Order **18:** 227–236, 2001.

k-Eulerian Posets

RICHARD EHRENBORG
Department of Mathematics, University of Kentucky, Lexington, KY 40506-0027, U.S.A.
E-mail: jrge@ms.uky.edu

(Received: 19 Sepember 2000; accepted 19 June 2001)

Abstract. A poset P is called k-Eulerian if every interval of rank k is Eulerian. The class of k-Eulerian posets interpolates between graded posets and Eulerian posets. It is a straightforward observation that a $2k$-Eulerian poset is also $(2k+1)$-Eulerian. We prove that the **ab**-index of a $(2k+1)$-Eulerian poset can be expressed in terms of $\mathbf{c} = \mathbf{a} + \mathbf{b}$, $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$ and $\mathbf{e}^{2k+1} = (\mathbf{a} - \mathbf{b})^{2k+1}$. The proof relies upon the algebraic approaches of Billera–Liu and Ehrenborg–Readdy. We extend the Billera–Liu flag algebra to a Newtonian coalgebra. This flag Newtonian coalgebra forms a Laplace pairing with the Newtonian coalgebra $\mathbf{k}\langle\mathbf{a}, \mathbf{b}\rangle$ studied by Ehrenborg–Readdy. The ideal of flag operators that vanish on $(2k + 1)$-Eulerian posets is also a coideal. Hence, the Laplace pairing implies that the dual of the coideal is the desired subalgebra of $\mathbf{k}\langle\mathbf{a}, \mathbf{b}\rangle$. As a corollary we obtain a proof of the existence of the **cd**-index which does not use induction.

Key words: **cd**-index, flag operators.

1. Introduction

A partially ordered set (poset) is *Eulerian* if every interval contains the same number of elements of even rank as of odd rank. Another formulation of the Eulerian property is every interval satisfies the Euler–Poincaré formula. The classical example of an Eulerian poset is the face lattice of a convex polytope. In the combinatorial study of polytopes there is great interest to understand the number of faces of different dimensions. More generally, a large open problem is to classify the flag f-vector of face lattices of convex polytopes, that is, the vector that enumerates chains (flags) of elements of the partially ordered set.

For the class of graded posets, there are no linear relations holding between the entries of the flag f-vector. However, for Eulerian posets there are linear relations. They are called the generalized Dehn-Sommerville relations and were discovered by Bayer and Billera [2]. Later Bayer and Klapper [5] proved that the flag f-vector information of an Eulerian poset can be efficiently encoded by a noncommutative polynomial called the **cd**-index. The **cd**-index has stirred a lot of interest in the field, since it yields itself to computations [3, 6, 7, 10, 11]. Moreover, inequalities have been proven for the coefficients of the **cd**-index for various classes of Eulerian posets [6, 15].

The proofs presented so far for the existence of the **cd**-index are *ad hoc*. They use that certain **ab**-polynomials can be written in terms of the variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$

and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. These proofs offer no reason why these **ab**-polynomials are **cd**-polynomials, that is, they do not explain why the **cd**-index is a polynomial in the variables **c** and **d**. Up to this point, the algebraic structure behind the scenes has not been explored in full.

Two articles have addressed the underlying algebraic structure of the **cd**-index. First, Billera and Liu [9] studied the generalized Dehn–Sommerville relations and were able to view them as an ideal in the algebra of flag operators. Secondly, Ehrenborg and Readdy [11] discovered the underlying coalgebra structure of the **cd**-index. This is the main tool for computations involving the **cd**-index; see [3, 6, 7, 10].

In the current paper we will join both of these approaches. By extending the flag algebra to a Newtonian coalgebra, the right algebraic setting emerges. The duality between the Billera–Liu and the Ehrenborg–Readdy approaches is formalized as a Laplace pairing. The fact the ideal encoding the generalized Dehn–Sommerville relations also is a coideal yields that the flag f-vectors of Eulerian posets form a subalgebra. This is the algebra generated by the noncommuting variables **c** and **d**, and hence gives a natural explanation of the algebraic structure of the **cd**-index.

