A Brief Introduction to Calculus

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Introduction

These notes are intended as a brief introduction to some of the main ideas and methods of calculus. They are very brief and are not intended as a mathematical exposition of the subject. They do not contain recipes for solving problems. Hence, you will not be able to solve homework problems by looking back through the notes and finding similar examples.

We feel that the only way one can really learn calculus (or any another subject) is to take basic ideas and apply those ideas to solve new problems. Hence the learning process is accomplished primarily by solving the problems.

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1. Functions and Graphs

A function is a rule that assigns one number to a given number. In general "number" will mean real number such as 1.25 or 6.498, $\sqrt{2}$, or π. The rule that defines the function can be described in several different ways. Perhaps the most common way of describing the function is an algebraic expression. For example

$$f(t) = t^2$$

is the function that assigns the square of number to any given number. The notation above is sometimes referred to as "functional notation". We may compute the value the function assigns to a given number by substitution the given number into the algebraic expression and then performing the algebra. For example, with the above function $f(t) = t^2$ we evaluate

$$f(2) = 2^2 = 4$$

$$f(3) = 3^2 = 9$$

$$f(4.1) = (4.1)^2 = 16.81$$

Functions may also be described by tables or lists. Suppose we measure some quantity such as weight or length at certain fixed times and express the results in a table. For example, suppose our measurements give the following results.

Time	Weight
1	3.2
2	4.5
3	6.2
4	7.8
5	9.3

We can then consider the table as a rule that assigns a number (weight) to a given time. Notice that this function is only defined for the values of time $1, 2, 3, 4,$ and 5. If we denote the function described by the table using functional notation we could write $W(t)$ to denote the weight at time t. In this case $W(t)$ only makes sense for the t values above.

In some cases it may be necessary to combine the two approaches above in order to adequately describe a function. For example

$$h(t) = t \text{ if } t \geq 0$$

$$h(t) = -t \text{ if } t < 0$$

describes the function $h(t)$ by two algebraic expressions and the expression that applies depends on the value of t. You may recognize the above function as the "absolute value" function.

Note that in the above examples, we have used the letter "t" to indicate the number or value with which we start, and $f(t)$ or $h(t)$ as the number assigned to this starting value. We could use other letters or symbols instead of "t" and you should get used to writing functions using a variety of symbols or letters. Basic texts in mathematics frequently use "x" to represent the starting value, but in applications (for example in finance or other business settings) more descriptive letters are usually used.

Algebraic Operations with Functions

We can perform algebraic operations with functions just as we do with numbers. Performing the algebraic operation results in a new function. For example if

$$f(t) = t \text{ and } g(t) = t^2 \text{ then}$$

$$h(t) = f(t) + g(t) = t + t^2$$

Another operation on functions is composition. For example

$$f(t) = t^2 + t + 1 \text{ and } g(t) = t^3 \text{ then}$$

$$f(g(t)) = (t^3)^2 + (t^3) + 1$$

Composition is also frequently written as

$$f(g(t)) = (f \circ g)(t)$$

It is very important to realize that the function is defined by the rule and not by the symbol used to describe the rule. For example we could have presented the composition example above as follows.

$$f(t) = t^2 + t + 1 \text{ and } g(s) = s^3$$

$$f(g(x)) = (x^3)^2 + (x^3) + 1$$

Note that if we use one symbol for a variable on the left side of an equation, we must use the same symbol on the right side. Thus the expression

$$F(t) = x^2$$

does not make sense.
The values of the variable for which the function is defined is called the *domain of the function*. If we write

$$F(t) = 3t + 4 \text{ for } t \geq 0$$

then the domain of the function, $F(t)$, is $t \geq 0$.
When we write an algebraic expression to define a function without putting specific conditions on the values of the variable, then the domain is assumed to be all values of the variable for which the expression makes sense. For example the domain of

$$G(x) = \sqrt{x}$$

is $x \geq 0$. The domain of the function

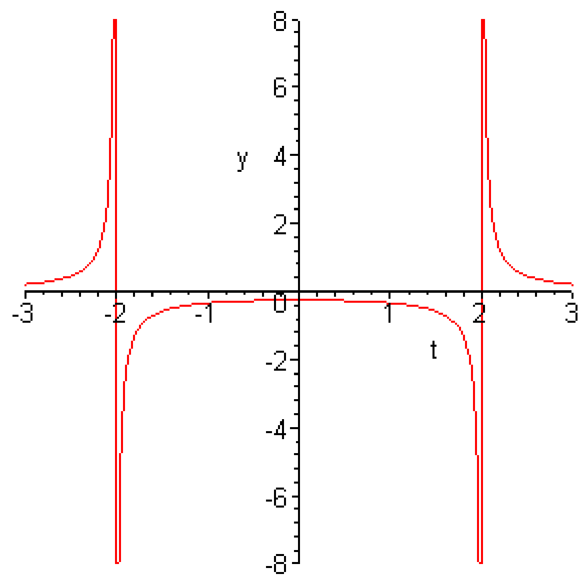
$$G(t) = \frac{1}{t}$$

is $t \neq 0$.

