a Cohen-Macaulay cubical complex is non-negative. An affirmative answer to his question would imply a new special case of Stanley's conjecture [26, Conjecture 2.1] about the non-negativity of the **cd**-index of Gorenstein* posets.

In Section 4 we give recursion formulas for the **cd**-indices of both semisuspended simplicial and cubical shelling components. These formulas will be useful in proving the results of Section 5.

Finally, in Section 5 we express the **cd**-index of the semisuspended cubical shelling components in terms of reduced variation polynomials of signed augmented André* permutations. Unsigned André* permutations may be obtained from André permutations by reversing the linear order of the letters and reading the permutation backwards. As first observed by Ehrenborg and Readdy in [13], this small twist allows one to handle the signed generalizations more easily. Our signed André* permutations generalize both the signed André permutations, introduced by Purtill in [22] and studied in greater generality by Ehrenborg and Readdy in [13], and the signed André-permutations introduced by Hetyei in [19]. We prove not only a signed analogue of Stanley's conjecture [26, Conjecture 3.1], but as an auxiliary result we also obtain a new description of the **cd**-index of semisuspended simplicial shelling components.

# 1 The cd-index and shellings

Let $P$ be a graded poset of rank $n+1$, that is, $P$ is ranked with rank function $\rho$, and $P$ has a distinguished minimum element $\widehat{0}$ and a distinguished maximum element $\widehat{1}$. The *flag $f$-vector* $(f_S : S \subseteq \{1, 2, \ldots, n\})$ is defined by

$$f_S \stackrel{\text{def}}{=} \left| \left\{ \{\widehat{0} < x_1 < \ldots < x_k < \widehat{1}\} \subseteq P \ : \ \{\rho(x_1), \ldots, \rho(x_k)\} = S \right\} \right|,$$

and the *flag $h$-vector* (also called the *beta-invariant*) is defined by the equation

$$h_S \stackrel{\text{def}}{=} \sum_{T \subseteq S} (-1)^{|S \setminus T|} \cdot f_T.$$

The **ab**-*index* $\Psi_P(\mathbf{a}, \mathbf{b}) = \Psi(P)$ of a poset $P$ is the following polynomial in the non-commuting variables $\mathbf{a}$ and $\mathbf{b}$:

$$\Psi_P(\mathbf{a}, \mathbf{b}) = \sum_{S \subseteq \{1,2,\ldots,n\}} h_S \cdot u_S, \tag{1.1}$$

where $u_S$ is the monomial $u_1 \cdots u_n$ satisfying

$$u_i = \begin{cases} \mathbf{a} & \text{if } i \notin S, \\ \mathbf{b} & \text{if } i \in S. \end{cases}$$

A poset $P$ is *Eulerian* if the Möbius function of any interval $[x, y]$ is given by $\mu(x, y) = (-1)^{\rho(y)-\rho(x)}$. Fine observed (see Bayer and Klapper's paper [6]) that the **ab**-index of an Eulerian poset can be written uniquely as a non-commutative polynomial in the variables $\mathbf{c} \stackrel{\text{def}}{=} \mathbf{a} + \mathbf{b}$ and $\mathbf{d} \stackrel{\text{def}}{=} \mathbf{ab} + \mathbf{ba}$. For an inductive proof of this fact, see Stanley [26]. In this case we call $\Psi_P(\mathbf{a}, \mathbf{b}) = \Phi_P(\mathbf{c}, \mathbf{d}) = \Phi(P)$ the **cd**-*index* of the Eulerian poset $P$.

Stanley [26] introduced the polynomial encoding the flag $f$-vector $\Upsilon_P(\mathbf{a}, \mathbf{b}) \stackrel{\text{def}}{=} \sum_{S \subseteq \{1,\ldots,n\}} f_S \cdot u_S$ and he observed that

$$\Psi_P(\mathbf{a}, \mathbf{b}) = \Upsilon_P(\mathbf{a} - \mathbf{b}, \mathbf{b}) \qquad \text{and} \qquad \Upsilon_P(\mathbf{a}, \mathbf{b}) = \Psi_P(\mathbf{a} + \mathbf{b}, \mathbf{b}). \tag{1.2}$$

**Definition 1.1** *Let $P$ be an Eulerian poset of rank $n+1$. To every chain $c = \{\widehat{0} < x_1 < \cdots < x_k < \widehat{1}\}$ in $P$ we associate a* weight $w(c) \stackrel{\text{def}}{=} z_1 \cdots z_n$, *where*

$$z_i = \begin{cases} \mathbf{b} & \text{if } i \in \{\rho(x_1), \ldots, \rho(x_k)\}, \\ \mathbf{a}-\mathbf{b} & \text{otherwise.} \end{cases}$$

Note that the chain $\{\widehat{0} < \widehat{1}\}$ gets the weight $(\mathbf{a}-\mathbf{b})^n$. By the first identity in (1.2), the **ab**-index $\Psi_P(\mathbf{a}, \mathbf{b})$ is the sum of the weights of all chains in $P$:

$$\Psi(P) = \Psi_P(\mathbf{a}, \mathbf{b}) = \sum_c w(c), \tag{1.3}$$

where $c$ ranges over all chains $c = \{\widehat{0} < x_1 < \cdots < x_k < \widehat{1}\}$ in the poset $P$.

A poset $P$ is called *near-Eulerian* if it may be obtained from an Eulerian poset $\widetilde{\Sigma}P$, called the *semisuspension* of $P$, by removing one coatom. The poset $\widetilde{\Sigma}P$ may be uniquely reconstructed from $P$ by adding a coatom $x$ which covers all $y \in P$ for which $[y, \widehat{1}]$ is the three element chain. By [23, Proposition 3.14.5] the Möbius function of $P$ is equal, up to a certain sign, to the Möbius function of $(\widetilde{\Sigma}P \setminus P) \cup \{\widehat{0}, \widehat{1}\}$. This complementary poset is a chain of length two, which has Möbius function equal to 0. Hence the Möbius function of a near-Eulerian poset $P$ is equal to zero. For more information and extended bibliography on Eulerian posets in general, see Stanley's survey [27].

The aim of this paper is to generalize Stanley's results about Eulerian simplicial posets to the cubical case. Originally, simplicial and cubical posets arose as generalizations of simplicial and cubical complexes, which we now define.

**Definition 1.2** *A* simplicial complex $\triangle$ *is a family of finite sets (called faces) on a vertex set $V$ closed under inclusion such that for each vertex $v \in V$ we have $\{v\} \in \triangle$. A* cubical complex $\mathcal{C}$ *is a family of finite sets (called faces) on a vertex set $V$ closed under intersection such that $\{v\} \in V$ for all $v \in V$ and for every face $\sigma \in \mathcal{C}$ the interval $[\emptyset, \sigma]$ is isomorphic to the lattice of faces of a cube.*

Simplicial and cubical posets may be intuitively regarded as generalized simplicial respectively cubical complexes whose faces are allowed to intersect in any subcomplex of their boundaries, rather than just a single face; see [25]. Hence a simplicial (respectively cubical) poset naturally has a distinguished minimum element $\widehat{0}$ (corresponding to the empty face), and for every element $x$ of a simplicial (respectively cubical) poset the interval $[\widehat{0}, x]$ is a Boolean algebra (respectively the face lattice of a cube). On the other hand, the existence of a maximum element $\widehat{1}$ is not required, since the highest dimensional faces (facets) of a simplicial or cubical complex need not be of the same dimension. If the complex is homeomorphic to a sphere (for instance, we are dealing with a simplicial or cubical polytope), it is natural to introduce a maximum element $\widehat{1}$ representing the "largest face" which is not necessarily a simplex or a cube anymore.

Based on this intuition, Stanley [26] defined an Eulerian simplicial poset as a graded poset for which for which $[\widehat{0}, x]$ is a Boolean algebra whenever $x < \widehat{1}$. We extend this definition to all graded simplicial or cubical posets as follows.

**Definition 1.3** *A graded poset $P$ is simplicial, respectively cubical, if for every $x \in P \setminus \{\widehat{1}\}$, the interval $[\widehat{0}, x]$ is a Boolean algebra, respectively the face lattice of a cube.*

Both simplicial and cubical complexes may be realized as regular CW-complexes. We call a poset $P$ with $\widehat{0}$ a *CW-poset* when for all $x > \widehat{0}$ in $P$ the geometric realization $|(\widehat{0}, x)|$ of the open interval $(\widehat{0}, x)$ is homeomorphic to a sphere. By [10], $P$ is a CW-poset if and only if it is the face poset $P(\Omega)$ of a regular CW-complex $\Omega$. Following Stanley in [26], we use $P_1(\Omega)$ to denote the face poset $P(\Omega)$ with $\widehat{1}$ adjoined. We will denote the closure of a cell $\sigma$ by $\overline{\sigma}$ or by $\mathrm{cl}(\sigma)$. If $\Omega$ is homeomorphic to a sphere then $P_1(\Omega)$ is Eulerian, while if $\Omega$ is homeomorphic to a ball then $P_1(\Omega)$ is near-Eulerian. For a regular CW-ball $\Omega$ the semisuspension $\widetilde{\Sigma} P_1(\Omega)$ of $P_1(\Omega)$ is of the form $P_1(\widetilde{\Sigma}\Omega)$ where $\widetilde{\Sigma}\Omega$ is the regular CW-sphere obtained from $\Omega$ by adding an extra facet, the boundary of which is identified with the boundary $\partial\Omega$ of $\Omega$.

Stanley observed the following; see [26, Lemma 2.1]. Let $\Omega$ be an $n$-dimensional CW-sphere, and $\sigma$ an (open) facet of $\Omega$. Let $\Omega'$ be obtained from $\Omega$ by subdividing $\overline{\sigma}$ into a regular CW-complex with two facets $\sigma_1$ and $\sigma_2$ such that $\partial\sigma$ remains the same and $\overline{\sigma_1} \cap \overline{\sigma_2}$ is a regular $(n-1)$-dimensional $CW$-ball $\Gamma$. Then we have

$$\Phi(P_1(\Omega')) \; - \; \Phi(P_1(\Omega)) \; = \; \Phi(P_1(\widetilde{\Sigma}\Gamma)) \cdot \mathbf{c} \; - \; \Phi(P_1(\partial\Gamma)) \cdot (\mathbf{c}^2 - \mathbf{d}). \tag{1.4}$$

In particular, if we take another $n$-dimensional CW-sphere and subdivide it isomorphically, the **cd**-index will change by the same amount, as described in the following lemma.

**Lemma 1.4** *Let $\Omega_1$ and $\Omega_2$ be $n$-dimensional CW-spheres. Assume that we subdivide a facet $\sigma^i$ of $\Omega_i$ ($i = 1, 2$) into two facets $\sigma^i_1$ and $\sigma^i_2$ such that $\partial(\sigma^i)$ is unchanged and $\mathrm{cl}\,(\sigma^i_1) \cap \mathrm{cl}\,(\sigma^i_2)$ is a regular $(n-1)$-dimensional $CW$-ball $\Gamma_i$. Then $P_1(\Gamma_1) = P_1(\Gamma_2)$ and $P_1(\partial\Gamma_1) = P_1(\partial\Gamma_2)$ imply*

$$\Phi(P_1(\Omega'_1)) \; - \; \Phi(P_1(\Omega_1)) \; = \; \Phi(P_1(\Omega'_2)) \; - \; \Phi(P_1(\Omega_2)).$$

Fine [6, Conjecture 3] conjectured that the **cd**-index of the face lattice of a convex polytope is non-negative. Stanley proved this in greater generality [26, Theorem 2.2] for *spherically shellable*, or $S$–shellable, regular CW-spheres.

**Definition 1.5** *Let $\Omega$ be an $n$-dimensional Eulerian regular CW-complex. A complex $\Omega$ or its face poset $P_1(\Omega)$ is called* spherically shellable *(or $S$–shellable) if either $\Omega = \{\emptyset\}$ (and so $P_1(\Omega)$ is the two-element chain $\{\widehat{0} < \widehat{1}\}$), or else we can linearly order the facets (open $n$-cells) $F_1, F_2, \ldots, F_m$ of $\Omega$ such that for all $1 \leq i \leq m$ the following two conditions hold:*

($S$-a) *$\partial\overline{F_1}$ is $S$–shellable of dimension $n-1$.*

($S$-b) *For $2 \leq i \leq m-1$, let $\Gamma_i \stackrel{\text{def}}{=} \mathrm{cl}[\partial\overline{F_i} - ((\overline{F_1} \cup \cdots \cup \overline{F_{i-1}}) \cap \overline{F_i})]$. Then $P_1(\Gamma_i)$ is near-Eulerian of dimension $n-1$, and the semisuspension $\widetilde{\Sigma}\Gamma_i$ is $S$–shellable, with the first facet of the shelling being the facet $\tau = \tau_i$ adjoined to $\Gamma_i$ to obtain $\widetilde{\Sigma}\Gamma_i$.*

As a consequence of Lemma 1.4, the **cd**-index of an $S$–shellable regular CW-sphere may be computed from just knowing $\partial\overline{F_1}$ and the complexes $\Gamma_i$.