We use these ideas to study posets with the property that every interval of rank k or less satisfies the Euler–Poincaré relation. We call such a poset *k-Eulerian*. This new class of posets can be seen as interpolating between the class of graded posets and the class of Eulerian posets. The interesting case to study is $(2k+1)$-Eulerian posets because a $2k$-Eulerian poset is necessarily $(2k+1)$-Eulerian. Since the linear relations that $(2k+1)$-Eulerian posets satisfy form a coideal, we directly obtain that flag f-vectors of $(2k+1)$-Eulerian posets form an algebra. Comparing the Hilbert series of this algebra with the algebra $\mathbf{k}\langle \mathbf{c}, \mathbf{d}, \mathbf{e}^{2k+1}\rangle$, we conclude that these two algebras are identical. Hence the flag f-vector of a $(2k+1)$-Eulerian posets can be written as a polynomial in the variables **c**, **d** and $\mathbf{e}^{2k+1}$. From the case $k = \infty$, we obtain as an immediate corollary a new proof of the existence of the **cd**-index for Eulerian posets.

This paper is organized as follows. In the next section we introduce flag vectors and the **cd**-index. In Section 3 we review the necessary algebraic tools. In Section 4 we develop the algebraic framework and study k-Eulerian posets. We end the paper with concluding remarks about other classes of posets that may fit in this algebraic setting.

2. Eulerian Posets and Flag Vectors

A partially ordered set (poset) P is *graded* if it has a unique minimal element $\hat{0}$, a unique maximal element $\hat{1}$ and there is rank function $\rho\colon P \to \mathbb{N}$ such that $\rho(\hat{0}) = 0$ and $\rho(y) = p(x)+1$ if the element y covers x. For $x \leq z$ the interval $[x, z]$ is the set $\{y\colon x \leq y \leq z\}$. The *rank* of an interval $[x, y]$ is $\rho(y) - \rho(x) = \rho(x, y)$. A graded poset is *Eulerian* if every interval of rank greater than or equal to one in the poset satisfies the Euler–Poincaré relation, that is, the number of elements of even rank equals the number of elements of odd rank. Equivalently, a poset is Eulerian if the Möbius function satisfies $\mu(x, y) = (-1)^{\rho(x,y)}$ for every interval $[x, y]$. The motivating example of Eulerian posets is face lattices of convex polytopes.

Let P be a poset of rank $n + 1$. For S a subset of $\{1, \ldots, n\}$ define $f_S(P)$ to be the number of chains (flags) in the poset having elements with ranks given by the set S. That is,

$$f_S(P) = \left|\left\{\left(\hat{0} = x_0 < x_1 < \cdots < x_{k+1} = \hat{1}\right): \left\{\rho(x_1), \ldots, \rho(x_k)\right\} = S\right\}\right|.$$

The 2^n values $f_S(P)$ constitute the *flag f-vector* of the poset P.

For S a subset $\{1, \ldots, n\}$ define the **ab**-polynomial v_S by $v_S = v_1 \cdots v_n$, where $v_i = \mathbf{b}$ if $i \in S$ and $v_i = \mathbf{a} - \mathbf{b}$ if $i \notin S$. The **ab**-*index* of a graded poset P of rank $n + 1$ is the noncommutative polynomial

$$\Psi(P) = \sum_S f_S(P) \cdot v_S,$$

where the sum ranges over all subsets S of $\{1, \ldots, n\}$.

When the poset P is Eulerian there are linear relations between the entries of the flag f-vector, namely the generalized Dehn–Sommerville relations [2]. Bayer and Billera also showed that the dimension of the subspace that satisfies the generalized Dehn–Sommerville relations is F_n, the nth Fibonacci number ($\sum_{n \geq 0} F_n \cdot t^n = 1/(1 - t - t^2)$). A different approach toward understanding the flag f-vector was taken by Fine, Bayer and Klapper. The following theorem was conjectured by Fine and proved by Bayer and Klapper [5].

THEOREM 2.1 (Bayer–Klapper). *The* **ab**-*index of an Eulerian poset can be written in terms of* $\mathbf{c} = \mathbf{a} + \mathbf{b}$ *and* $\mathbf{d} = \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a}$.

When $\Psi(P)$ is written in terms of **c** and **d** it is called the **cd**-*index*. The existence of the **cd**-index is equivalent to the generalized Dehn–Somerville relations. Hence the **cd**-index of an Eulerian poset encodes the flag f-vector without any redundancies. Another way to see this is that the **cd**-monomials form a basis for the subspace satisfying the generalized Dehn–Somerville relations. A different proof of this result was given by Stanley [15]. An improved version of this proof due to Hetyei appears in [6, Section 3].