Graphs

Graphs give us a visual way of understanding functions. The function rule is presented in a manner that allows us to see the properties of the function quickly if not precisely. The graph of a function (of one variable) is a two dimensional object, and the graph is usually described algebraically as an equation. Here is an example.

$$y = f(t) = \frac{1}{t^2 - 4}$$



The symbol t is called the *independent variable* and the symbol y is called the *dependent variable*. Points in the two dimensional drawing are indicated by a pair of numbers, (a, b), with a as the value of the independent variable, t, and b the value of the dependent variable, y. The horizontal axis is in this case the t axis and the vertical axis is the y axis. Note that one may use y as a horizontal axis.

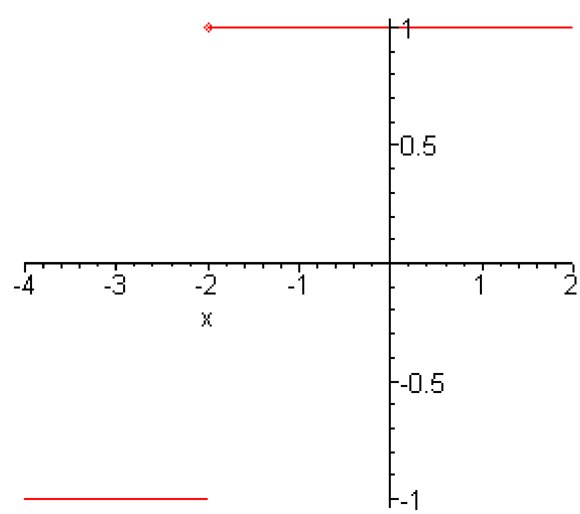
In this example the domain of the function is the set of all $t \neq \pm 2$. We see immediately from the graph the behavior of the function for large positive or negative values of t as well as the behavior for values of t near 2 and -2. Hence for getting an overall idea of the behavior of the function, the graph is often more useful than the algebraic expression. Of course, for computations we need the algebraic expression.

When we graph a function that is defined by different algebraic expressions for different values of the (independent) variable, we must be careful to indicate how the function is defined at the break. For example if

$$f(x) = -1 \text{ if } x < -2 \text{ and}$$

$$f(x) = 1 \text{ if } x \geq -2$$

we can place a solid dot at the point $(-2, 1)$ to indicate that the value of the function at -2 is 1.



Often an open dot or circle is used at the point $(-2, -1)$ to indicate that the value at -2 is not -1.

If the graph of a function crosses an axis, for example the t axis, then the point of crossing is called a t intercept.

2. Linear Functions, Lines, and Linear Equations

Lines are the simplest geometric object after a point, and linear functions are the simplest algebraic functions. The approximation of complicated functions by linear functions is one of the basic tools in mathematics and its applications. Hence, an absolutely firm understanding of linear functions is essential to an understanding of more complicated functions.

A linear function of the variable x is a function of the form

$$L(x) = a + bx$$

where a and b are constants. For example,

$$L(x) = 3 + 5x$$

is a linear function. Note that the function

$$L(x) = A + B(x - c)$$

is also a linear function, although it is written in a slightly different form. For example,

$$L(x) = 8 + 5(x - 1)$$

can be rewritten as

$$\begin{aligned} L(x) &= 8 + 5(x - 1) \\ &= 8 + 5x - 5 \\ &= 8 - 5 + 5x \\ &= 3 + 5x \end{aligned}$$

This particular form of a linear function, $L(x) = A + B(x - c)$, will be useful later.

Lines and Linear Equations

The graph of a *linear function* is a *line*. If $L(x)$ is a linear function and we introduce the dependent variable y, then

$$y = L(x)$$

is a *linear equation*. Note the terminology we use here. Be sure you understand the difference between a linear function, line, and linear equation, since they represent three different mathematical objects. If

$L(x) = a + bx$ then the equation of the line (the graph of the function $L(x)$) can be written as

$$y = L(x)$$

$$y = a + bx$$

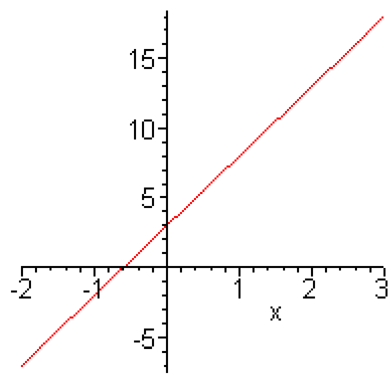
If $L(x) = A + B(x - c)$ then the equation of the line representing the graph of $L(x)$ is