The definition of $S$–shellability is different from the usual notion of shellability (given e.g. in [10, Definition 4.1]), which is called $C$–shellability in [26], and which may be stated as follows.

**Definition 1.6** *Let $\Omega$ be an $n$-dimensional regular CW-complex. A complex $\Omega$ or its face poset $P_1(\Omega)$ is called $C$–*shellable *if either $\Omega = \{\emptyset\}$ (and so $P_1(\Omega)$ is the two-element chain $\{\widehat{0} < \widehat{1}\}$), or else we can linearly order its facets (open $n$-cells) $F_1, F_2, \ldots, F_m$ such that the following two conditions hold:*

($C$-a) *$\partial\overline{F_1}$ is $C$–shellable of dimension $n-1$.*

($C$-b) *For $2 \le i \le m$, the complex $\partial\overline{F_i}$ is $C$–shellable such that the $C$–shelling begins with the facets of $\partial\overline{F_i}$ contained in $\overline{F_1} \cup \cdots \cup \overline{F_{i-1}}$.*

In general, neither notion of shellability implies the other [26, page 494]. It is trivially true, however, that the two notions of shellability coincide for the geometric realizations of simplicial spheres. We will show in Section 2 that the same holds for cubical complexes.

## 2 Equivalence of usual and spherical shellability for cubical spheres

Let $\mathcal{C}^n$ denote the complex of faces of an $n$-cube with vertex set $V(\mathcal{C}^n)$. Any $n$-cube may be geometrically realized in $\mathbb{R}^n$ as the the convex hull of the vertex set $\{0,1\}^n$. We call such a realization $\phi : V(\mathcal{C}^n) \longrightarrow \mathbb{R}^n$ of a cube a *standard geometric realization.* By abuse of notation, we will also denote by $\phi$ the map associating the convex hull of $\{\phi(v) \,:\, v \in \sigma\}$ to a face $\sigma \in \mathcal{C}^n$. Thus $\phi$ associates to a face of a cubical complex a closed face of a CW-complex. Using $\phi$ we may define the boundary $\partial\mathcal{C}^n$ as the inverse image under $\phi$ of the boundary of $[0,1]^n$.

Following Metropolis and Rota [21], given a standard geometric realization $\phi$, we encode the nonempty faces $\sigma$ of the $n$-cube with vectors $(u_1, u_2, \ldots, u_n) \in \{0, 1, *\}^n$ such that for every $i \in \{1, 2, \ldots, n\}$ we set $u_i = 0$ or 1 respectively if the $i$-th coordinate of every element of $\phi(\sigma)$ is 0 or 1 respectively and $u_i = *$ otherwise. Using this coding, the facets of $\partial\mathcal{C}^n$ will correspond to those vectors $(u_1, \ldots, u_n)$ for which exactly one $u_i$ is not the $*$-sign.

**Definition 2.1** *Let $A_i^0$, respectively $A_i^1$, denote the facet $(u_1, u_2, \ldots, u_n)$ with $u_i = 0$, respectively $u_i = 1$ and $u_k = *$ for $k \neq i$. Let $\{F_1, \ldots, F_k\}$ be a collection of facets of $\partial(\mathcal{C}^n)$. Let $r$ be the number of indices $i$ such that exactly one of $A_i^0$ and $A_i^1$ belongs to $\{F_1, \ldots, F_k\}$, and let $s$ be the number of indices $j$ such that such that both $A_j^0$ and $A_j^1$ belong to $\{F_1, \ldots, F_k\}$. We call $(r, s)$ the* type *of $\{F_1, \ldots, F_k\}$.*

Note that when the type of $\{F_1, \ldots, F_k\}$ is $(r, s)$ then there are exactly $n-r-s$ coordinates $i$ such that neither $A_i^0$ nor $A_i^1$ belong to $\{F_1, \ldots, F_k\}$. Clearly the type does not depend on the choice of the standard geometric realization.

The following lemma plays a crucial role in the proof of several statements in this section.

**Lemma 2.2** *Let $\mathcal{C}$ be the family of all faces contained in a collection of facets of $\mathcal{C}^n$ of type $(r, s)$, where $r > 0$. Then there is a convex polytope $\mathcal{P}$ and a point $p$ outside it such that the collection of those faces of $\mathcal{P}$ which can be seen from the point $p$ is a geometric representation of $\mathcal{C}$.*

**Proof:** Without loss of generality, we may assume that the facet $A_n^0$ belongs to $\mathcal{C}$ while the facet $A_n^1$ does not. Let us represent $\mathcal{C}^n$ by a polytope $\mathcal{P}$, given by the following facet inequalities:

1. $x_n \geq 0$. This is the facet $A_n^0$.
2. $x_n \leq 1$. We associate $A_n^1$ to this facet.
3. For $1 \leq i \leq n-1$ such that $A_i^0 \in \mathcal{C}$ we have $x_i + \varepsilon \cdot x_n \geq 0$. We associate $A_i^0$ to this facet.
4. For $1 \leq i \leq n-1$ such that $A_i^0 \notin \mathcal{C}$ we have $x_i - \varepsilon \cdot x_n \geq 0$. We associate $A_i^0$ to this facet.
5. For $1 \leq i \leq n-1$ such that $A_i^1 \in \mathcal{C}$ we have $x_i - \varepsilon \cdot x_n \leq 1$. We associate $A_i^1$ to this facet.
6. For $1 \leq i \leq n-1$ such that $A_i^1 \notin \mathcal{C}$ we have $x_i + \varepsilon \cdot x_n \leq 1$. We associate $A_i^1$ to this facet.

It is easy to verify that the face lattice of $\mathcal{P}$ is isomorphic to the face lattice of $\mathcal{C}^n$, i.e., we get a skewed cube. To do so, observe that the vertex set of $\mathcal{P}$ is

$$\{0,1\}^{n-1} \times \{0\} \;\cup\; \{a_1, b_1\} \times \{a_2, b_2\} \times \cdots \times \{a_{n-1}, b_{n-1}\} \times \{1\},$$

where the numbers $a_1, \ldots, a_{n-1}$ and $b_1, \ldots, b_{n-1}$ are given by

$$a_i \stackrel{\text{def}}{=} \begin{cases} -\varepsilon & \text{if } A_i^0 \in \mathcal{C} \\ \varepsilon & \text{if } A_i^0 \notin \mathcal{C} \end{cases}, \quad \text{and} \quad b_i \stackrel{\text{def}}{=} \begin{cases} 1+\varepsilon & \text{if } A_i^1 \in \mathcal{C} \\ 1-\varepsilon & \text{if } A_i^1 \notin \mathcal{C} \end{cases}.$$

Let us place ourselves at the point $p = (\frac{1}{2}, \ldots, \frac{1}{2}, -R)$, where $R > 1/2\varepsilon$. A facet of $\mathcal{P}$ is visible from $p$ if and only if its defining inequality is not satisfied by the coordinates of $p$. Hence the facet associated to $A_i^j$ where $1 \leq i \leq n$ and $j \in \{0,1\}$ is visible from $p$ if and only if $A_i^j$ belongs to $\mathcal{C}$. A lower-dimensional face is visible from the point $p$ if and only if it is contained in a visible facet. □

The following observation is originally due to Ron Adin and Clara Chan, who used this fact without proof. (See part (iii) of Theorem 5 and Section 4 in [2], or the implicit assumption in the proof of Theorem 1 in [12].) The only place where a (more complicated than the present) proof appears is [17].

**Lemma 2.3** *Let $\{F_1, \ldots, F_k\}$ be a collection of facets of $\partial\mathcal{C}^n$ and $\phi$ a standard geometric realization of $\mathcal{C}^n$. Then $\phi(F_1) \cup \cdots \cup \phi(F_k)$ is an $(n-1)$-sphere if and only if it has type $(0, n)$ and it is an $(n-1)$-ball if and only if its type $(r, s)$ satisfies $r > 0$.*

**Proof:** Consider first the case $r > 0$. Then according to Lemma 2.2, a collection of facets of type $(r, s)$ may be geometrically represented as the set of all facets of a polytope $\mathcal{P}$ visible from a given point $p$. The union of these facets is homeomorphic to their central projection from $p$ onto a hyperplane separating $\mathcal{P}$ and $p$. The projection is an $(n-1)$-polytope, hence it is homeomorphic to an $(n-1)$-dimensional ball.

Note next that $\{F_1, \ldots, F_k\}$ has type $(0, n)$ if and only if it is the collection of all facets of the $n$-cube. The surface of an $n$-cube is an $(n-1)$ dimensional sphere and so $\{F_1, \ldots, F_k\}$ must be a sphere.

It only remains to be shown that for all other types $(r, s)$, the set $\bigcup_{i=1}^{k} \phi(F_i)$ is not homeomorphic to an $(n-1)$-ball or an $(n-1)$-sphere. Here the map $\phi$ is a standard geometric realization of the $n$-cube. The types not listed above are of the form $(0, s)$ with $0 \leq s < n$. Let us fix such a type. Consider all those coordinates $i$ for which neither $A_i^0$ nor $A_i^1$ belong to $\{F_1, \ldots, F_k\}$. Without loss of generality, we may assume that these coordinates are $i = 1, 2, \ldots, n-s$. Consider the continuous map $\psi : \mathbb{R}^n \times [0, 1] \longrightarrow \mathbb{R}^n$, defined by

$$((x_1, x_2, \ldots, x_{n-s}, x_{n-s+1}, \ldots, x_n), t) \longmapsto (t \cdot x_1, t \cdot x_2, \ldots, t \cdot x_{n-s}, x_{n-s+1}, \ldots, x_n).$$

This map retracts $\bigcup_{i=1}^{k} \phi(F_i)$ to a collection of all facets of an $s$-cube, i.e. an $(s-1)$-dimensional sphere. Now the lemma follows from the fact that an $(s-1)$-sphere with $s < n$ is not homotopy equivalent to an $(n-1)$-ball or an $(n-1)$ sphere, because its homology groups are different. $\square$

It is easy to see by induction on the dimension that there exists a $C$–shelling of the boundary of $[0, 1]^n$ starting with the facets $\{\phi(F_1), \ldots, \phi(F_k)\}$ if and only if $\phi(F_1) \cup \cdots \cup \phi(F_k)$ is an $(n-1)$-ball or an $(n-1)$-sphere (in the latter case $k = 2n$ and $\phi(F_1) \cup \cdots \cup \phi(F_k) = \partial[0, 1]^n$). Thus we may rephrase the definition of $C$–shellability for finite cubical complexes in a purely combinatorial way as follows.

**Lemma 2.4** *Let $\mathcal{C}$ be an $n$-dimensional pure cubical complex, that is, a cubical complex whose maximal faces are of the same dimension. An ordering $F_1, \ldots, F_m$ of the facets of $\mathcal{C}$ induces a $C$-shelling of its geometric realization if and only if for every $k \in \{2, \ldots, m\}$ the following two conditions hold:*

*(i) The set of faces contained in $F_k \cap (F_1 \cup \cdots \cup F_{k-1})$ is a pure complex of dimension $(n-1)$.*

*(ii) The collection of the facets of $\partial F_k$ contained in $F_1 \cup \cdots \cup F_{k-1}$ has type $(r, s)$ with $r > 0$ or $(r, s) = (0, n-1)$.*

**Definition 2.5** *We call the cubical complex of faces contained in $F_k \cap (F_1 \cup \cdots \cup F_{k-1})$ the $k$th* shelling component*, and the type $(r, s)$ associated to it the* type of the shelling component*. We will also consider the empty cubical complex as a shelling component and say that the first shelling component has type $(0, 0)$.*

Using Lemma 2.4, the equivalence of $C$–shellability and $S$–shellability for cubical spheres depends on the following two key lemmas.

**Lemma 2.6** *Let $\mathcal{C}$ be the family of all faces contained in a collection of facets of $\mathcal{C}^n$ of type $(r, s)$, where $r > 0$. Then the boundary complex of $\mathcal{C}$ is $C$–shellable.*

**Proof:** As in the proof of Lemma 2.3, we use Lemma 2.2 to represent $\mathcal{C}$ as the subdivision of an $(n-1)$-dimensional polytope $\mathcal{Q}$. Here the facets of $\mathcal{C}$ are geometrically represented as the set of all facets of a polytope $\mathcal{P}$ visible from a given point $p$, and $\mathcal{Q}$ is their central projection from $p$ onto a hyperplane. The boundary complex of $\mathcal{Q}$ is isomorphic to the boundary complex of $\mathcal{C}$, which is thus $C$–shellable by the theorem of Bruggesser and Mani [11]. □

**Lemma 2.7** *Let $F_1, \ldots, F_{2s}$ be a collection of facets of $\partial C^n$ of type $(0, s)$ with $0 < s < n$, and $\Omega$ the regular CW-complex representing the cubical complex of faces contained in $F_1 \cup \cdots \cup F_{2s}$. Then the partially ordered set $P_1(\Omega)$ is not near-Eulerian.*

**Proof:** According to the proof of Lemma 2.3, $\phi(F_1) \cup \cdots \cup \phi(F_{2s})$ is homotopy equivalent to a sphere of dimension $s - 1 \geq 0$, and so it has reduced Euler characteristic $(-1)^{s-1}$. By [23, Proposition 3.8.9], we obtain $\mu(P_1(\Omega)) = (-1)^{s-1} \neq 0$. Therefore $P_1(\Omega)$ cannot be near-Eulerian. □

**Theorem 2.8** *Let $\Omega$ be the geometric realization of a cubical complex $\mathcal{C}$ as a regular CW-complex. Assume that $\Omega = \{\emptyset\}$ or it is an $n$-sphere. Then an ordering $F_1, \ldots, F_m$ of the facets of $\mathcal{C}$ induces a $C$-shelling if and only if it induces an $S$–shelling.*

**Proof:** Assume first that $F_1, \ldots, F_m$ is an $S$–shelling. Then by the definition of $S$–shellability and by Lemma 2.7, conditions (i) and (ii) of Lemma 2.4 are satisfied. Thus $F_1, \ldots, F_m$ is a $C$–shelling.