Let P and Q be two posets. Define the star product $P * Q$ to be the poset on the set $(P - \{\hat{1}\}) \cup (Q - \{\hat{0}\})$. The order relation is given by $x \leq_{P*Q} y$ if one of the following three conditions is satisfied: (i) $x, y \in P$ and $x \leq_P y$, (ii) $x, y \in Q$ and $x \leq_Q y$, (iii) $x \in P$ and $y \in Q$. Stanley observed that the **ab**-index of $P * Q$ is the product of the respective **ab**-indices, that is, $\Psi(P * Q) = \Psi(P) \cdot \Psi(Q)$; see [15, Lemma 1.1].

3. Newtonian Coalgebras and Laplace Correspondence

We begin by recalling some notation. Let **k** be a field and A a vector space over **k**. A *product* on the vector space A is a linear map $\mu\colon A \otimes A \to A$. The product μ

is associative if $\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu)$. Similarly, a *coproduct* on the vector space A is a linear map $\Delta \colon A \to A \otimes A$. The coproduct Δ is coassociative if $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$.

DEFINITION 3.1. Let A be a vector space with an associative product μ, and a coassociative coproduct Δ. We call the triplet (A, μ, Δ) a *Newtonian coalgebra* if it satisfies the identity

$$\Delta \circ \mu = (\mathrm{id} \otimes \mu) \circ (\Delta \otimes \mathrm{id}) + (\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta).$$

The definition of Newtonian coalgebra originated from Joni and Rota [13] under the name infinitesimal coalgebra. The first major example was the Newtonian coalgebra of polynomials. This was mentioned in [13] and studied in-depth by Hirschhorn and Raphael [12], who used it to study the algebra of divided differences. Ehrenborg and Readdy [11] discovered the **ab**-index may be viewed as a Newtonian coalgebra homomorphism between the Newtonian coalgebra spanned by graded posets and the Newtonian coalgebra $\mathbf{k}\langle \mathbf{a}, \mathbf{b} \rangle$. In the case of Eulerian posets, their result implies the **cd**-index is a Newtonian coalgebra homomorphism from the Newtonian coalgebra spanned by Eulerian posets to $\mathbf{k}\langle \mathbf{c}, \mathbf{d} \rangle$.

Let $*$ denote the product $(x_1 \otimes \cdots \otimes x_m) * (y_1 \otimes \cdots \otimes y_n) = x_1 \otimes \cdots \otimes x_m \cdot y_1 \otimes \cdots \otimes y_n$. The Newtonian condition can be reformulated as $\Delta(x \cdot y) = \Delta(x) * y + x * \Delta(y)$, that is, the coproduct behaves like a derivation.

A *subalgebra* S of a Newtonian coalgebra A is a linearly closed subset of A such that $S \cdot S \subseteq S$ and $\Delta(S) \subseteq S \otimes S$. An ideal I of a Newtonian coalgebra A is a linearly closed subset such that $A \cdot I, I \cdot A \subseteq A$ and $\Delta(I) \subseteq A \otimes I + I \otimes A$. The Newtonian coalgebra A is graded of *type* (p, q) if $A = \bigoplus_{n \geq 0} A_n$ and the following inclusions hold: $A_i \cdot A_j \subseteq A_{i+j+p}$ and $\Delta(A_n) \subseteq \bigoplus_{i+j=n+q} A_i \otimes A_j$. Similarly, one defines subalgebras and ideals to be graded. Observe that if I is a graded ideal of A then the quotient A/I is also a graded Newtonian coalgebra.

DEFINITION 3.2. Let A and B be two graded Newtonian coalgebras over the field $\mathbf{k}$. A graded *Laplace pairing* between A and B is a bilinear form $\langle \cdot \mid \cdot \rangle \colon B \times A \to \mathbf{k}$ such that $\langle B_m \mid A_n \rangle = 0$ for $m \neq n$, the restriction $\langle \cdot \mid \cdot \rangle \colon B_n \times A_n \to \mathbf{k}$ is non-degenerate, and the following two identities hold:

$$\langle x \cdot y \mid u \rangle = \sum_u \langle x \mid u_{(1)} \rangle \cdot \langle y \mid u_{(2)} \rangle, \tag{3.1}$$

$$\langle x \mid u \cdot v \rangle = \sum_x \langle x_{(1)} \mid u \rangle \cdot \langle x_{(2)} \mid v \rangle, \tag{3.2}$$

where $x, y \in B$ and $u, v \in A$.