$$y = A + B(x - c)$$

For example if $L(x) = 8 + 5(x - 1)$ we have

$$y = 8 + 5(x - 1)$$

If we graph the linear function $L(x) = 8 + 5(x - 1)$ we get



We can obtain this graph by plotting a couple of points and then drawing the line through the two points. For example, a table of values for $L(x)$ is

x	$L(x)$
1	8
2	13

If we plot the two points $(1, 8)$ and $(2, 13)$ and then draw the line through the two points, we obtain the graph of $L(x)$.Notice that when x increases by one unit (from 1 to 2) the value of $L(x)$ (or y) increases by 5 units. The number 5 appeared as the coefficient of $(x - 1)$ in the expression for $L(x)$. We would have seen exactly the same thing if we had used different values for x. For example,

x	y
5	28
6	33

The increase in $L(x)$ or y for a unit increase in x always appears as the coefficient of x in the expression for $L(x)$. The value is called the **slope** of the line $y = L(x)$. In applications this value is also called the **marginal value of L**. If, for example, $L(x)$ represents the cost of producing x items and $L(x) = 8 + 5(x - 1)$ then the **marginal cost** is 5. It represents the cost of producing one additional unit. The slope of a line has a nice geometric interpretation. It measures the steepness of the graph of the linear function. The larger the value of the slope, the steeper the line. If the slope is negative, then the linear function decreases as the value of the independent variable increases, and the graph " heads downhill".

The above example gives us two ways to determine the expression for a linear function or equivalently, to find the equation of a line. First suppose we are given two points on the line - perhaps by a table. We will use the values above as an example since we already know what we should get as an expression for $L(x)$. So, for example, suppose the two points are $(1, 8)$ and $(2, 13)$. The value of $L(x)$ for $x = 1$ must be 8, so we write

$$L(x) = 8 + m(x - 1)$$

where m represents some constant that we must determine. (Think of m as the marginal value). How do we determine m? When $x = 2$, $L(x) = 13$ so we substitute these values into the expression for $L(x)$. We get

$$L(2) = 13$$
$$8 + m(2 - 1) = 13$$

Now we can solve this last equation for m. In two steps we get

$$8 + m(2 - 1) = 13$$
$$8 + m = 13$$
$$m = 5$$

Hence the expression for $L(x)$ is $L(x) = 8 + 5(x - 1)$ which is as expected.

From the above calculation, it should be clear that a linear function is completely determined by its value at two points (two x values).

Now suppose we are given a point and the slope of the linear function. How do we determine the expression for the linear function? It is even easier than in the above case (given two points). For example suppose the point is $(6, 33)$ and the slope is 5. Then the expression for $L(x)$ is

$$L(x) = 33 + m(x - 6)$$

since we must get 33 when $x = 6$. But we are given the value of the slope, $m = 5$. Hence the expression for $L(x)$ is

$$L(x) = 33 + 5(x - 6)$$

So we really did not need any calculation at all! You should now see the advantage of the form

$$L(x) = A + B(x - c)$$

for a linear equation.
Note that we could rewrite $L(x)$ as

$$\begin{aligned} L(x) &= 33 + 5(x - 6) \\ &= 33 + 5(x - 1 - 5) \\ &= 33 + 5(x - 1) + 5(-5) \\ &= 33 + 5(x - 1) - 25 \\ &= 8 + 5(x - 1) \end{aligned}$$

If we multiplied out the 5 and the $(x - 1)$ we would obtain

$$\begin{aligned} L(x) &= 8 + 5x - 5 \\ &= 3 + 5x \end{aligned}$$

This is the form we had at the very beginning.

The above method of finding the expression for a linear function may be slightly different than what you are familiar with from algebra. However it has several advantages and you should try to understand this method. First of all, given a point and slope, the method allows you to write down immediately the expression for the linear function. Second, you are much less likely to get mixed up if symbols other than x and y are used as the independent and dependent variables respectively. For example, suppose you know the relationship between cost, $c,$ and interest rate, $r,$ is linear. You are given a cost rate table.

c	r
300	7
400	8

Suppose you want to find c as a linear function of r. Write

$$\begin{aligned} c &= L(r) \\ &= 300 + m(r-7) \end{aligned}$$

Now find m.

$$\begin{aligned} 400 &= 300 + m(8-7) = 300 + m \\ 100 &= m \end{aligned}$$

Hence

$$c = L(r) = 300 + 100(r-7)$$

Of course you can work this problem using the other methods for finding the equation of a line, but it is easy to get things backwards (usually the mistake that is made is getting the reciprocal of the slope).

We should finally mention one particular line that is not the graph of a function. Recall again that a line is a geometric object. Suppose we have drawn a vertical line in the (x,y) plane. Then there is no linear function $L(x)$ such that the vertical line is the graph of the function $L(x)$, since a function $L(x)$ may take on at most one value for a given x, However, it is possible to write an equation that represents the line. For example, consider the vertical line through the point $(2,5)$. Since the line is vertical, the x value of any point on the line must be 5, There is no restriction on the y value of a point on the line. Hence the equation of the line is

$$x = 2$$

Finally, we summarize three different forms for the equation of a line. Each has an advantage in some setting.