We show the other implication by induction on dimension. For $n = -1$ we get that $\Omega = \{\emptyset\}$ is both $S$–shellable and $C$–shellable. Assume now we are given a $C$–shelling $F_1, \ldots, F_m$ of the cubical sphere $\Omega$ of dimension $n \geq 0$. The complex $\partial\overline{F_1}$ is $C$–shellable by the definition of a $C$–shelling, and since it is homeomorphic to a sphere of dimension $n - 1$, it is also $S$–shellable by the induction hypothesis. Hence condition ($S$-a) of Definition 1.5 is satisfied. For condition ($S$-b) we must first check that, for $2 \leq k \leq m - 1$, the collection of the facets of $\partial\overline{F_k}$ contained in $F_1 \cup \cdots \cup F_{k-1}$ cannot have type $(0, n-1)$. In fact, as long as no $C$–shelling component of type $(0, n-1)$ occurs, the union of the enumerated cells is homeomorphic to an $n$-ball, and the first time we attach a $C$–shelling component of type $(0, n-1)$, we obtain an $n$-sphere, to which no further cells can be added without getting an $\Omega$ properly containing an $n$-sphere and thus not homeomorphic to it. Hence by Lemma 2.4, we may assume that for $2 \leq k \leq m - 1$, the collection of the facets of $\partial\overline{F_k}$ contained in $F_1 \cup \cdots \cup F_{k-1}$ has type $(r, s)$ with $r > 0$. Thus for $\Gamma_k \stackrel{\text{def}}{=} \text{cl}[\partial\overline{F_k} - ((\overline{F_1} \cup \cdots \cup \overline{F_{k-1}}) \cap \overline{F_k})]$, the partially ordered set $P_1(\Gamma_k)$ is near-Eulerian of dimension $n - 1$. We only need to show that $\widetilde{\Sigma}\Gamma_k$ is $S$–shellable, with the first facet of the shelling being the facet $\tau = \tau_k$ adjoined to $\Gamma_k$ to obtain $\widetilde{\Sigma}\Gamma_k$.

Without loss of generality, we may assume that $F_k$ is represented as a standard $n$-cube and exactly the facets $G_1 \stackrel{\text{def}}{=} A_1^0, G_2 \stackrel{\text{def}}{=} A_1^1, \ldots, G_{2s-1} \stackrel{\text{def}}{=} A_s^0, G_{2s} \stackrel{\text{def}}{=} A_s^1, G_{2s+1} \stackrel{\text{def}}{=} A_{s+1}^0, G_{2s+2} \stackrel{\text{def}}{=}$

$A^0_{s+2}, \ldots, G_{2s+r} \stackrel{\text{def}}{=} A^0_{s+r}$ of $\partial\overline{F_k}$ are contained in $F_1 \cup \cdots \cup F_{k-1}$. We claim that when we put the semisuspending facet $G_0 \stackrel{\text{def}}{=} \tau$ in front of this collection, we obtain an $S$–shelling of $\widetilde{\Sigma}\Gamma_k$. It is easy to see again that for $1 \leq i \leq 2s+r-1$, the collection $(\overline{G_0} \cup \cdots \cup \overline{G_{i-1}}) \cap \overline{G_i}$ of facets of $\overline{G_i}$ is of type $(r_i, s_i)$ where $r_i > 0$. Hence we may use the induction hypothesis to show that the condition imposed on these facets by the definition of $S$–shellability is satisfied. Finally, the boundary complex of $G_0$ is $C$-shellable by Lemma 2.6 and thus $S$-shellable by the induction hypothesis. □

Thus in the case of cubical spheres, we may simply speak about shellings without any reference to $C$-shellings or $S$-shellings.

**Definition 2.9** *Given a shellable cubical $n$-sphere $\mathcal{C}$ and a shelling $F_1, \ldots, F_m$ of it, we denote the number of shelling components of type $(r, s)$ by $c_{r,s}$. In particular, we have $c_{0,0} = c_{0,n} = 1$. We call the vector $(\ldots, c_{r,s}, \ldots)$ the* $c$-vector *of the shelling.*

Similar to the way Stanley treated the simplicial case in [26], we may express the **cd**-index of a shellable cubical sphere in terms of the numbers $c_{r,s}$, and the **cd**-indices of (semisuspended) shelling components of one dimension higher. For this purpose, we introduce the following notation.

**Definition 2.10** *Let $B_n$ be the Boolean algebra and $C_n$ the cubical lattice of rank $n$. That is, $B_n$ is the face lattice of the $(n-1)$-dimensional simplex $\triangle^{n-1}$ while $C_n$ is that of the cube $\mathcal{C}^{n-1}$. We denote $\Phi(B_n)$ and $\Phi(C_n)$ by $U_n$ and $V_n$, respectively. In particular, for $n = 1$ we have $U_1 = V_1 = 1$.*

**Definition 2.11** *Given a collection $F_1, \ldots, F_k$ of $k \leq n-1$ facets of $\partial\triangle^{n-1}$, we denote the semisuspension of the poset $[\widehat{0}, F_1] \cup \cdots \cup [\widehat{0}, F_k] \cup \{\widehat{1}\} \subset B_n$ by $B_{n,k}$ and its* **cd***-index by $U_{n,k}$. Given a collection $F_1, \ldots, F_{r+2s}$ of facets of $\partial\mathcal{C}^{n-1}$ of type $(r, s)$, where $r$ is positive, we denote the semisuspension of the poset $[\widehat{0}, F_1] \cup \cdots \cup [\widehat{0}, F_{r+2s}] \cup \{\widehat{1}\} \subset C_n$ by $C_{n,r,s}$ and its* **cd***-index by $V_{n,r,s}$.*

**Example 2.12** $C_{n+1,1,0}$ is obtained from $C_n$ by adding an extra $\widehat{1}$ above the maximal element and adding a new coatom covering all coatoms of the original lattice. Hence we have $V_{n+1,1,0} = V_n \cdot \mathbf{c}$.

We have the following cubical analogue for a special case of [26, Theorem 3.1].

**Proposition 2.13** *Let $\mathcal{C}$ be an $(n-1)$-dimensional shellable cubical sphere which has a shelling with $c$-vector $(\ldots, c_{r,s}, \ldots)$. Then the* **cd***-index of $P_1(\mathcal{C})$ is given by*

$$\Phi(P_1(\mathcal{C})) = V_{n+1,1,0} + \sum_{\substack{r,s \\ r>0}} c_{r,s} \cdot (V_{n+1,r+1,s} - V_{n+1,r,s}). \tag{2.1}$$

**Proof:** For every $(r, s)$ with $0 < r$ and $r + s \leq n-1$ we may take a shelling $F_1, \ldots, F_{2n}$ of the boundary of an $n$-cube such that the collection $\{F_1, \ldots, F_{r+2s}\}$ has type $(r, s)$ and the collection

$\{F_1, \ldots, F_{r+2s+1}\}$ has type $(r+1, s)$. Then the $(r+2s+1)$st shelling component has type $(r, s)$ and the difference between the **cd**-indices of the semisuspensions of the two collections is $V_{n+1,r+1,s} - V_{n+1,r,s}$.

As a consequence of Lemma 1.4, if the $k$-th shelling step of $\mathcal{C}$ has type $(r, s)$ then the difference between the **cd**-indices of the semisuspensions of the first $k+1$ facets and of the first $k$ facets is the same as above, i.e., $V_{n+1,r+1,s} - V_{n+1,r,s}$. Finally, $P_1(\mathcal{C})$ is the semisuspension of the complex obtained just before the last shelling step. $\square$

# 3 Adin's cubical $h$-vector and the cd-index of a cubical Eulerian poset

A fundamental difference between simplicial complexes and cubical complexes is that, in the simplicial case, the number of shelling components of each type is determined by the $h$-vector; whereas, in the cubical case, this number is not even an invariant of the complex.

**Definition 3.1** *Let $P$ be a graded simplicial or cubical poset, either without a distinguished maximum element $\widehat{1}$ and of rank $n$, or with a distinguished maximum element, and of rank $n+1$.*[1] *For $i = -1, 0, \ldots, n-1$ we denote the number of elements of rank $i+1$ in $P$ by $f_i$. The vector $(f_{-1}, f_0, \ldots, f_{n-1})$ is called the $f$-*vector *of $P$. When $P$ is simplicial we define its $h$-*vector *by*

$$\sum_{i=0}^{n} h_i \cdot x^{n-i} \stackrel{\text{def}}{=} \sum_{j=0}^{n} f_{j-1} \cdot (x-1)^{n-j}.$$

It is well known that when $P$ is the face poset of an $(n-1)$-dimensional $C$–shellable simplicial complex, $h_i$ is the number of facets $F_j$ in any shelling $F_1, \ldots, F_m$ for which $F_j \cap (F_1 \cup \cdots \cup F_{j-1})$ is a collection of $i$ facets of $\partial F_j$. In this sense the $c$-vector of a shelling of a cubical sphere is an analogue of the $h$-vector. It must be noted, however, that a cubical sphere may have several $c$-vectors corresponding to different shelling orders, as shown in the following example.

**Example 3.2** For the three dimensional cube there are essentially two different shellings. These two shellings have different $c$-vectors:

| Shelling order | $c_{0,0}$ | $c_{1,0}$ | $c_{2,0}$ | $c_{1,1}$ | $c_{0,2}$ |
|---|---|---|---|---|---|
| $A_1^0, A_2^0, A_3^0, A_1^1, A_2^1, A_3^1$ | 1 | 1 | 2 | 1 | 1 |
| $A_1^0, A_2^0, A_1^1, A_3^0, A_2^1, A_3^1$ | 1 | 2 | 0 | 2 | 1 |

Let us note here the following straightforward relationship between the $f$-vector and the $c$-vector of a shelling for a $C$-shellable $(n-1)$-dimensional cubical complex $\mathcal{C}$.

$$\sum_{k=0}^{n-1} f_k \cdot x^k = \sum_{r,s} c_{r,s} \cdot (x+2)^{n-1-r-s} \cdot (x+1)^r \cdot x^s. \tag{3.1}$$

[1] With these distinctions we get the exact same definitions for simplicial posets as the ones given by Stanley in [25] and [26].