Observe if B is graded of type (p, q) then the type of A is $(-q, -p)$. Moreover, $\dim(B_n) = \dim(A_n)$ for all non-negative integers n.

Let $\langle \cdot \mid \cdot \rangle : B \times A \to \mathbf{k}$ be a Laplace pairing and V a graded subspace of B. That is, we can write $V = \bigoplus_{n \geq 0} V_n$, where $V_n \subseteq B_n$. Let $V_n^{\perp}$ be the subspace of A_n given by $V_n^{\perp} = \{u \in A_n : \forall x \in V_n \ \langle x \mid u \rangle = 0\}$. Let $V^{\perp}$ be the direct sum $\bigoplus_{n \geq 0} V^{\perp}$. The following proposition is straightforward and thus the proof is omitted.

PROPOSITION 3.3. *Let A and B be a graded Laplace pairing. If S is a graded subalgebra of B then $S^{\perp}$ is a graded ideal of A. If I is a graded ideal of B then $I^{\perp}$ is a graded subalgebra of A. Moreover, $I^{\perp}$ and the quotient Newtonian algebra B/I form a graded Laplace pairing.*

4. *k*-Eulerian Posets

DEFINITION 4.1. A poset P is k-Eulerian if all its intervals of rank k or less are Eulerian.

Observe every poset is 1-Eulerian and that a poset is 2-Eulerian if every 2-interval in the Hasse diagram is a diamond. We claim that the condition on a poset being $2k$-Eulerian is equivalent to the condition of being $(2k+1)$-Eulerian. Hence it is sufficient to study k-Eulerian posets when k is odd. The proof of the claim is included in the arguments leading up to the proof of the main result.

We now state the main theorem of this section.

THEOREM 4.2. *The linear span of the* **ab**-*indices of* $(2k+1)$*-Eulerian posets is the algebra* $\mathbf{k}\langle \mathbf{c}, \mathbf{d}, \mathbf{e}^{2k+1} \rangle$ *for* $k \geq 0$.

We first begin with a lemma that shows one direction of the theorem.

LEMMA 4.3. *The linear span of the* **ab**-*indices of* $(2k+1)$*-Eulerian posets contains the algebra* $\mathbf{k}\langle \mathbf{c}, \mathbf{d}, \mathbf{e}^{2k+1} \rangle$ *for* $k \geq 0$.

Proof. Let B_n denote the Boolean algebra on n elements. Then $\Psi(B_2) = \mathbf{c}$ and $\Psi(B_3) - \Psi(B_2 * B_2) = (\mathbf{c}^2 + \mathbf{d}) - \mathbf{c} \cdot \mathbf{c} = \mathbf{d}$. Let T_n be the Eulerian poset B_2^{*n}. Observe that $\Psi(T_n) = \mathbf{c}^n$. Let U_n be the poset of rank $n+1$ obtained by taking two copies of T_n and identifying the minimal and maximal elements. We have $\Psi(U_n) = 2 \cdot \mathbf{c}^n - \mathbf{e}^n$. Observe that U_{2k+1} is $(2k+1)$-Eulerian and $2\Psi(T_{2k+1}) - \Psi(U_{2k+1}) = \mathbf{e}^{2k+1}$. Now by applying [15, Lemma 1.1] the containment follows. □

Continuing the work in [9, 11], we introduce two Newtonian coalgebras and a Laplace pairing between them. This will give us the appropriate algebraic framework to prove Theorem 4.2.

Let $\mathbf{k}\langle \mathbf{a}, \mathbf{b} \rangle$ be the algebra of noncommutative polynomials in the variables $\mathbf{a}$ and $\mathbf{b}$. We enrich $\mathbf{k}\langle \mathbf{a}, \mathbf{b} \rangle$ with a coalgebra structure by defining the coproduct of a monomial by $\Delta(u_1 \cdots u_n) = \sum_{i=1}^{n} u_1 \cdots u_{i-1} \otimes u_{i+1} \cdots u_n$. Hence $\mathbf{k}\langle \mathbf{a}, \mathbf{b} \rangle$ is a graded Newtonian coalgebra of type $(0, -1)$. This is the Newtonian coalgebra that is central in the work of Ehrenborg and Readdy [11]. Since $\Delta(\mathbf{c}) = 2 \cdot 1 \otimes 1$ and $\Delta(\mathbf{e}) = 0$, $\mathbf{k}\langle \mathbf{c}, \mathbf{d}, \mathbf{e}^{2k+1}\rangle$ is a graded subcoalgebra of $\mathbf{k}\langle \mathbf{a}, \mathbf{b}\rangle$.