$y = a + m(x-b),$ the equation of the line through (a,b) with slope m

$y = c + mx,$ the equation of the line through $(0,c)$ with slope m

$Ax + By + C = 0,$ the equation of a line that can be used to describe vertical lines

3. Limits

The concept of **limit** is one idea that allows calculus to solve problems that are impossible to solve with algebra alone. The combination of the ideas of limit and linear approximation together provide most of the results of calculus. We are not going to give a precise mathematical definition of limit here, since we believe it is important that you first develop a feel for the idea through computations and examples before attempting to understand a precise definition. Here are a couple of examples of problems that require the use of limit.

Suppose you deposit $1000 in an account that pays 10% interest compounded one time per year. After one year the account contains $1100. Now suppose the interest is compounded two times per year. The amount in the account after one year is now $1102.50. If the interest is compounded four times per year the amount after one year is $1103.80. Here is a table that gives the resulting amount in the account after one year as a function of the number of times per year that the interest is compounded.

Times Compounded	Amount (in dollars)
1	1100
2	1102.50
4	1103.80
12	1104.70

(You may not be familiar with compound interest, but your bank and every bank uses this method to compute interest. Don’t worry about the details of this now, but just keep in mind that it is a simple algebraic computation that you can do on your calculator.) We see that as the number of times we compound the interest increases, the amount in the account at the end of the year increases. An obvious question is this; does this amount get larger and larger without bound. For example, if we compound enough times per year might we end up with $2000 in the account at the end of the year? The answer is no. In fact no matter how many times we compound per year we would never end up with more than $1,105.20 (approximately). To obtain this result we could try increasing the number of rows in our table above, and compute the amount in the account at the end of the year. If you do this with a calculator you will quickly come to the **limiting value** of \$1,105.20. In fact if you compounded 1000 times per year and then 2000 times per year the difference in the amount in the account at the end of the year would be so small that it could not be indicated with the number of digits available on the calculator.

Here is a second example of a problem that requires the concept of limit to solve. Find the area of a circle of radius two (2). You might object and say that every high school student knows the answer: $\pi 2^2$ or just 4π. This is of course correct, but one might ask - what is the decimal value of π. The answer to this question requires the concept of limit.

Finally, here is a third example of the use of limit. In this case the resulting limit is rather obvious, but the example still might help you understand the concept. Suppose I start with \$1000 in an account (the account does not pay interest). I spend 1/2 of the money on 1 January. The next day I spend 1/2 of the remaining amount (\$500). The day after that, I spend 1/2 of the amount remaining. I continue this process day after day. What is the limiting amount in my account? The answer is clearly \$0. It should also be clear that this will never happen! Even after 10 years I would have some minuscule amount remaining in the account (assuming we can subdivide currency as small as we like).

With these three examples, we are ready to consider an example using functional notation. Suppose $F(x) = 1/x$. Since this function is not defined (does not make sense) if $x = 0$, we consider only positive values of x. consider the following question. What is the limiting value of $F(x)$ as x gets larger and larger. (You should see the connection with the previous example). If we substitute larger and larger values for x into the expression for $F(x)$ it becomes clear that the limiting value is 0. We want to introduce some notation that will summarize the question and the answer. First we need notation for the concept of “x gets larger and larger”. You are probably already familiar with the notation that will be used, but it can create a great deal of confusion. So be aware that the concept of “**infinity**” is a very difficult one to define and understand. The notation we use for “x gets larger and larger” is

$$x \to \infty$$

and we say this as ”x tends to infinity”. Since we are considering larger and larger positive values of x we can be more precise and say "x tends to positive infinity". Now using this notation and the idea of limit that we have intuitively introduced above we have the notation

$$\lim_{x \to \infty} F(x) = \lim_{x \to \infty}(1/x) = 0$$

How do we compute limits? In the first example above we suggested computing the limit of the amount in the account at the end of the year by doing computations using more and more compounding periods. Frequently this may be necessary to evaluate the limit. It can obviously be quite time consuming, and we would like to have easier methods. In many cases algebraic simplification can be used to put a function expression in a form so the limiting value becomes obvious. For example, suppose

$$G(x) = \frac{1+x}{x+x^2}$$

and we want to compute

$$\lim_{x \to \infty} G(x)$$

If we rewrite

$$G(x) = \frac{1+x}{x+x^2} = \frac{1+x}{(1+x)x} = \frac{1}{x}$$

(which we can do if the value of x is positive) then we see

$$\lim_{x \to \infty} G(x) = \lim_{x \to \infty}(1/x) = 0$$

With the example $F(x) = 1/x$ we could also consider what happens to the values of $F(x)$ for positive values of x that get smaller and smaller; in other words as these positive values get closer and closer to 0. The notation for this is