In fact, it is easy to verify that, whenever we attach a shelling component of type $(r, s)$, every newly added face is at least $s$-dimensional, and for $k \geq s$ the number of new $k$-dimensional faces is

$$\sum_{i+j=k-s} \binom{n-1-r-s}{i} \cdot 2^{n-1-r-s-i} \cdot \binom{r}{j}.$$

Fortunately there are some linear relations between the polynomials $V_{n,r,s}$. As a consequence of Lemma 1.4 we have

$$V_{n,r+1,s} - V_{n,r,s} = V_{n,r,s+1} - V_{n,r+1,s} \quad \text{for } n > 1,\ r > 0,\ s \geq 0 \text{ and } r+s \leq n-1. \tag{3.2}$$

Repeated use of this equation allows us to express all polynomials $V_{n,i,j}$ as linear combinations of the polynomials $V_{n,1,0}, V_{n,1,1}, V_{n,1,2}, \ldots, V_{n,1,n-2}$ as follows.

$$V_{n,r,s} = \frac{1}{2^{r-1}} \cdot \sum_{k=0}^{r-1} \binom{r-1}{k} \cdot V_{n,1,s+k} \quad \text{for } n > 1,\ r > 0,\ s \geq 0 \text{ and } r+s \leq n-1. \tag{3.3}$$

As a consequence of equation (3.2) we also have

$$V_{n,r+1,s} - V_{n,r,s} = \frac{1}{2} \cdot (V_{n,r,s+1} - V_{n,r,s}) \quad \text{for } n > 1,\ r > 0,\ s \geq 0 \text{ and } r+s \leq n-1. \tag{3.4}$$

Substituting equations (3.3) and (3.4) into equation (2.1), we obtain the following formula for the **cd**-index of an $(n-1)$-dimensional shellable cubical sphere $\mathcal{C}$.

$$\begin{aligned}
\Phi(P_1(\mathcal{C})) &= V_{n+1,1,0} + \sum_{\substack{r,s \\ r>0}} c_{r,s} \cdot (V_{n+1,r+1,s} - V_{n+1,r,s}) \\
&= V_{n+1,1,0} + \sum_{\substack{r,s \\ r>0}} c_{r,s} \cdot \frac{1}{2} \cdot (V_{n+1,r,s+1} - V_{n+1,r,s}) \\
&= V_{n+1,1,0} + \sum_{\substack{r,s \\ r>0}} c_{r,s} \cdot \left( \frac{1}{2^r} \cdot \sum_{k=0}^{r-1} \binom{r-1}{k} \cdot V_{n+1,1,s+k+1} - \frac{1}{2^r} \cdot \sum_{k=0}^{r-1} \binom{r-1}{k} \cdot V_{n+1,1,s+k} \right) \\
&= V_{n+1,1,0} + \sum_{l=1}^{n-1} \left( \sum_{\substack{r,s \\ r \geq \max(1,l-s)}} \frac{c_{r,s}}{2^r} \cdot \binom{r-1}{l-1-s} \right) \cdot (V_{n+1,1,l} - V_{n+1,1,l-1}).
\end{aligned}$$

Observe that by the proof of Stanley's theorem about the non-negativity of the coefficients of an $S$–shellable CW-sphere in [26], the polynomials $V_{n+1,1,l} - V_{n+1,1,l-1}$ have non-negative coefficients. Keeping in mind Stanley's [26, Theorem 3.1], our last equality suggests to define $h_0 \stackrel{\text{def}}{=} 1$ and

$$h_l \stackrel{\text{def}}{=} \sum_{\substack{r,s \\ r \geq \max(1,l-s)}} \frac{c_{r,s}}{2^r} \cdot \binom{r-1}{l-1-s} \quad \text{for } 1 \leq l \leq n-1 \tag{3.5}$$

to be the first $n$ entries of the cubical $h$-vector. This is equivalent to setting

$$\sum_{l=0}^{n-1} h_l \cdot x^l = 1 + \sum_{\substack{r,s \\ r>0}} \frac{c_{r,s}}{2^r} \cdot (1+x)^{r-1} \cdot x^{s+1},$$

which, by $c_{0,0} = c_{0,n-1} = 1$, is equivalent to

$$(1+x) \cdot \sum_{l=0}^{n-1} h_l \cdot x^l = 1 - x^n + x \cdot \sum_{r,s} \frac{c_{r,s}}{2^r} \cdot (1+x)^r \cdot x^s.$$

**Proposition 3.3** *Given a shellable cubical $(n-1)$-sphere $\mathcal{C}$ and the c-vector of one of its shellings, we have*

$$\sum_{r,s} \frac{c_{r,s}}{2^r} \cdot (1+x)^r \cdot x^s = \sum_{k=0}^{n-1} f_k \cdot x^k \left(\frac{1-x}{2}\right)^{n-1-k},$$

*and so the expression (3.5) independent of the choice of the shelling.*

**Proof:** Substituting $\dfrac{2 \cdot x}{1-x}$ for $x$ in the equation (3.1) we obtain

$$\begin{aligned}
\sum_{k=0}^{n-1} f_k \cdot \frac{2^k \cdot x^k}{(1-x)^k} &= \sum_{r,s} c_{r,s} \cdot \left(\frac{2 \cdot x}{1-x}\right)^s \cdot \left(\frac{2 \cdot x}{1-x} + 1\right)^r \cdot \left(\frac{2 \cdot x}{1-x} + 2\right)^{n-1-r-s} \\
&= \frac{2^{n-1}}{(1-x)^{n-1}} \sum_{r,s} \frac{c_{r,s}}{2^r} \cdot x^s \cdot (x+1)^r,
\end{aligned}$$

and hence the claim follows. □

**Definition 3.4** *Let $P$ be an Eulerian cubical poset of rank $n+1$, with $f$-vector $(f_{-1}, f_0, \ldots, f_n)$. We define the $h$-*vector *of $P$ by the following polynomial equation.*

$$\sum_{l=0}^{n} h_l \cdot x^l \stackrel{\text{def}}{=} \frac{1 + x^{n+1} + \sum_{k=0}^{n-1} f_k \cdot x^{k+1} \cdot \left(\frac{1-x}{2}\right)^{n-1-k}}{1+x}$$

Observe that the right hand side of the definition is a polynomial because of the Eulerian equation $\sum_{j=-1}^{n} (-1)^j \cdot f_j = 0$. By Proposition 3.3, for a shellable cubical $(n-1)$-sphere the $h_l$'s given by this definition satisfy $h_0 = 1$ and equation (3.5). In particular, the $h$-vector of a shellable cubical sphere is non-negative. We also added $h_n = 1$, and it follows from the cubical Dehn-Sommerville equations that for every Eulerian cubical poset of rank $n+1$, we have $h_i = h_{n-i}$ for all $i$. Using the Eulerian equation $\sum_{j=-1}^{n} (-1)^j \cdot f_j = 0$ it is also easy to confirm that whenever a cubical Eulerian poset $P$ of rank $n+1$ is a lattice, this $h$-vector is identical with the $1/2^{n-1}$-multiple of the *cubical $h$-vector* suggested by Adin in [2] for the $(n-1)$-dimensional cubical complex with face poset $P \setminus \{\widehat{1}\}$. Using this definition, for the case when $P \setminus \{\widehat{1}\}$ is the face poset of the boundary complex of an $n$-cube, we obtain that the $h$-vector is $(1, 1, \ldots, 1)$, in close analogy to the simplicial case. On the other hand, after the normalization, our $h$-vector does not necessarily have integer entries.

**Example 3.5** Let $P$ be the Eulerian cubical poset for which $P \setminus \{\widehat{1}\}$ is the face poset of the two-dimensional cubical sphere represented in Figure 1. (The facets of the complex are ABCD, ABFE,

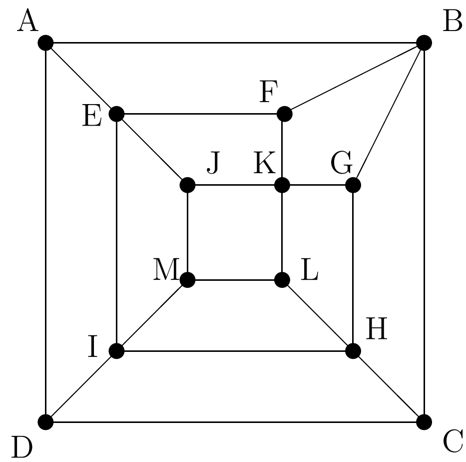


Figure 1: A two-dimensional cubical sphere

BGKF, BCHG, CDIH, AEID, EFKJ, GHLK, HIML, EJMI, and JKLM.) For this complex we have

$$f_{-1} = 1, \quad f_0 = 13, \quad f_1 = 22, \quad \text{and} \quad f_2 = 11.$$

Hence the cubical $h$-vector is given by

$$\begin{aligned} \sum_{l=0}^{3} h_l \cdot x^l &= \frac{1 + x^4 + \sum_{k=0}^{2} f_k \cdot x^{k+1} \cdot \left(\frac{1-x}{2}\right)^{2-k}}{1+x} = \frac{1 + x^4 + 13 \cdot x \cdot \left(\frac{1-x}{2}\right)^2 + 22 \cdot x^2 \cdot \left(\frac{1-x}{2}\right) + 11 \cdot x^3}{1+x} \\ &= \frac{x^4 + \frac{13}{4} \cdot x^3 + \frac{9}{2} \cdot x^2 + \frac{13}{4} \cdot x + 1}{1+x} = x^3 + \frac{9}{4} \cdot x^2 + \frac{9}{4} \cdot x + 1. \end{aligned}$$

Using the cubical $h$-vector, we have the following cubical analogue of Stanley's [26, Theorem 3.1].

**Theorem 3.6** *Let $P$ be an Eulerian cubical poset of rank $n+1$, with $h$-vector $(h_0, h_1, \ldots, h_n)$. Then we have*

$$\Phi(P) = h_0 \cdot V_{n+1,1,0} + \sum_{l=1}^{n-1} h_l \cdot (V_{n+1,1,l} - V_{n+1,1,l-1})$$

**Proof:** Our previous calculations in this section show that the theorem holds in the case when $P = P_1(\mathcal{C})$ where $\mathcal{C}$ is a shellable cubical sphere. Just like in the simplicial case, the **cd**-index of an Eulerian cubical poset depends linearly on the $f$-vector, and so also on the $h$-vector. It is sufficient to prove that the $h$-vectors of the shellable cubical $(n-1)$-spheres linearly span the vector space of all $h$-vectors of cubical posets of rank $n+1$, since then the result extends by linearity to all Eulerian cubical spheres.

This fact is the consequence of two results. On the one hand, Adin has proved that the $h$-vector of an Eulerian cubical poset of rank $n+1$ satisfies the equalities

$$h_i = h_{n-i} \quad \text{for } i = 1, \ldots, n.$$

This is (ii) of Theorem 5 in [2].[2] Hence the dimension of the vector space spanned by the flag $h$-vectors of Eulerian cubical posets of rank $n+1$ is at most $\lfloor \frac{n}{2} \rfloor + 1$. On the other hand, one may give $\lfloor \frac{n}{2} \rfloor + 1$ cubical polytopes of dimension $n$, whose $f$-vectors (and hence also their $h$-vectors) are linearly independent. See, for instance, Part 1 of section 9.4 in Grünbaum's book [16]. The face complex $\mathcal{C}$ of a cubical polytope is a cubical sphere, and it is shellable according to the famous result of Bruggesser and Mani [11]. □

As we have seen, the $h_i$'s do not have to be integers, yet when we multiply them with the **cd**-polynomials in Theorem 3.6, the sum must be a **cd**-polynomial with integer coefficients, since the **cd**-index of every Eulerian poset must be integral. Actually, for the **cd**-index of a cubical Eulerian poset even more can be said.

**Proposition 3.7** *Let $P$ be an Eulerian cubical poset of rank $n+1$. Let $w$ be a* **cd***-monomial of degree $n$, which contains $m$ factors equal to* **d***. If $m \geq 1$ then the coefficient of $w$ in the* **cd***-index $\Psi(P)$ is divisible by $2^{m-1}$.*

For zonotopes, i.e., polytopes which are Minkowski sums of line segments, a sharper results holds. Namely the coefficient of the **cd**-monomial $w$ is divisible by $2^m$ where $m$ is the number of **d**'s in $w$. This was first proven in [8]. A more explicit formula for the **cd**-index of zonotopes (and more generally, oriented matroids) appears in [9]. For more results on the parity of the $f$-vector of cubical polytopes see [4].

**Proof of Proposition 3.7:** Our proof will follow the proof in [8]. First observe that for a cubical poset $P$ and a non-empty subset $S$ of $\{1, \ldots, n\}$ we have that $2^{|S|-1}$ divides $f_S$. Indeed, to choose such a chain in the face lattice, first choose a face of rank $s$, where $s$ is the largest element in the set $S$. This face is a cube. The remainder of the chain will be chosen from this cube. But the cube is centrally symmetric, hence every time we choose an element the number of choices is divisible by 2. Thus the observation follows.

A subset $S$ is called *sparse* if it does not contain two consecutive integers. In [8, Section 6] Billera, Ehrenborg, and Readdy define the sparse $k$-vector by

$$k_S = \sum_{T \subseteq S} (-2)^{|S-T|} \cdot f_T,$$

for a sparse subset $S$ of $\{1, \ldots, n\}$. Let $w$ be a **cd**-monomial containing exactly $m$ factors equal to **d**. Then the coefficient of $w$ in the **cd**-index $\Psi(P)$ is an integer linear combination of $k_S$, where $S$ ranges over the sparse sets of cardinality $m$; see [8, Proposition 6.6]. A more explicit formula was given later [7].

Since $2^{|T|-1}$ divides $f_T$ we obtain that $2^{|S|-1}$ divides $k_S$. Now using [8, Proposition 6.6] the result follows. □

[2] Although Adin stated his theorem only for Eulerian cubical posets $P = P_1(\mathcal{C})$ obtained from (not necessarily shellable) cubical spheres $\mathcal{C}$, in his proof he only uses equalities holding for every Eulerian cubical poset. (He observes this fact in a note at the end of section 4 of [2].)

As noted earlier, the differences $V_{n+1,1,l} - V_{n+1,1,l-1}$ have non-negative coefficients. Hence the non-negativity of the $h$-vector of an Eulerian cubical poset $P$ implies the non-negativity of $\Phi(P)$. Adin asked whether the cubical $h$-vector of a Cohen-Macaulay cubical complex is non-negative ([2, Question 1]). Because of the analogy to the simplicial case [25, Theorem 3.10] we make the following conjecture.

**Conjecture 3.8** *The $h$-vector of every Gorenstein$^*$ cubical poset is non-negative.*

For the definition of Cohen-Macaulay and Gorenstein$^*$ posets we refer the reader to section 2 in [26]. This conjecture, if true, implies the special case of [26, Conjecture 2.1] when the poset is a cubical poset. The simplicial case is [26, Corollary 3.1].