Let $\mathcal{A}$ be the *algebra of flag operators*, that is, the vector space spanned by the symbols f_S^n where n is a positive integer and S is a subset of $\{1, \ldots, n-1\}$. Define a grading on $\mathcal{A}$ by $\deg(f_S^n) = n - 1$. Hence we can write $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n$. Observe that $\dim(\mathcal{A}_n) = 2^n$. The algebra structure on the algebra of flag operators is given by $f_S^n \cdot f_T^m = f_{S \cup \{n\} \cup (T+n)}^{n+m}$ and is extended by linearity. This algebra is associative, but does not have a unit. The product was first suggested by Kalai [14] and the algebra it defines was first studied rigorously by Billera and Liu [9].

We enrich the algebra of flag operators $\mathcal{A}$ with the coproduct

$$\Delta(f_S^n) = \sum_{\substack{i+j=n+1 \\ i,j \geq 1}} f_{S \cap \{1,\ldots,i-1\}}^i \otimes f_{S \cap \{i,\ldots,n-1\} - i + 1}^j,$$

and extend Δ by linearity. It is straightforward to see that $\mathcal{A}$ forms a graded Newtonian coalgebra of type $(1, 0)$ and has a counit.

Define a bilinear form on $\mathcal{A} \times \mathbf{k}\langle \mathbf{a}, \mathbf{b}\rangle \to \mathbf{k}$ by $\langle f_S^n \mid v_T\rangle = \delta_{S,T}$ where v_T is the **ab**-polynomial of degree $n - 1$ defined in Section 2. This bilinear form restricts to a non-degenerate form $\mathcal{A}_n \times \mathbf{k}\langle \mathbf{a}, \mathbf{b}\rangle_n \to \mathbf{k}$. We claim that $\mathcal{A}$ and $\mathbf{k}\langle \mathbf{a}, \mathbf{b}\rangle$ form a graded Laplace pair with this bilinear form. The first relation of a Laplace pairing, Equation (3.1), was verified by the author and appears in [9, Proposition 5.2]. The second relation, Equation (3.2), is equally direct to prove.

Observe for a poset P of rank n and S a subset of $\{1, \ldots, n-1\}$ we have $\langle f_S^n \mid \Psi(P)\rangle = f_S(P)$. Following [9] define the nth *Euler form* χ_n in $\mathcal{A}_{n-1}$ by

$$\chi_n = f_\emptyset^n - f_{\{1\}}^n + f_{\{2\}}^n - \cdots + (-1)^{n-1} \cdot f_{\{n-1\}}^n + (-1)^n \cdot f_\emptyset^n.$$

Hence a poset P of rank n satisfies the Euler–Poincaré relation if $\langle \chi_n \mid \Psi(P)\rangle = 0$.

Let I be the two-sided ideal in $\mathcal{A}$ generated by the elements $\chi_1, \chi_2, \ldots$. Similarly, let I_k be the ideal in $\mathcal{A}$ generated by the elements $\chi_1, \chi_2, \ldots, \chi_k$. Observe that the ideal I is the union $\bigcup_{k \geq 1} I_k$. More importantly, for a k-Eulerian poset P we have that $\langle I_k \mid \Psi(P)\rangle = 0$.