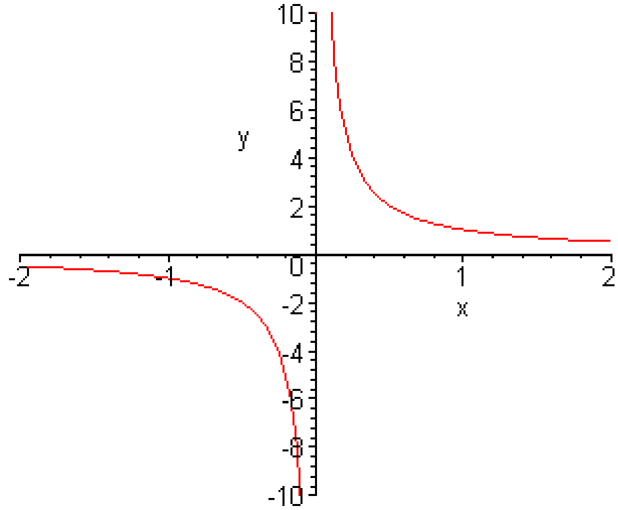
$$\lim_{x \to 0}(1/x)$$

A couple of computations should convince you that the values become larger and larger positive numbers. Note that if you did the same computations with negative values of x very close to zero, you would get more and more negative values of the function. To distinguish these various possibilities we can augment our notation in the following way. The meaning of the following should be clear from our discussion.

$$\lim_{x \to 0^+}(1/x) = +\infty$$

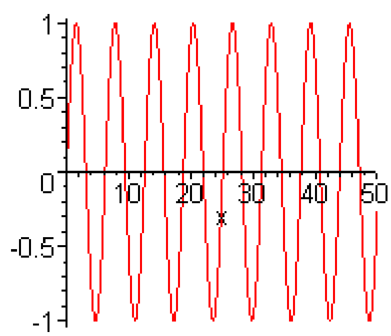
$$\lim_{x \to 0^-}(1/x) = -\infty$$

It will help to understand this notation by graphing the function $1/x$.



The above notation $x \to 0^+$ can be extended in an obvious way. If we write $x \to 3^+$ we mean that "x tends to 3 from the right". Similarly, $x \to 3^-$ means that "x tends to 3 from the left". This will become important when we consider continuity in the next section.

It can happen that the limit of a certain function may not exist, that is it may not make sense to talk about the limit. For example suppose a function oscillates back and forth between -1 and $+1$. An example is the function $\sin(x)$ whose graph looks like



In this case we write

$$\lim_{x\to\infty} \sin(x) \text{ does not exist.}$$

Note that

$$\lim_{x\to 0}(1/x) \text{ does not exit (be sure you can explain why not)}$$

Limits of functions satisfy certain nice properties that make computations easier than one might guess. If $f(x), g(x),$ and $h(x)$ are functions, a is a number, and the limits

$$\lim_{x \to a} f(x),\ \lim_{x \to a} g(x),\ \text{and}\ \lim_{x \to a} h(x)$$

all exist and are not zero, then the following rules hold

$$\begin{aligned}
\lim_{x \to a}[f(x) + g(x)] &= \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \\
\lim_{x \to a}[f(x)g(x)] &= \lim_{x \to a} f(x) \lim_{x \to a} g(x) \\
\lim_{x \to a}[f(x)/h(x)] &= \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} h(x)}
\end{aligned}$$

If one or more of the original limits does not exist or equals zero, then one must be very careful in evaluating limits of sums, products , or quotients.

4. Continuity

The concept of continuity is closely related to limits. One can understand the concept by considering a couple of examples. First, let $f(x)$ be defined by

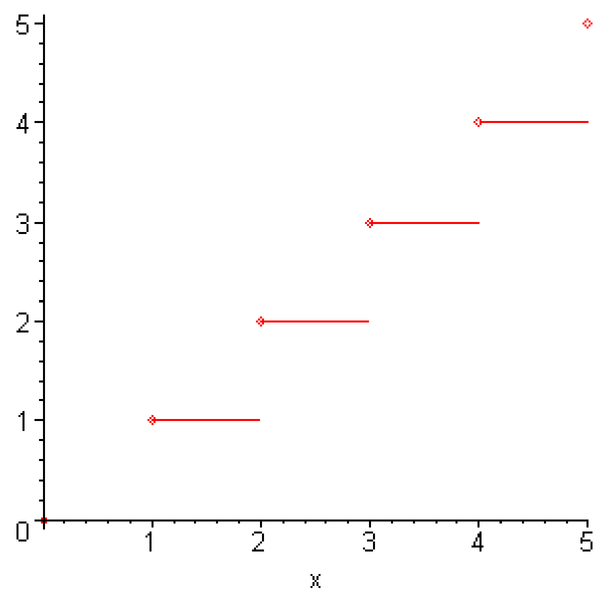
$$f(x) = 0 \text{ if } x < 0$$

$$f(x) = 1 \text{ if } x \geq 0$$

This function is discontinuous in the sense that the value of $f(x)$ jumps as x goes from negative to positive. Here is a slightly more complicated example. Let