# 4 The cd-index of semisuspended shelling components

In the remaining part of this paper we develop a cubical analogue of Stanley's conjecture [26, Conjecture 3.1] to describe the polynomials $V_{n,i,j}$ combinatorially. In the process, we will also obtain a new combinatorial description of the polynomials $U_{n,k}$. In this section we give a formula for $V_{n+2,i,j}$ and for $U_{n+2,k}$, using the chain weight calculation method.

Before starting, we wish to remind the reader of the following formulas; see [13, Section 3, equation (1) and Section 5, equation (3)] or [19, equations (5) and (16)].

**Proposition 4.1 (Purtill)** *We have*

$$U_{n+2} = \sum_{i=1}^{n} \binom{n}{i} \cdot U_i \cdot \mathbf{d} \cdot U_{n+1-i} + \mathbf{c} \cdot U_{n+1} \quad \text{for } n \geq 1,$$

*and*

$$V_{n+2} = \sum_{i=0}^{n-1} \binom{n}{i} \cdot 2^{n-i} \cdot V_{i+1} \cdot \mathbf{d} \cdot U_{n-i} + V_{n+1} \cdot \mathbf{c} \quad \text{for } n \geq 1.$$

Proposition 4.1 gives the recursion formulas for the special polynomials $U_{n,n-1} = U_n$ and $V_{n,1,n-2} = V_n$. In the following theorems these special cases will not be covered. The next result is analogous to [19, Proposition 6]. See also [14, Theorem 8.1].

**Theorem 4.2** *For $2 \leq k \leq n$ we have*

$$\begin{aligned} U_{n+2,k} &= \sum_{\substack{i \leq k-2 \\ j \leq n-k}} \binom{k-1}{i} \binom{n-k+1}{j} \cdot U_{i+j+1} \cdot \mathbf{d} \cdot U_{n-i-j,k-i-1} \\ &+ \sum_{i \leq k-2} \binom{k-1}{i} \cdot U_{n+2-k+i} \cdot \mathbf{d} \cdot U_{k-i+1} + U_{n+1} \cdot \mathbf{c}. \end{aligned}$$

**Proof:** We calculate the total weight of the chains in $B_{n+2,k} \setminus \{\widehat{0}, \widehat{1}\}$. We assume that $B_{n+2,k}$ was obtained by adding an extra coatom $E$ to the poset $\bigcup_{i=1}^{k} [\widehat{0}, \{1, 2, \dots, n+2\} \setminus \{i\}] \cup \{\widehat{1}\}$. Recall that $(x, z]$ denotes the half open interval $\{y \ : \ x < y \le z\}$.

Consider first those chains $c$ for which every element is either the coatom $E$ or a set not containing the element 1. All the elements of such chains belong to the set $(\widehat{0}, \{2, \dots, n+2\}] \cup \{E\}$. A proper subset $\sigma$ of $\{2, \dots, n+2\}$ is not less than $E$ if and only if $\sigma$ contains the set $\{k+1, \dots, n+2\}$. Let us provisorily add these missing $<$ relations between a proper subset of $\{2, \dots, n+2\}$ and $E$. The total weight of chains in $(\widehat{0}, \{2, \dots, n+2\}] \cup \{E\}$ with all such $<$ relations added is $U_{n+1} \cdot \mathbf{c}$. From this we need to subtract the total weight of all chains which use one of the provisorily added $<$ relations. These chains are exactly those which contain $E$ as their largest element, and a superset $\sigma$ of $\{k+1, \dots, n+2\}$. There are $\binom{k-1}{i}$ ways to choose such a $\sigma$ with $n+2-k+i$ elements, and the total weight of all chains to be subtracted is

$$\sum_{i=0}^{k-2} \binom{k-1}{i} \cdot U_{n+2-k+i} \cdot \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})^{k-i-2} \cdot \mathbf{b}.$$

Here the first $\mathbf{b}$ is produced by $\sigma$ and the second $\mathbf{b}$ by $E$. Hence the total weight of all chains belonging to this case is

$$U_{n+1} \cdot \mathbf{c} - \sum_{i=0}^{k-2} \binom{k-1}{i} \cdot U_{n+2-k+i} \cdot \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})^{k-i-2} \cdot \mathbf{b}. \tag{4.1}$$

From now on we assume that every chain considered contains a set $\lambda$ with $1 \in \lambda$.

Let us now compute the total weight of all chains $c$ which contain a set $\lambda$ with $1 \in \lambda$ and $n+2 \notin \lambda$. We may assume that $\lambda$ is the largest element of $c$ with respect to $n+2 \notin \lambda$. We denote the cardinality $|\lambda \cap \{2, \dots, k\}|$ by $i$ and the cardinality $|\lambda \cap \{k+1 \dots, n+1\}|$ by $j$. So the size of $\lambda$ is $i+j+1$, and thus the open interval $(\widehat{0}, \lambda)$ is isomorphic to $B_{i+j+1} \setminus \{\widehat{0}, \widehat{1}\}$. When $j \le n-k$ holds then all elements of $c$ above $\lambda$ belong to the half-closed interval $[\lambda \cup \{n+2\}, \widehat{1})$, which is isomorphic to $B_{n-i-j,k-i-1} \setminus \{\widehat{1}\}$. When $j = n-k+1$ then every element above $\lambda$ belongs either to the half-closed interval $[\lambda \cup \{n+2\}, \widehat{1})$ which is now isomorphic to $B_{k-i-1} \setminus \{\widehat{1}\}$, or the only element above $\lambda$ is $E$. Hence the total weight of all such chains is

$$\begin{aligned} &\sum_{\substack{i \le k-2 \\ j \le n-k}} \binom{k-1}{i} \binom{n-k+1}{j} \cdot U_{i+j+1} \cdot \mathbf{b} \cdot \mathbf{a} \cdot U_{n-i-j,k-i-1} \\ &+ \sum_{i=0}^{k-2} \binom{k-1}{i} \cdot U_{n+2-k+i} \cdot \mathbf{b} \cdot \mathbf{a} \cdot U_{k-i-1} + \sum_{i=0}^{k-2} \binom{k-1}{i} \cdot U_{n+2-k+i} \cdot \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})^{k-i-2} \cdot \mathbf{b}. \end{aligned} \tag{4.2}$$

The binomial coefficients in the above sum account for the number of ways of choosing the $i$ elements of $\lambda$ belonging to $\{2, \dots, k\}$ and the $j$ elements of $\lambda$ belonging to $\{k+1 \dots, n+1\}$. The first $\mathbf{b}$ in every summand is produced by $\lambda$, the subsequent factors $\mathbf{a} = (\mathbf{a} - \mathbf{b}) + \mathbf{b}$ in the first two sums are produced by the set $\lambda \cup \{n+2\}$ which may be included in a chain or not, independently of all other decisions. The third summand corresponds to the case when $j = n-k+1$ and the only element of the chain above $\lambda$ is $E$.

For all the remaining chains the smallest set $\lambda \in c$ with $1 \in \lambda$ contains $n+2$. We denote again $|\lambda \cap \{2, \dots, k\}|$ by $i$ and $|\lambda \cap \{k+1 \dots, n+1\}|$ by $j$. A similar but easier reasoning to the previous case shows that the total weight of these chains is

$$\begin{aligned}&\sum_{\substack{i\le k-2\\ j\le n-k}}\binom{k-1}{i}\binom{n-k+1}{j}\cdot U_{i+j+1}\cdot \mathbf{a}\cdot\mathbf{b}\cdot U_{n-i-j,k-i-1}\\ &+\sum_{i=0}^{k-2}\binom{k-1}{i}\cdot U_{n+2-k+i}\cdot\mathbf{a}\cdot\mathbf{b}\cdot U_{k-i-1}.\end{aligned} \tag{4.3}$$

Adding the weights (4.1), (4.2), and (4.3) we obtain the statement of the theorem. □

**Theorem 4.3** *For $1 \le i$ we have*

$$\begin{aligned}V_{n+2,i,j} &= \sum_{\substack{i_0,i_1,i_*,j_*,k_*\\ i_0+j-j_*>0\\ i_1+k-k_*>0}}\binom{i-1}{i_0\; i_1\; i_*}\binom{j}{j_*}\binom{k}{k_*}\cdot 2^{j-j_*+k-k_*}\cdot V_{i_*+j_*+k_*+1}\cdot\mathbf{d}\cdot U_{n-i_*-j_*-k_*,i_0+j-j_*}\\ &+\sum_{\substack{i_0,j_*\\ i_0+j-j_*>0}}\binom{i-1}{i_0}\binom{j}{j_*}\cdot 2^{j-j_*}\cdot V_{n+2-i_0+j_*-j}\cdot\mathbf{d}\cdot U_{i_0+j-j_*}+V_{n+1}\cdot\mathbf{c}.\end{aligned}$$

*where $k = n+1-i-j$.*

**Proof:** We use the notation introduced in Definition 2.1. Let us assume that we obtained $C_{n+2,i,j}$ from the collection of facets $A^0_1, A^0_2, \ldots, A^0_i, A^0_{i+1}, \ldots, A^0_{i+j}, A^1_{i+1}, A^1_{i+2}, \ldots, A^1_{i+j}$ of $\partial C^{n+1}$, by adding an extra facet $E$.

We apply equation (1.3) to compute $V_{n+2,i,j}$ by calculating the total weight of all strictly increasing chains of $C_{n+2,i,j} \setminus \{\widehat{0}, \widehat{1}\}$.

Consider first those chains $c$ which contain an element of the form $\tau = (1, u_2, \ldots, u_{n+1})$. We may assume that $\tau$ is the largest such face. Let us denote the number of 1's 0's, and $*$'s among $u_2, \ldots, u_i$ by $i_0, i_1$, and $i_*$ respectively. Let $j_*$ stand for the number of $*$'s among $u_{i+1}, \ldots, u_j$. Finally, let $k_*$ stand for the number of $*$'s among $u_{i+j+1}, u_{i+j+2}, \ldots, u_{n+1}$. Because $\tau$ is in our choice of $C_{n+2,i,j}$, we must have $i_0 + j - j_* > 0$. The elements of $c$ below $\tau$ form an arbitrary chain of the open interval $(\widehat{0}, \tau)$, which is isomorphic to $C_{i_*+j_*+k_*+1} \setminus \{\widehat{0}, \widehat{1}\}$. When $i_1 + k - k_* > 0$ then the elements of $c$ above $\tau$ form an arbitrary chain of the half-open interval $[(*, u_2, \ldots, u_{n+1}), \widehat{1})$ which is isomorphic to the interval $B_{n-i_*-j_*-k_*,i_0+j-j_*} \setminus \{\widehat{1}\}$. When $i_1 = k - k_* = 0$ then the elements of $c$ above $\tau$ form either an arbitrary chain of the half-open interval $[(*, u_2, \ldots, u_{n+1}), \widehat{1})$ which is now isomorphic to $B_{i_0+j-j_*} \setminus \{\widehat{1}\}$, or the only element above $\tau$ in $c$ is the extra facet $E$. Taking also into account the numbers of ways of choosing $\tau$ for fixed $i_0, i_1, i_*, j_*$, and $k_*$, we obtain that the total weight of all chains containing a face with $u_1 = 1$ is

$$\begin{aligned}
&\sum_{\substack{i_0,i_1,i_*,j_*,k_* \\ i_0+j-j_*>0 \\ i_1+k-k_*>0}} \binom{i-1}{i_0\ i_1\ i_*}\binom{j}{j_*}\binom{k}{k_*} \cdot 2^{j-j_*+k-k_*} \cdot V_{i_*+j_*+k_*+1} \cdot \mathbf{b} \cdot \mathbf{a} \cdot U_{n-i_*-j_*-k_*, i_0+j-j_*} \\
&+ \sum_{\substack{i_0,j_* \\ i_0+j-j_*>0}} \binom{i-1}{i_0}\binom{j}{j_*} \cdot 2^{j-j_*} \cdot V_{n+2-i_0+j_*-j} \cdot \mathbf{b} \cdot \mathbf{a} \cdot U_{i_0+j-j_*} \\
&+ \sum_{\substack{i_0,j_* \\ i_0+j-j_*>0}} \binom{i-1}{i_0}\binom{j}{j_*} \cdot 2^{j-j_*} \cdot V_{n+2-i_0+j_*-j} \cdot \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{i_0+j-j_*-1} \cdot \mathbf{b}.
\end{aligned} \tag{4.4}$$

Observe that the factor $V_{i_*+j_*+k_*+1}$ gives the correct contribution even when $i_* + j_* + k_* = 0$. In the first two summands the factors $\mathbf{b}$ are produced by the faces $\tau = (1, u_2, \ldots, u_{n+1})$, and the factors $\mathbf{a} = (\mathbf{a}-\mathbf{b}) + \mathbf{b}$ are produced by the faces $(*, u_2, \ldots, u_{n+1})$ which may be included in the chain or not, independently of all other decisions. The third summand corresponds to the case when $i_1 = k - k_* = 0$, and the only element of $c$ above $\tau$ is $E$.