LEMMA 4.4. *If P is a $2k$-Eulerian poset of rank $2k+1$ then P is Eulerian. Hence the two ideals I_{2k} and I_{2k+1} are identical.*

Proof. Among the Euler forms there is the relation

$$\chi_{2n+1} = -\frac{1}{2} \sum_{i=1}^{2n} \left(f_\emptyset^i \cdot \chi_{2n+1-i} + (-1)^i \cdot \chi_{2n+1-i} \cdot f_\emptyset^i\right). \tag{4.1}$$

(See [9, Proposition 3.3].) If P is a $2k$-Eulerian poset of rank $2k + 1$ then (4.1) implies P satisfies the Euler–Poincaré relations, that is, P is an Eulerian poset. Hence we conclude that every $2k$-Eulerian poset is also a $(2k + 1)$-Eulerian poset. Also, the two ideals I_{2k} and I_{2k+1} are generated by the elements $\chi_2, \chi_4, \ldots, \chi_{2k}$. □

PROPOSITION 4.5. *The ideals I_k and I are ideals of the Newtonian coalgebra $\mathcal{A}$.*

Proof. Since I is the union of the I_k's, it is enough to prove that I_k is a coideal. First, we have

$$\Delta(\chi_n) = \sum_{\substack{i+j=n+1 \\ i,j\geq 1}} \left(\chi_i \otimes f_\emptyset^j + (-1)^{i+1} \cdot f_\emptyset^i \otimes \chi_j\right).$$

Hence for $n \leq k$ we obtain $\Delta(\chi_n) \in I_k \otimes \mathcal{A} + \mathcal{A} \otimes I_k$. Now consider the element $\Delta(x \cdot \chi_n \cdot y) = \Delta(x) * \chi_n \cdot y + x * \Delta(\chi_n) * y + x \cdot \chi_n * \Delta(y)$. Hence we have that $\Delta(x \cdot \chi_n \cdot y) \in I_k \otimes \mathcal{A} + \mathcal{A} \otimes I_k$. By linearity $\Delta(I_k) \subseteq I_k \otimes \mathcal{A} + \mathcal{A} \otimes I_k$, which completes the proof. □

PROPOSITION 4.6. *The Hilbert series of the algebra $I_{2k+1}^{\perp}$ is given by*

$$\mathcal{H}\left(I_{2k+1}^{\perp}\right) = 1 + (1+t) \cdot \frac{1}{1 - t - t^2 - t^{2k+2}} \cdot \left(t + t^{2k+1}\right).$$

Proof. At this point we will prove half of this proposition, that is, the coefficient-wise inequality $\mathcal{H}(I_{2k+1}^{\perp}) \leq 1 + (1+t)(t + t^{2k+1})/(1 - t - t^2 - t^{2k+2})$. Equality will follow from the proof of Theorem 4.2.

The ideal I_{2k+1} encodes the linear relations holding among the flag f-vector entries of a $(2k+1)$-Eulerian poset. Hence the dimension of the nth component of $I_{2k+1}^{\perp}$ is given by the number of flag operators f_S^{n+1} that form a basis. We present such a natural basis, but only show that it is a spanning set, thus proving the inequality. With a more careful argument one can show that it is a basis, and hence the equality of the proposition follows.

Let n be a non-negative integer and S a subset of $\{1, 2, \ldots, n\}$. We call the pair (n, S) a *permissible pair* if it satisfies the following condition:

> Assume that $\alpha, \beta \in S \cup \{0, n+1\}$ and that $\{\alpha+1, \ldots, \beta-1\} \cap S = \{\beta - 1\}$. Then $\beta - \alpha \geq 2k+2$.

If n is given, we call S a permissible set.

A subset S of $\{1, 2, \ldots, n\}$ is called *sparse* if $\{i, i+1\} \not\subseteq S$ for all i and $n \notin S$. Observe the class of permissible sets extends the notion of sparse sets by setting k to be infinity. Moreover, knowing the entries of the flag f-vector of an Eulerian poset for sparse sets is equivalent to knowing the whole flag f-vector; see [2].

To complete the proof of Proposition 4.6, we need the next two lemmas.

LEMMA 4.7. *Let P be a $(2k+1)$-Eulerian poset of rank $n+1$. The flag f-vector entries for permissible sets completely determine the full flag f-vector. That is, for any set S one can express f_S as a linear combination of permissible entries of the flag f-vector.*

Proof. Given the set S look for the obstruction to permissibility with the largest value of β. Since the intervals of P between ranks α and β are Eulerian, we have the linear relation

$$\left(1 + (-1)^{\beta-\alpha}\right) \cdot f_T = \sum_{i=\alpha+1}^{\beta-1} (-1)^{i-\alpha-1} \cdot f_{T\cup\{i\}},$$

where $T = S - \{\beta - 1\}$. This relation allows us to replace f_S with a linear combination of entries of the flag f-vector satisfying the property that their obstruction to permissibility has strictly smaller β-values. Hence by continuing in this manner, the result follows. □

It remains to enumerate the permissible sets. Let x and y be noncommuting variables. For a pair n and S, let $m_{n,S}$ denote the monomial $x_1 \cdots x_n$, where $x_i = x$ if $i \notin S$ and $x_i = y$ otherwise.