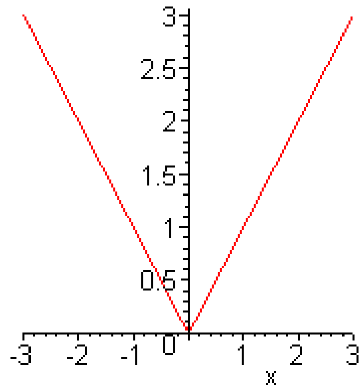
$$gif(x) = \text{ the greatest integer less than or equal to } x$$

So, for example $gif(3.2) = 3$. This function is sometimes denoted by brackets, $gif(x) = [x]$. This function is discontinuous at the integers. Here is a plot.



On the other hand, the function

$$g(x) = |x|,$$



the absolute value function, is continuous since there are no jumps in the value of the function. We can be more precise in talking about continuity or discontinuity by specifying the points at which we have discontinuity. The function $f(x)$ is discontinuous at $x = 0$ but is continuous everywhere else. The function, $gif(x)$ is discontinuous at the integers but continuous everywhere else. The absolute value function is continuous for all values of x. The connection with limits can be expressed as follows.

$$\lim_{t \to 0^-} f(x) = 0$$

$$\lim_{t \to 0^+} f(x) = 1$$

Since the left and right limits are not equal, the function $f(x)$ is not continuous at 0. In general, for any function $F(x)$, if

$$\lim_{t \to a^-} F(x) = \lim_{t \to a^+} F(x) = F(a)$$

then $F(x)$ is continuous at a. If $F(x)$ is continuous at all points, we simply say F is continuous. In some cases, a function may be discontinuous at a point because the function is not defined at the point. For example,

$$h(x) = \frac{1}{x - 7}$$

is not continuous at $x = 7$, but it is continuous for all other values of x.

Why is the concept of continuity important?

First of all, it is generally much easier to work with functions that are continuous. Hence, we might have a preference for continuous functions when we model economic or physical events. However, in doing so, we should be sure that this is reasonable. For example, suppose you wanted to model stock prices of a certain company from 10:00 am to 11:00 am on a business day. It is reasonable to assume the stock price varies continuously in this time period. Now suppose you wish to model a stock price between 3:00 PM of one business day to 10:00 am of the next business day. It would be unreasonable to model the stock price during this time period by a continuous function since rather drastic events may occur between the close of business one day and the opening of business the next day. Since essentially all potential stock buyers would have time to react to the events, stock prices could change dramatically overnight.

Continuity can also tell us something about existence of solutions of equations. A very important example is the *Intermediate Value Theorem*. If the function $F(x)$ is continuous on the interval $[a, b]$ and $F(a) = m,\ F(b) = M,$ then f takes on all values between m and $M.$

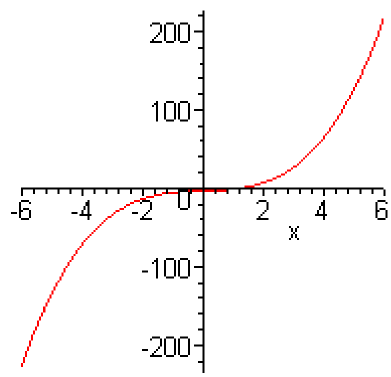
5. Linear Approximation

The concept of limit applied to the tool of linear approximation is the foundation of calculus and modern analysis. In the second section we discussed linear functions. We will now apply linear functions to make approximations of more complicated functions. We start with some examples.

Suppose

$$F(x) = x^3 + x - 3$$

Here is the plot.



Now suppose we want to solve the equation

$$F(x) = 0 \text{ or}$$

$$x^3 + x - 3 = 0$$

From the graph, it appears the solution should be a number between 1 and 2. In fact, $F(1) = -1$ and $F(2) = 7$, so by continuity there must be some x value between 1 and 2 where the function is 0. Note that this is an application of the Intermediate Value Theorem mentioned above. Let's use these two values to find a linear function, $L(x)$ that approximates $F(x)$ for x between 1 and 2. With $L(1) = -1 = F(1)$ and $L(2) = 7 = F(2)$, we have