We are left with those chains which contain only elements from the set $\{E\} \cup \{(u_1, \ldots, u_{n+1}) : u_1 = 0 \text{ or } *\}$. We divide these chains further into two subclasses.

Assume first that there is an element $\tau$ in the chain $c$, which is of the form $(*, u_2, \ldots, u_{n+1})$. We suppose that $\tau$ is the smallest in $c$ having this property. Then every element below $\tau$ in $c$ belongs to the open interval $(\widehat{0}, (0, u_2, \ldots, u_n))$. A very similar but somewhat simpler reasoning to the previous case yields that the total weight of all such chains is

$$\begin{aligned}
&\sum_{\substack{i_0,i_1,i_*,j_*,k_* \\ i_0+j-j_*>0 \\ i_1+k-k_*>0}} \binom{i-1}{i_0\ i_1\ i_*}\binom{j}{j_*}\binom{k}{k_*} \cdot 2^{j-j_*+k-k_*} \cdot V_{i_*+j_*+k_*+1} \cdot \mathbf{a} \cdot \mathbf{b} \cdot U_{n-i_*-j_*-k_*, i_0+j-j_*} \\
&+ \sum_{\substack{i_0,j_* \\ i_0+j-j_*>0}} \binom{i-1}{i_0}\binom{j}{j_*} \cdot 2^{j-j_*} \cdot V_{n+2-i_0+j_*-j} \cdot \mathbf{a} \cdot \mathbf{b} \cdot U_{i_0+j-j_*}.
\end{aligned} \tag{4.5}$$

The chains not taken into account in the sums (4.4) and (4.5) are exactly the chains contained in $(\widehat{0}, (0, *, *, \ldots, *)] \cup (\widehat{0}, E]$. Observe that exactly those elements $\sigma = (0, u_2, \ldots, u_{n+1})$ of the interval $(\widehat{0}, (0, *, *, \ldots, *))$ are not less than $E$ which satisfy

$$u_{i+j+1} = u_{i+j+2} = \cdots = u_{n+1} = * \quad \text{and} \quad 1 \notin \{u_1, \ldots, u_i\}. \tag{4.6}$$

Therefore, if we add all missing $<$ relations between $E$ and the proper faces of $(0, *, *, \ldots, *)$, we add chains of total weight

$$\sum_{\substack{i_0,j_* \\ i_0+j-j_*>0}} \binom{i-1}{i_0}\binom{j}{j_*} \cdot 2^{j-j_*} \cdot V_{n+2-i_0+j_*-j} \cdot \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{i_0+j-j_*-1} \cdot \mathbf{b}.$$

Here the first $\mathbf{b}$ is produced by the largest $\sigma$ with property (4.6) in every added chain, and the second $\mathbf{b}$ is produced by $E$. The total weight of the chains in the poset with the $<$ relations added is $V_{n+1} \cdot \mathbf{c}$, and so we obtain the contribution

$$V_{n+1} \cdot \mathbf{c} - \sum_{\substack{i_0,j_* \\ i_0+j-j_*>0}} \binom{i-1}{i_0}\binom{j}{j_*} \cdot 2^{j-j_*} \cdot V_{n+2-i_0+j_*-j} \cdot \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{i_0+j-j_*-1} \cdot \mathbf{b}. \tag{4.7}$$

in the last case.

Adding the sums (4.4), (4.5), and (4.7) we obtain the stated formula for $V_{n+2,i,j}$. □

It would be interesting to find a more compact recursion for the polynomials $V_{n,i,j}$. Could there be a recursion using derivations in analogy to the results in [14]?

## 5 Augmented André* signed permutations

Let $X$ be a finite (possibly empty) linearly ordered set with $m$ elements and linear order $\Lambda$. A *permutation* on $X$ is a list $(\tau_1, \ldots, \tau_m)$ such that every letter of $X$ occurs exactly once. We will say that $i \in \{2, \ldots, m\}$ is a *descent* of $\tau$ if we have $\tau_{i-1} > \tau_i$ (otherwise $i$ is an *ascent*). The *descent set* $D_\Lambda(\tau)$ of $\tau$ is the set

$$D_\Lambda(\tau) = \{i \ : \ \tau_{i-1} > \tau_i\}.$$

We say that $\tau$ has a *double descent* if there is an index $i$, where $2 \le i \le m-1$ such that $\tau$ has a descent at the $i$th and $(i+1)$st positions. In other words, both $i$ and $i+1$ belong to $D_\Lambda(\tau)$. Given a (possibly empty) subinterval $[i,j] \subseteq \{1, 2, \ldots, m\}$, we define the *restriction of $\tau$ to $[i,j]$* to be the permutation $\tau\Big|_{[i,j]} = (\tau_i, \tau_{i+1}, \ldots, \tau_j)$.

**Definition 5.1** *Let $X$ be a finite linearly ordered set with linear order $\Lambda$. A permutation $\tau = (\tau_1, \ldots, \tau_m)$ on $X$ is an* André* permutation *if it satisfies the following:*

1. *The permutation $\tau$ has no double descents.*
2. *For all $2 \le i < j \le m$, if $\tau_{i-1} = \max_\Lambda\{\tau_{i-1}, \tau_i, \tau_{j-1}, \tau_j\}$ and $\tau_j = \min_\Lambda\{\tau_{i-1}, \tau_i, \tau_{j-1}, \tau_j\}$, then there exists a $k$, with $i < k < j$ such that $\tau_{i-1} <_\Lambda \tau_k$.*

*We call an André* permutation* augmented *if its first letter is $\min_\Lambda X$. We denote the set of augmented André* permutations by $\mathcal{A}(X)$.*

Observe that we obtain the usual definition of André permutations (as it is given in [15] or in [22]) if we read the permutations backwards and reverse the linear order. This modified approach was first introduced by Ehrenborg and Readdy in [13], and will allow us to have a more comfortable description of the polynomials $V_{n,i,j}$.

In analogy with [22, Corollary 5.6] we have the following recursive description of augmented André* permutations.

**Proposition 5.2** *Let $X$ be a finite set with linear order $\Lambda$ and $|X| = n$. A permutation $\tau = (\tau_1, \ldots, \tau_n)$ on $X$ is an augmented André* permutation if and only if for $m \stackrel{\text{def}}{=} \tau^{-1}(\max_\Lambda X)$ the permutations $\tau\Big|_{[1,m-1]}$ and $\tau\Big|_{[m+1,n]}$ are augmented André* permutations and $\tau_1 = \min_\Lambda X$.*

Whenever $X$ has at least two elements, $\tau_1 = \min_\Lambda X$ guarantees $m > 1$, and the length of $\tau\Big|_{[1,m-1]}$ is less than the length of $\tau$. The recursion starts with observing that for $X = \emptyset$ or $|X| = 1$ every permutation of $X$ is an augmented André* permutation. The fact that the empty word is an augmented André* permutation is also used in the extreme case when $m = n$.

**Definition 5.3** *Let $N$ be a subset of $\mathbb{P}$ of cardinality $n$. Define $-N = \{-i \;:\; i \in N\}$. A* (non-augmented) signed permutation *$\sigma$ on the set $N$ is a list of the form*

$$(\sigma_1, \sigma_2, \ldots, \sigma_n)$$

*such that for all $i$, $\sigma_i \in N \cup -N$ and $(|\sigma_1|, |\sigma_2|, \ldots, |\sigma_n|)$ is a permutation on $N$. An* augmented signed permutation *$\sigma$ on $N$ is a list*

$$(0, \sigma_1, \sigma_2, \ldots, \sigma_n)$$

*such that $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a signed permutation on $N$. We will write $\sigma_0 = 0$.*

As in the signless case, we use the notation $\sigma\Big|_{[i,j]}$ to denote the restricted permutation $\sigma\Big|_{[i,j]} = (\sigma_i, \sigma_{i+1}, \ldots, \sigma_j)$.

Let $\Lambda$ be a linear order on the set $N \cup \{0\} \cup -N$. The *descent set* of a signed permutation $\sigma$ (augmented or non-augmented) with respect to $\Lambda$ is the set

$$D_\Lambda(\sigma) = \{i \;:\; \sigma_{i-1} >_\Lambda \sigma_i\}.$$

Here $D_\Lambda(\sigma)$ is a subset of $\{1, 2, \ldots, n\}$ for augmented permutations, and it is a subset of $\{2, \ldots, n\}$ for non-augmented permutations.

As before, we say that a signed permutation $\sigma$ as a *double descent* if there is an $i$ such that both $i$ and $i+1$ belong to the descent set $D_\Lambda(\sigma)$ of $\sigma$.

Assume from now on that $i >_\Lambda 0$ and $i >_\Lambda -i$ holds for all $i \in N$.

**Definition 5.4** *Let $N$ be a subset of the positive integers $\mathbb{P}$ of cardinality $n$. If $n = 0$ then $(0)$ is the only* augmented André* signed permutation *on $N$. If $n > 0$ then an augmented signed permutation $\sigma = (0 = \sigma_0, \sigma_1, \ldots, \sigma_n)$ on the set $N$ is an* augmented André* signed permutation *if the following three conditions are satisfied:*

1. *The permutation $\sigma$ has no double descents.*
2. *For all $1 \le i < j \le n$, if $\sigma_{i-1} = \max_\Lambda\{\sigma_{i-1}, \sigma_i, \sigma_{j-1}, \sigma_j\}$ and $\sigma_j = \min_\Lambda\{\sigma_{i-1}, \sigma_i, \sigma_{j-1}, \sigma_j\}$, then there exists a $k$, with $i < k < j$ such that $\sigma_{i-1} <_\Lambda \sigma_k$.*
3. *For $x = \max N$, there exists $1 \le m \le n$ such that $\sigma_m = x$ and that $\sigma\Big|_{[0,m-1]}$ is an augmented André* signed permutation on the set $J$, where $J = \{|\sigma_k| \;:\; 1 \le k \le m-1\}$.*

In particular, condition 3 implies that the letter $x = \max N$ appears with positive sign in an augmented André* signed permutation.

Observe that conditions 1 and 2 of Definition 5.4 are equivalent to the following

*1'. The permutation $(0 = \sigma_0, \sigma_1, \ldots, \sigma_n)$ is an André* permutation on the set $\{0 = \sigma_0, \sigma_1, \ldots, \sigma_n\}$ linearly ordered by the restriction of $\Lambda$.*

A non-augmented signed permutation satisfying conditions 1 and 2 in Definition 5.4 is called a *non-augmented André* signed permutation*. (For non-augmented permutations we need to rewrite condition 2 as: "For all $2 \leq i < j \leq n \ldots$".) Clearly, a signed permutation $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a non-augmented signed André* permutation, if and only if it is an André* permutation on the set $\{\sigma_1, \ldots, \sigma_n\}$ linearly ordered by the restriction of $\Lambda$. We denote the set of all augmented André* signed permutations on the set $N$ by $\mathcal{A}^{\pm}(N)$ and the set of all non-augmented André* signed permutations on the set $N$ by $\mathcal{N}^{\pm}(N)$. Furthermore, we denote the set of those non-augmented André* signed permutations which begin with their smallest element (with respect to the linear order $\Lambda$) by $\mathcal{N}_0^{\pm}(N)$. That is,

$$\mathcal{N}_0^{\pm}(N) \stackrel{\text{def}}{=} \{(\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathcal{N}^{\pm}(N) \ : \ \sigma_1 = \min_{\Lambda}\{\sigma_1, \sigma_2, \ldots, \sigma_n\}\}.$$

In particular, for $N = \emptyset$, the set $\mathcal{N}_0^{\pm}(\emptyset)$ consists only of the empty permutation.

**Example 5.5** Let $N \stackrel{\text{def}}{=} \{1, 2, \ldots, n\}$ and consider the linear order

$$-n <_{\Lambda} -n+1 <_{\Lambda} \cdots <_{\Lambda} -1 <_{\Lambda} 0 <_{\Lambda} 1 <_{\Lambda} \cdots <_{\Lambda} n-1 <_{\Lambda} n$$

on $-N \cup \{0\} \cup N$. Then $\mathcal{A}^{\pm}(N)$, $\mathcal{N}^{\pm}(N)$, and $\mathcal{N}_0^{\pm}(N)$ are the same as the similarly denoted sets of augmented (respectively non-augmented) **r**-signed André-permutations studied in [13] for $\mathbf{r} = (2, 2, \ldots, 2)$. The set $\mathcal{A}^{\pm}(N)$ may be obtained from Purtill's set of augmented signed André permutations defined in [22] by reversing the permutations and replacing each $k \in -N \cup N$ with $-k$. [3]

**Example 5.6** Let $N \stackrel{\text{def}}{=} \{1, 2, \ldots, n\}$ and consider the linear order

$$0 <_{\Lambda} -1 <_{\Lambda} 1 <_{\Lambda} -2 <_{\Lambda} 2 <_{\Lambda} \cdots <_{\Lambda} -n <_{\Lambda} n$$

on $-N \cup \{0\} \cup N$. Then $\mathcal{A}^{\pm}(N)$ and $\mathcal{N}^{\pm}(N)$ may be obtained from the corresponding sets of augmented (respectively non-augmented) signed André permutations defined in [19] on the set $\{1, 2, \ldots, n+1\}$ by reading each permutation backwards, and replacing each letter $k$ with $n+1-k$, while keeping its sign.