LEMMA 4.8. *The generating function of permissible sets is given by*

$$\sum_{(n,S)} m_{n,S} = 1 + (1 + y) \cdot \frac{1}{1 - (x + xy + x^{2k}y^2)} \cdot \left(x + x^{2k}y\right), \tag{4.2}$$

where the sum ranges over permissible pairs (n, S).

Proof. It is straightforward to see that all monomials that appear after expanding the right-hand side of (4.2) are permissible. Moreover, a permissible monomial $w \neq 1$ can be expressed as product $w_1 \cdots w_m$, $m \geq 2$, where the first term w_1 is either 1 or y, the interior terms $w_2, \ldots, w_{m-1}$ are either x, xy or $x^{2k}y^2$ and the last term w_m is either x or $x^{2k}y$. This proves the lemma. □

By setting both x and y to be t in the rational generating function in Lemma 4.8, we obtain the desired Hilbert series of Proposition 4.6. □

The free product of two Newtonian coalgebras is also a Newtonian coalgebra. For graded Newtonian coalgebras with unit elements, the Hilbert series of the free product is described in the next lemma.

LEMMA 4.9. *Let A and B be two graded algebras both containing a unit element. Assume that their Hilbert series are given respectively by $1 + f(t)$ and $1 + g(t)$. Then the Hilbert series of the free (noncommutative) product $A * B$ is given by*

$$\left(1 + f(t)\right) * \left(1 + g(t)\right) = 1 + \frac{f(t) + g(t) + 2f(t) \cdot g(t)}{1 - f(t) \cdot g(t)}.$$

LEMMA 4.10. *The Hilbert series of* $\mathbf{k}\langle \mathbf{c}, \mathbf{d}, \mathbf{e}^{2k+1}\rangle$ *is given by*

$$\frac{1}{1-t} * \left(\frac{1-t^{2k}}{1-t^2} + \frac{t^{2k}}{1-t}\right).$$

Proof. The algebra $\mathbf{k}\langle \mathbf{c}, \mathbf{d}, \mathbf{e}^{2k+1}\rangle = \mathbf{k}\langle \mathbf{c}, \mathbf{e}^2, \mathbf{e}^{2k+1}\rangle$ is the free product of $\mathbf{k}\langle \mathbf{c}\rangle$ and $\mathbf{k}\langle \mathbf{e}^2, \mathbf{e}^{2k+1}\rangle$ and hence the result follows from Lemma 4.9. □

We are now ready to complete the proof of Theorem 4.2.

Proof of Theorem 4.2. We claim that the two Newtonian algebras $I_{2k+1}^{\perp}$ and $\mathbf{k}\langle \mathbf{c}, \mathbf{d}, \mathbf{e}^{2k+1}\rangle$ are identical. We already know that the subalgebra $\mathbf{k}\langle \mathbf{c}, \mathbf{d}, \mathbf{e}^{2k+1}\rangle$ is contained in $I_{2k+1}^{\perp}$. By a direct calculation it follows that the two Hilbert series in Lemma 4.10 and Proposition 4.6 are identical. Hence the algebras are equal. This also completes the proof of Proposition 4.6. □

Theorem 4.2 and its proof also extend to the case when $k = \infty$. We then obtain the classical statement of the existence of the **cd**-index.

COROLLARY 4.11. *The linear span of the* **ab**-*indices of Eulerian posets is the algebra* $\mathbf{k}\langle \mathbf{c}, \mathbf{d}\rangle$.

It is important to note that this proof of the existence of the **cd**-index differs from previous proofs of Bayer and Klapper [5], Stanley [15], and Billera and Liu [9]. The main difference is that their proofs are all based on induction on the underlying poset, whereas this proof avoids induction.