$$L(x) = -1 + m(x - 1).$$

Now using $L(2) = 7$, we have

$$L(2) = 7 = -1 + m(2 - 1)$$

Solving for m, we get

$$7 = -1 + m(2 - 1)$$

$$8 = m$$

Hence,

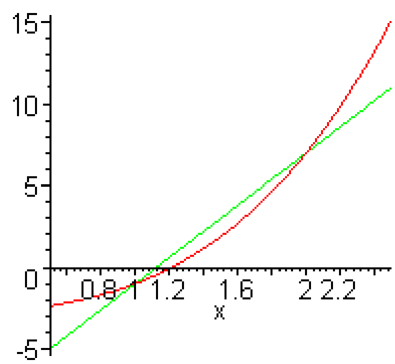
$$L(x) = -1 + 8(x - 1)$$

Now, instead of solving the equation $F(x) = 0$, we solve $L(x) = 0$. This should give us an approximation to the solution to $F(x) = 0$. Solving $L(x) = 0$ gives

$$L(x) = -1 + 8(x - 1) = 0$$

$$x = 1.125$$

If you use a calculator to solve the original equation, $F(x) = 0$, you should get a value for x of approximately 1.2134. Hence, our approximation is off by about $.1$. In fact, the calculator obtains the value it does by essentially continuing the linear approximation process. Here is a plot of the function $F(x)$ and the linear approximation.



Here is a summary of the idea of linear approximation applied to a function using two values.

Given a function $F(x)$

We want a linear function $L(x)$ that approximates $F(x)$

Select two values of x, say a and b, and compute $F(a)$ and $F(b)$

Find $L(x)$, the linear function through the points $(a, F(a))$ and $(b, F(b))$

$L(x) = F(a) + m(x - a)$ with $m = \dfrac{F(b) - F(a)}{b - a}$

This one simple formula above is extremely powerful and useful. It gives the linear approximation to a function using two points. The expression above for m is a **difference quotient**, and has important geometric and practical interpretations. First, the difference quotient represents the slope of the line defined by the equation

$$y = L(x) = F(a) + m(x - a).$$

It has the practical interpretation as the average rate of increase of the function $F(x)$ between the x values a and b. The difference quotient will play an important role in the next section when we consider *derivatives*, but for now we give one example of an application.

Rate of Increase Example. The annual growth rate of a business is often used to measure investment desirability. Suppose a business has sales of $1,500,000 ($1.5 m) during January and $1.6 m during February. What is the annual growth rate for the business? What are the expected sales during July (assuming linear growth rate)? Since the increase is $.1m per month, you should see that the expected July sales will be $2.1 m. Let's express expected sales as a function of time, t.

Let's first decide on units. Use millions of dollars for sales amount during a month and call it S. Use years for time with the end of January corresponding to $t = 1/12$. Now using the linear model for growth we have

$$S = 1.5 + m(t - 1/12)$$

We obtain this since at the end of January (1/12 year form the beginning of January) we have sales of $1.5 m. Now find the value of m.

$$1.6 = 1.5 + m(2/12 - 1/12)$$

$$m = \frac{1.6 - 1.5}{(2/12) - (1/12)}$$

Note that this is a difference quotient. If we compute this value, we get

$$m = \frac{1.6 - 1.5}{(2/12) - (1/12)} = 1.2$$

What are the units? Since we used millions of dollars for sales and years for time, we have

$$m = \$1.2 \text{ million per year}$$

This is the annual growth rate. An expression for S in terms of t is

$$S = 1.5 + 1.2(t - 1/12)$$

Since the end of July corresponds to $t = 7/12,$ we have expected sales for July will

$$S = 1.5 + 1.2(7/12 - 1/12) = 2.1$$

or sales of $\$2.1$ million during July. Note that in this example, S itself is a rate since S represents monthly sales or sales per month. Thus it makes perfect sense to consider any t value greater than 0. Note that S does not represent total sales from the first of January. How would you compute total sales? We will come back to this problem later, but you should be able to figure out how you get the total sales for the year, for example.

Summary Linear Approximation is one of the most useful tools of applied mathematics. The above discussion shows how one obtains a linear approximation, and it introduces the difference quotient. The difference quotient represents a rate of increase.

6. Introduction to the Derivative

In the last section we introduced the linear approximation to a function F, and we saw that finding a linear approximation involved a difference quotient. This difference quotient was (geometric interpretation) the slope of the line through the points

$$(a,F(a)) \text{ and } (b,F(b))$$

Suppose now that the value b is just a little to the right of a on the x axis. Write b as

$$b = a + h$$

where we think of h as taking on a small positive value. The difference quotient then becomes

$$\frac{F(b) - F(a)}{b - a} = \frac{F(a+h) - F(a)}{(a+h) - a} = \frac{F(a+h) - F(a)}{h}$$

What is the limiting value of the difference quotient as $h \to 0$ (or as b approaches a). If F is continuous, then both the numerator and denominator get closer and closer to zero so the limit may not exist. However, in many cases the limit does exist, and it has the geometric interpretation as the slope of the tangent line to the graph of

$$y = F(x)$$

at the point $(a,F(a))$. In case the limit does exist it is called the **derivative of the function F at the point** a. There are several different notations used for this limit; we will use the notation $F'(a)$. Hence we have

$$F'(a) = \lim_{h \to 0} \frac{F(a+h) - F(a)}{h}$$

provided this limit exists.