The following two lemmas describe how André* signed permutations behave under restriction.

**Lemma 5.7** *Let $\sigma = (0, \sigma_1, \ldots, \sigma_n)$ be an augmented André* signed permutation on an index set $N$ of cardinality $n$. Let $0 \leq j \leq n$, and let $J$ be the index set $\{|\sigma_k| \ : \ 1 \leq k \leq j\}$. Then the restriction*

[3] Note that the same transformation does not give an exact correspondence between the definitions of non-augmented André-permutations given in [13] and in [22].

$\sigma\Big|_{[0,j]}$ *is an augmented André* signed permutation on the index set* $J$*. Furthermore, let* $1 \le i \le j \le n$ *and let* $K$ *be the index set* $\{|\sigma_k| \,:\, i \le k \le j\}$*. Then the restriction* $\sigma\Big|_{[i,j]}$ *is a non-augmented André* signed permutation on the index set* $K$*.*

*Similarly, let* $\sigma = (\sigma_1, \ldots, \sigma_n)$ *be an non-augmented André* signed permutation on an index set* $N$ *of cardinality* $n$*. Let* $1 \le i \le j \le n$ *and let* $K$ *be the index set* $\{|\sigma_k| \,:\, i \le k \le j\}$*. Then restriction* $\sigma\Big|_{[i,j]}$ *is a non-augmented André* signed permutation on the index set* $K$*.*

The proof of Lemma 5.7 follows from the definitions. In particular, Lemma 5.7 gives for $j = 1$ the following:

**Corollary 5.8** *If* $\sigma = (0, \sigma_1, \ldots, \sigma_n)$ *is an augmented André* signed permutation, then* $0 <_\Lambda \sigma_1$*. In other words, every augmented André* signed permutation begins with an ascent.*

**Lemma 5.9** *Let* $\sigma = (0, \sigma_1, \ldots, \sigma_n)$ *be an augmented André* signed permutation on an index set* $N$ *of cardinality* $n$*. Assume that* $x = \max N$*, and* $\sigma_m = x$*. Let* $I$ *be the index set* $\{|\sigma_k| \,:\, m+1 \le k \le n\}$*. Then the restriction* $\sigma\Big|_{[m+1,n]} = (\sigma_{m+1}, \ldots, \sigma_n)$ *is an non-augmented André* signed permutation on the index set* $I$*. Moreover,* $\sigma\Big|_{[m+1,n]}$ *belongs to the set* $\mathcal{N}_0^{\pm}(I)$*.*

*Similarly, let* $\sigma = (\sigma_1, \ldots, \sigma_n)$ *be a non-augmented André* signed permutation on an index set* $N$ *of cardinality* $n$*. Assume that* $\sigma_m = \max_\Lambda\{\sigma_1, \ldots, \sigma_n\}$*. Let* $I$ *be the index set* $\{|\sigma_k| \,:\, m+1 \le k \le n\}$*. Then the restriction* $\sigma\Big|_{[m+1,n]} = (\sigma_{m+1}, \ldots, \sigma_n)$ *is an non-augmented André* signed permutation on the index set* $I$*. Moreover,* $\sigma\Big|_{[m+1,n]}$ *belongs to the set* $\mathcal{N}_0^{\pm}(I)$*.*

**Proof:** It is enough to consider the case when $\sigma$ is augmented. By Lemma 5.7, we know that $\sigma\Big|_{[m+1,n]} \in \mathcal{N}^{\pm}(I)$. To prove the additional requirement, we will proceed by proof by contradiction. Thus assume that $\sigma_{m+1} \neq \min_\Lambda\{\sigma_{m+1}, \ldots, \sigma_{n+1}\}$. If $\sigma_{m+1} >_\Lambda \sigma_{m+2}$ then $\sigma$ has a double descent, namely $\sigma_m >_\Lambda \sigma_{m+1} >_\Lambda \sigma_{m+2}$, which contradicts the fact that $\sigma$ is an augmented André* signed permutation. Thus we have $\sigma_{m+1} <_\Lambda \sigma_{m+2}$. Since $\sigma\Big|_{[m+1,n]} \notin \mathcal{N}_0^{\pm}(I)$, there exists an index $j$ which is greater than $m+1$ such that $\sigma_j <_\Lambda \sigma_{m+1}$. We may choose the smallest such $j$, thus obtaining the following inequalities: $\sigma_{j-1} >_\Lambda \sigma_{m+1} >_\Lambda \sigma_j$. Apply now condition 2 in Definition 5.4, with $i = m+1$. Thus there exist $k$ such that $m+1 < k < j$ and $\sigma_k >_\Lambda \sigma_m$. This would contradict that $\sigma_m = x$ is the largest element. Hence, we conclude that $\sigma_{m+1} = \min_\Lambda\{\sigma_{m+1}, \ldots, \sigma_n\}$, and $\sigma\Big|_{[m+1,n]} \in \mathcal{N}_0^{\pm}(I)$. □

We define the *variation* $U(\pi)$ of a signed or unsigned permutation $\pi$ as $U(\pi) = u_S$, where $S$ is the descent set of $\pi$ and $u_S$ is the **ab**-word defined in Section 1. In the case when $\pi$ contains no double descents (e.g., when $\pi$ is a signed or unsigned, augmented or non-augmented André* permutation), the *reduced variation* of $\pi$, which we denote by $V(\pi)$, is formed by replacing each **ab** in $U(\pi)$ with **d** and then replacing each remaining letter by a **c**. Given a set $\mathcal{P}$ of signed or unsigned permutations, we denote the sums $\sum_{\pi \in \mathcal{P}} U(\pi)$ and $\sum_{\pi \in \mathcal{P}} V(\pi)$ respectively by $U(\mathcal{P})$ and $V(\mathcal{P})$.

**Remark** The variation (respectively reduced variation) of an unsigned André* permutation $\tau$ is the reverse of the variation (respectively reduced variation) of the André permutation $\tau^{\text{rev}}$ obtained by reversing $\tau$ and the underlying linear order $\Lambda$. The reverse of a variation is obtained by writing all monomials involved in the reverse order.

Taking into account this remark, and the fact that reversing all words in a **cd**-index corresponds to taking the **cd**-index of the dual poset, we have the following variation of Purtill's [22, Theorem 6.1].

**Proposition 5.10** *$U_n = V(\mathcal{A}(\{1, 2, \ldots, n\}))$ holds for all $n \in \mathbb{P}$.*

Let $N$ be an $n$-element subset of $\mathbb{P}$. As noted after Definition 5.4, a non-augmented signed permutation $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a non-augmented André* permutation if and only if it is André* as an unsigned permutation on the set $\{\sigma_1, \ldots, \sigma_n\}$, linearly ordered by the restriction of $\Lambda$. Clearly $\sigma$ belongs to $\mathcal{N}_0^{\pm}(N)$ as well if and only if we have $\sigma \in \mathcal{A}(\{\sigma_1, \ldots, \sigma_n\})$. Thus from Proposition 5.10 and from the fact that there are $2^{|N|}$ ways to choose the signs of $\sigma_1, \sigma_2, \ldots,$ and $\sigma_n$, we have the following corollary.

**Corollary 5.11** *We have $V\left(\mathcal{N}_0^{\pm}(N)\right) = 2^{|N|} \cdot U_{|N|}$.*

**Remark** The reduced variation $V(\sigma)$ of an augmented André* signed permutation $\sigma$ on the set $N$ has the following recursive description. Assume that $N$ has cardinality $n$. If $\sigma_m = x$, where $x = \max N$, then

$$V(\sigma) = \begin{cases} V(\sigma\Big|_{[0,m-1]}) \cdot \mathbf{d} \cdot V(\sigma\Big|_{[m+1,n]}) & \text{if } m < n \\ V(\sigma\Big|_{[0,n-1]}) \cdot \mathbf{c} & \text{if } m = n, \end{cases}$$

with $V(0) = 1$. This recursion is well defined since $\sigma\Big|_{[m+1,n]}$ belongs to the set $\mathcal{N}_0^{\pm}([m+1, n])$ the reduced variation of which is described in Corollary 5.11.

The following description of the polynomials $U_{n,k}$ is analogous to Stanley's [26, Conjecture 3.1] proved in [19, Theorem 2].

**Theorem 5.12** *Let $\mathcal{A}_{n,k}$ denote the set $\{\tau \in \mathcal{A}(\{0, 1, \ldots, n-1\}) \,:\, \tau_n \in \{n-1, n-2, \ldots, n-k\}\}$. Then we have $U_{n,k} = V(\mathcal{A}_{n,k})$.*

**Proof:** Let us denote $\mathcal{A}(\{1, 2, \ldots, n-1\})$ by $\mathcal{A}_{n+2}$. It is sufficient to show that $V(\mathcal{A}_{n+2,k})$ satisfies the recursion formula given in Theorem 4.2 for $2 \leq k \leq n$. In fact, we may use the recursion formula to show our theorem by induction, where our induction basis is formed by the following two results:

(i) Proposition 5.10 which implies our statement for $U_{n,n-1} = U_n$ and $V(\mathcal{A}_{n,n-1}) = V(\mathcal{A}_n)$,

(ii) the relations $U_{n+1,1} = U_n \cdot \mathbf{c}$ and $V(\mathcal{A}_{n+1,1}) = V(\mathcal{A}_{n,1}) \cdot \mathbf{c}$ which may be easily seen directly.

We use the recursive description given in Proposition 5.2 to show the recursion formula for $V(\mathcal{A}_{n+2,k})$. Assume we are given a permutation $\tau = (0 = \tau_0, \tau_1, \dots, \tau_{n+1}) \in \mathcal{A}_{n+2,k}$. Let us count the letters which occur before the maximum letter $n+1$ in $\tau$: let $i$ denote the number of such letters from the set $\{n-k+2, n-k+3, \dots, n\}$, let $j$ count the number of such letters from the set $\{1, 2, \dots, n-k+1\}$. Then we have $\tau_{i+j} = n+1$. In the case when $i \leq k-2$ and $j \leq n-k$ holds, we have that $\tau$ belongs to $\mathcal{A}_{n+2,k}$ if and only if $(0 = \tau_0, \tau_1, \dots, \tau_{i+j})$ is an augmented André* permutation on $\{0 = \tau_0, \tau_1, \dots, \tau_{i+j}\}$ and $(\tau_{i+j+1}, \dots, \tau_{n+2})$ is an augmented André* permutation on $\{\tau_{i+j+1}, \dots, \tau_{n+2}\}$ ending with one of the $k-i-1$ largest letters. Thus the total reduced variation of such permutations is

$$\sum_{\substack{i \leq k-2 \\ j \leq n-k}} \binom{k-1}{i} \binom{n-k+1}{j} \cdot V(\mathcal{A}_{i+j+1}) \cdot \mathbf{d} \cdot V(\mathcal{A}_{n-i-j,k-i-1}). \tag{5.1}$$

When $i = k-1$ then $n+1$ must be the last letter, and the total reduced variation of such permutations is $V(\mathcal{A}_{n+1}) \cdot \mathbf{c}$. Finally, when $j = n-k+1$ and $i \leq k-2$ then there is no restriction on the ending letter of $(\tau_{i+j+1}, \dots, \tau_{n+2})$, and the total reduced variation of such permutations is

$$\sum_{i \leq k-2} \binom{k-1}{i} \cdot V(\mathcal{A}_{i+n-k+2}) \cdot \mathbf{d} \cdot V(\mathcal{A}_{k-i-1}). \tag{5.2}$$

Adding the reduced variations (5.1) and (5.2) to $V(\mathcal{A}_{n+1}) \cdot \mathbf{c}$ we obtain the desired expression for $V(\mathcal{A}_{n+2,k})$. □

**Remark** Observe that, in terms of "usual" André permutations, Theorem 5.12 expresses the polynomials $U_{n,k}$ as the reduced variation of augmented André permutations *starting* with given letters, while Stanley's [26, Conjecture 3.1] (shown in [19, Theorem 2]) partitions the augmented André permutations depending on their *second to last letter*.

Proposition 5.2 has the following signed analogue.