5. Concluding Remarks

Theorem 4.2 describes the linear span of flag f-vectors of $(2k+1)$-Eulerian posets. There are two natural questions which arise. First, can the integer span of $(2k+1)$-Eulerian posets be described? We conjecture that this ring is

$$\mathbb{Z}\left\langle \mathbf{c}, \mathbf{d}, \frac{\mathbf{c}^{2k+1} - \mathbf{e}^{2k+1}}{2}\right\rangle.$$

Secondly, can the cone of positive linear combinations of flag f-vectors of $(2k+1)$-Eulerian posets be described? The answer to this question would interpolate between the cones studied in [4, 8]. Namely, Billera and Hetyei [8] have a complete description of the cone of flag f-vectors of graded posets. Bayer and Hetyei [4] have similarly studied Eulerian posets and have a complete description of the cone generated by Eulerian posets up to rank seven.

It is worth mentioning that a different coalgebraic approach to the **cd**-index has been taken by Aguiar [1]. He shows that every Newtonian coalgebra A together with a linear map $f: A \to \mathbf{k}$ contains a canonical Eulerian subcoalgebra $\mathcal{E}(A)$. Using this structure, he concludes the existence of the **cd**-index. Generalizing his ideas, we deduce that there is a sequence of canonical subcoalgebras $A = \mathcal{E}_1(A) \supseteq \mathcal{E}_3(A) \supseteq \mathcal{E}_5(A) \supseteq \cdots \supseteq \mathcal{E}(A)$, such that $\mathcal{E}_{2k+1}(A)$ corresponds to the $(2k+1)$-Eulerian posets.

Are there other classes of posets that would fit into the algebraic framework developed in Section 4? One suggestion is to consider posets where every interval of rank k or less is a Boolean algebra. It would be interesting to determine the corresponding Hilbert series.

Acknowledgements

The author thanks Gábor Hetyei and Margaret Readdy for inspiring discussions. The author was supported by National Science Foundation, DMS 97-29992, and NEC Research Institute, Inc. while at the Institute for Advanced Study and partially supported by Swedish Natural Science Research Council DNR 702-238/98. This paper was completed at Cornell University under support by National Science Foundation grants DMS 98-00910 and DMS 99-83660.

References

1. Aguiar, M., Infinitesimal Hopf algebras and the **cd**-index of polytopes, in preparation.
2. Bayer, M. and Billera, L. (1985) Generalized Dehn-Sommerville relations for polytopes, spheres and Eulerian partially ordered sets, *Invent. Math.* **79**, 143–157.
3. Bayer, M. and Ehrenborg, R. (2000) The toric h-vectors of posets, *Trans. Amer. Math. Soc.* **352**, 4515–4531.
4. Bayer, M. and Hetyei, G. (2001) Flag vectors of Eulerian partially ordered sets, *European J. Combin.* **22**, 5–26.
5. Bayer, M. and Klapper, A. (1991) A new index for polytopes, *Discrete Comput. Geom.* **6**, 33–47.
6. Billera, L. J. and Ehrenborg, R. (2000) Monotonicity of the **cd**-index for polytopes, *Math. Z.* **233**, 421–441.
7. Billera, L. J., Ehrenborg, R. and Readdy, M. (1997) The **c**-2**d**-index of oriented matroids, *J. Combin. Theory Ser. A* **80**, 79–105.
8. Billera, L. J. and Hetyei, G. (2000) Linear inequalities for flags in graded partially ordered sets, *J. Combin. Theory Ser. A* **89**, 77–104.
9. Billera, L. J. and Liu, N. (2000) Noncommutative enumeration in graded posets, *J. Algebraic Combin.* **12**, 7–24
10. Ehrenborg, R. and Fox, H., Inequalities for **cd**-indices of joins and products of polytopes, to appear in *Combinatorica.*
11. Ehrenborg, R. and Readdy, M. (1998) Coproducts and the **cd**-index, *J. Algebraic Combin.* **8**, 273–299.
12. Hirschhorn, P. S. and Raphael, L. A. (1992) Coalgebraic foundation of the method of divided differences, *Adv. Math.* **91**, 75–135.
13. Joni, S. A. and Rota, G.-C. (1979) Coalgebras and bialgebras in combinatorics, *Stud. Appl. Math.* **61**, 93–139.
14. Kalai, G. (1988) A new basis of polytopes, *J. Combin. Theory Ser. A* **49**, 191–209.
15. Stanley, R. P. (1994) Flag f-vectors and the **cd**-index, *Math. Z.* **216**, 483–499.