**Proposition 5.13** *There exists a bijection between the two sets*

$$\mathcal{A}^{\pm}([n+1]) \quad \textit{and} \quad \dot{\bigcup_{I+J=[n]}} \mathcal{A}^{\pm}(J) \times \mathcal{N}_0^{\pm}(I),$$

*where all the unions are disjoint and* $\times$ *is the Cartesian product.*

**Proof:** We break the augmented André* signed permutations at the point where the largest element $n+1$ occurs. By doing so, we have the following map by Lemma 5.7:

$$F : \mathcal{A}^{\pm}([n+1]) \longrightarrow \mathcal{A}^{\pm}([n]) \,\dot{\cup}\, \dot{\bigcup_{\substack{I+J=[n] \\ I \neq \emptyset}}} \mathcal{A}^{\pm}(J) \times \mathcal{N}_0^{\pm}(I).$$

This map is well defined because of Lemmas 5.7 and 5.9. To see that $F$ is bijective, it is enough to prove that $F$ has an inverse. Given $\sigma' = (0, \sigma_1, \dots, \sigma_{m-1}) \in \mathcal{A}^{\pm}(J)$ and $\sigma'' = (\sigma_{m+1}, \dots, \sigma_{n+1}) \in \mathcal{N}_0^{\pm}(I)$, let $\sigma = (\sigma', n+1, \sigma'')$. It is easy to see that $\sigma$ satisfies conditions 1 and 3 in Definition 5.4. To show that $\sigma$ also satisfies condition 2, it is enough to consider the following two cases. First, when $i < m$ and $m < j$, let $k = m$ in condition 2. The remaining case is when $i - 1 = m$ and $m < j$. However, this case will not occur since $\sigma''$ belongs to $\mathcal{N}_0^{\pm}(I)$. Hence $\sigma$ is an augmented André* permutation, and lies in the set $\mathcal{A}^{\pm}([n+1])$. Thus we conclude that $F$ is a bijection. □

**Theorem 5.14** *We have $V_n = V(\mathcal{A}^{\pm}([n]))$.*

**Proof:** It is enough to show that $V(\mathcal{A}^{\pm}([n]))$ satisfies the same recurrence as the one given for $V_n$ in Proposition 4.1. This formula will follow by the bijection given in Proposition 5.13. Summing the reduced variation over $\mathcal{A}^{\pm}([n+1])$ we find

$$V\left(\mathcal{A}^{\pm}([n+1])\right) = V\left(\mathcal{A}^{\pm}([n])\right)\cdot\mathbf{c} + \sum_{\substack{I+J=[n]\\ I\neq\emptyset}} V\left(\mathcal{A}^{\pm}(J)\right)\cdot\mathbf{d}\cdot V\left(\mathcal{N}_0^{\pm}(I)\right). \qquad \square$$

Finally, we arrived at a description of the polynomials $V_{n,i,j}$ in terms of reduced variation of signed augmented André* permutations.

**Theorem 5.15** *Let $N \subset \mathbb{P}$ be an $n$-element set and $\Lambda$ a linear order on $N \cup -N \cup \{0\}$ such that $0 <_\Lambda i$ and $-i <_\Lambda i$ for all $i \in N$. Assume that $A$ and $B$ are disjoint subsets of $N$ such that $A \cup B \cup -B$ is an upper segment in $N \cup -N$, and all the elements of $A$ are larger than the elements of $B \cup -B$ with respect to $\Lambda$. Let us denote $|A|$ by $i$ and $|B|$ by $j$, where we assume $i > 0$ or $j = n$. Then $V_{n+1,i,j}$ is the total reduced variation of all those signed augmented André*-permutations with respect to $\Lambda$ which end with a letter from $A \cup B \cup -B$.*

**Proof:** Observe first that in the case when $j = n$, we must have $i = 0$, $V_{n+1,0,n} = V_n$ and our theorem reduces to Theorem 5.14. Thus we may assume $i > 0$. For $i = 1$ and $j = 0$ we have $n \geq 2$ and $V_{n+1,1,0} = V_n \cdot \mathbf{c}$, hence the statement is again an easy consequence of Theorem 5.14. Thus we only need to consider the case when $i > 0$ and $i + j \geq 2$, and so $n \geq 2$. In order to get a better match with the notation of Theorem 4.3, we denote from now on $|N| - 1$ by $n$. Now it is sufficient to show that the reduced variation polynomial $V\left(\{\sigma = (0 = \sigma_0, \sigma_1, \ldots, \sigma_{n+1}) \,:\, \sigma \in \mathcal{A}^{\pm}(N),\, \sigma_{n+1} \in A \cup B \cup -B\}\right)$ satisfies the formula given in Theorem 4.3.

Suppose we are given an augmented signed André* permutation $\sigma = (0 = \sigma_0, \sigma_1, \ldots, \sigma_{n+1})$ with $\sigma_{n+1} \in A \cup B \cup -B$. Assume $i_*$ letters of $A$, $j_*$ letters of $B$ and $k_*$ letters of $C \stackrel{\text{def}}{=} N \setminus (A \cup B)$ occur before the letter $n+1$. Among the letters after $n+1$ there are $i_0$ belonging to $A$ and having positive sign, and $i_1$ letters from $A$ occur with a negative sign. Thus we have $i = i_0 + i_1 + i_* + 1$. There are $j - j_*$ letters of $B$ and $n + 1 - i - j - k_* = k - k_*$ letters of $N \setminus (A \cup B)$ after $n + 1 = \sigma_{i_*+j_*+k_*}$. Let us denote the intersection of $\{|\sigma_0|, |\sigma_1| \ldots, |\sigma_{n+1}|\} \setminus \{|\sigma_0|, |\sigma_1| \ldots, |\sigma_{i_*+j_*+k_*}|\}$ with $A$, $B$, and $C$ respectively by $A_1$, $B_1$, and $C_1$ respectively. Clearly we have $|B_1| = j - j_*$ and $|C_1| = i_1 + k - k_*$. Observe that $A_1 \cup B_1 \cup -B_1$ is an upper segment in $A_1 \cup B_1 \cup C_1 \cup -A_1 \cup -B_1 \cup -C_1$ with respect to the restriction of $\Lambda$.

Consider first those $\sigma$'s for which both $i_0+j-j_*$ and $i_1+k-k_*$ are positive. By Proposition 5.13, $\sigma$ is an augmented signed André* permutation, ending with a letter from $A \cup B \cup -B$ if and only if we have $(0 = \sigma_0, \sigma_1, \ldots, \sigma_{i_*+j_*+k_*-1}) \in \mathcal{A}^{\pm}(\{|\sigma_1|, \ldots, |\sigma_{i_*+j_*+k_*}|\})$ and $(\sigma_{i_*+j_*+k_*+1}, \ldots, \sigma_{n+1}) \in \mathcal{N}_0^{\pm}(\{|\sigma_{i_*+j_*+k_*+1}|, \ldots, |\sigma_{n+1}|)$ where $\sigma_{n+1}$ must belong to $A_1 \cup B_1 \cup -B_1$. We observed in the proof of Corollary 5.11 that $(\sigma_{i_*+j_*+k_*+1}, \ldots, \sigma_{n+1}) \in \mathcal{N}_0^{\pm}(\{|\sigma_{i_*+j_*+k_*+1}|, \ldots, |\sigma_{n+1}|)$ is equivalent to $(\sigma_{i_*+j_*+k_*+1}, \ldots, \sigma_{n+1}) \in \mathcal{A}(\{\sigma_{i_*+j_*+k_*+1}, \ldots, \sigma_{n+1})$. Thus by the Theorems 5.14 and 5.12 the total reduced variation of all such $\sigma$'s is

$$\sum_{\substack{i_0,i_1,i_*,j_*,k_* \\ i_0+j-j_*>0 \\ i_1+k-k_*>0}} \binom{i-1}{i_0\ i_1\ i_*} \binom{j}{j_*} \binom{k}{k_*} \cdot 2^{j-j_*+k-k_*} \cdot V_{i_*+j_*+k_*+1} \cdot \mathbf{d} \cdot U_{n-i_*-j_*-k_*, i_0+j-j_*} \tag{5.3}$$

Observe that the factor $2^{j-j_*+k-k_*}$ comes from the number of possible signings of the set $B_1 \cup C_1$.

Consider next the case $i_0 + j - j_* = 0$. Then $n+1$ must be the last letter, i.e., we must have $i_* + j_* + k_* = n+1$. The contribution of all such $\sigma$'s is $V_{n+1} \cdot \mathbf{c}$ by Theorem 5.14.

Assume finally that $i_0 + j - j_*$ is positive but $i_1 + k - k_*$ is zero. Then there is no restriction of the last letter of $\sigma$, and the contribution of all such $\sigma$'s is

$$\sum_{\substack{i_0,j_* \\ i_0+j-j_*>0}} \binom{i-1}{i_0} \binom{j}{j_*} \cdot 2^{j-j_*} \cdot V_{n+2-i_0+j_*-j} \cdot \mathbf{d} \cdot U_{i_0+j-j_*} \tag{5.4}$$

Adding the sums (5.3) and (5.4) to $V_{n+1} \cdot \mathbf{c}$ we obtain the desired formula for $V_{n+2,i,j}$. □

## References


[1] R. M. Adin, *A new cubical h-vector,* Conference Proceedings at the 6th International Conference on Formal Power Series and Algebraic Combinatorics, DIMACS/Rutgers 1994.

[2] R. M. Adin, *A new cubical h-vector*, Disc. Math. 157 (1996), pp. 3–14.

[3] E. K. Babson, L. J. Billera, and C. S. Chan, *Neighborly Cubics and a Cubical Lower Bound Conjecture*, Israel J. Math. 102 (1997), pp. 297–315.

[4] E. K. Babson and C. S. Chan, *Counting faces of cubical spheres modulo two,* to appear in Disc. Math.

[5] M. Bayer and L. Billera, *Generalized Dehn-Sommerville relations for polytopes, spheres and Eulerian partially ordered sets*, Invent. Math. 79 (1985), pp. 143–157.

[6] M. Bayer and A. Klapper, *A new index for polytopes*, Discrete Comput. Geom. 6 (1991), pp. 33–47.

[7] L. J. Billera and R. Ehrenborg, *Monotonicity of the* **cd**-*index for polytopes,* to appear in Math. Z.

[8] L. J. Billera, R. Ehrenborg, and M. Readdy, The **cd**-index of zonotopes and arrangements, *Mathematical essays in honor of Gian-Carlo Rota* (B. E. Sagan and R. P. Stanley, eds.), Birkhäuser, Boston, 1998, pp. 23–40.

[9] L. J. Billera, R. Ehrenborg, and M. Readdy, *The* **c**-2**d**-*index of oriented matroids*, J. Combin. Theory Ser. A 80 (1997), pp. 79–105.

[10] A. Björner, *Posets, regular CW-complexes and Bruhat order*, European J. Combin. 5 (1984), pp. 7–16.

[11] H. Bruggesser and P. Mani, *Shellable decompositions of cells and spheres*, Math. Scand. 29 (1971), pp. 197–205.

[12] C. Chan, *Plane Trees and H-Vectors of Shellable Cubical Complexes*, SIAM J. Disc. Math. 4 (1991), pp. 568–574.

[13] R. Ehrenborg and M. Readdy, *The* **r**-*cubical lattice and a generalization of the* **cd**-*index*, European J. Combin. 17 (1996), pp. 709–725.

[14] R. Ehrenborg and M. Readdy, *Coproducts and the* **cd**-*index*, J. Algebraic Combin. 8 (1998), pp. 273–299.

[15] D. Foata and M. P. Schützenberger, *Nombres d'Euler et permutations alternantes,* in: J.N. Srivastava et al., "A Survey of Combinatorial Theory," Amsterdam, North–Holland, 1973, pp. 173–187.

[16] B. Grünbaum, "*Convex Polytopes,"* John Wiley & Sons, Interscience Division, 1967.

[17] G. Hetyei, *"Simplicial and Cubical Complexes: Analogies and Differences,"* Doctoral dissertation, Massachusetts Institute of Technology, 1994.

[18] G. Hetyei, *On the Stanley ring of a cubical complex*, Discrete Comput. Geom. 14 (1995), pp. 305–330.

[19] G. Hetyei, *On the cd-variation polynomials of André and simsun permutations*, Discrete Comput. Geom. 16 (1996), pp. 259–276.

[20] G. Hetyei, *Invariants des complexes cubiques*, Ann. Sci. Math. Québec 20 (1996), pp. 35–52.

[21] N. Metropolis and G.-C. Rota, *Combinatorial structure of the faces of the $n$-cube*, SIAM J. Appl. Math. 35 (1978), pp. 689–694.

[22] M. Purtill, *André permutations, lexicographic shellability and the cd-index of a convex polytope*, Trans. Amer. Math. Soc. 338 (1993), pp. 77–104.

[23] R. Stanley, "*Enumerative Combinatorics, Vol. I,"* Wadsworth and Brooks/Cole, Pacific Grove, 1986.

[24] R. Stanley, *Generalized h-vectors, intersection cohomology of toric varieties, and related results,* in: "Commutative Algebra and Combinatorics," M. Nagata, H. Matsumura, eds., Advanced Studies in Pure Math. 11, North-Holland, New York, 1987, pp. 187–213.

[25] R. Stanley, *$f$-vectors and $h$-vectors of simplicial posets*, J. Pure Appl. Alg. 71 (1991), pp. 319–331.

[26] R. Stanley, *Flag $f$-vectors and the cd-index*, Math. Z. 216 (1994), pp. 483–499.

[27] R. Stanley, A survey of Eulerian posets, in: "Polytopes: Abstract, Convex, and Computational," T. Bisztriczky, P. McMullen, R. Schneider, A. I. Weiss, eds., NATO ASI Series C, vol. 440, Kluwer Academic Publishers, 1994.