Here is a simple example. To be sure you do not become dependent on our calling the independent variable "x", let

$$f(s) = s^2$$

Let's find $f'(3)$. We first form the difference quotient and simplify.

$$\frac{f(3+h) - f(3)}{h} = \frac{(3+h)^2 - 3^2}{h} = \frac{9 + 6h + h^2 - 9}{h}$$

$$= \frac{6h + h^2}{h} = 6 + h$$

Now we take the limit as h tends to 0 of this difference quotient.

$$f'(3) = \lim_{h\to0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to0}(6+h) = 6$$

Suppose we wanted to find $f'(4)$ instead. We could obviously go through the same steps to get the result. However, it would make more sense to try to get an expression for the derivative *in general*. Compute $f'(s)$ where s represents any value. Going through the same steps with s replacing 3 above we have

$$f'(s) = \lim_{h\to0} \frac{f(s+h) - f(s)}{h}$$

$$\lim_{h\to0} \frac{(s+h)^2 - s^2}{h} = \lim_{h\to0} \frac{s^2 + 2sh + h^2 - s^2}{h} = \lim_{h\to0}(2s+h) = 2s$$

Hence if

$$f(s) = s^2 \text{ then}$$

$$f'(s) = 2s$$

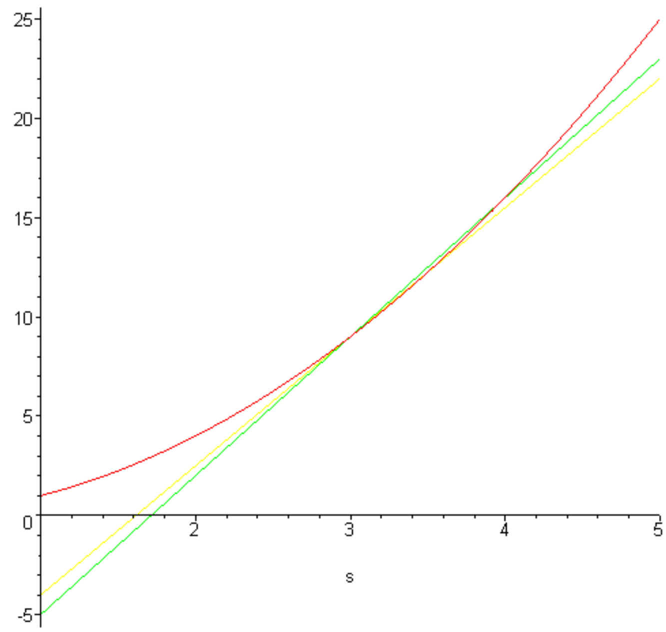
Let's check this with $s = 3$. We get $f'(3) = 2 \times 3 = 6$ as before. Similarly $f'(4) = 8$. In fact the above steps, compute the difference quotient and take the limit, can be applied in a very general setting. Although the algebra gets a little more complicated, it is fairly easy to see that if n is any integer other than 0 and if

$$f(s) = s^n \text{ then}$$

$$f'(s) = ns^{n-1}$$

What about the case when $n = 0$? In fact the same formula holds - you should figure out why it holds.

We can see what is going on by plotting the function $f(s) = s^2$ and two linear approximations. We will make the linear approximations using the points $s = 3$ and $s = 4$ for the first linear approximation and $s = 3$ and $s = 3.5$ for the second. Here is the plot.



As you can see, we get lines that are very close to the tangent in both cases.

What does the value $f'(3)$ represent? It is the slope of the tangent line to the graph of $f(s)$ at the point $s = 3$. Let's write down the expression for the linear function, $L(s)$, that defines the tangent line. We already have the slope; it is $f'(3) = 6$ as we computed above. Thus using our nice form for linear functions we have immediately

$$L(s) = 9 + 6(s - 3)$$

This linear function is the *linear approximation to* $f(s)=s^2$ *at the point* $s=3$. Note that now we are using only one point (in this case $s = 3$) to define the linear approximation. Of course we need either two points or one point and a slope to define a linear function. Hence, here we use one point and the slope (the derivative of the function) to get the linear approximation. Be sure you understand the difference between the two methods of getting a linear approximation. We can write down an expression for the linear approximation of an arbitrary function, $F(s)$, at a point $s = a$ as follows. Since the derivative of F at $s = a$ is $F'(a)$, we see that the linear approximation to $F(s)$ at $s = a$ is

$$L(s) = F(a) + F'(a)(s - a)$$

Again, you should compare this with the linear approximation using two points. These two versions of linear approximation are probably the two most important things to learn in calculus. Of course "learn" does not mean memorize the formula!

We used the rather simple function $f(s) = s^2$ as an example and had no trouble computing the limit of the difference quotient. In fact, in many cases it is not only difficult to compute the limit of the difference quotient, but the limit may not exist. Why not? The geometric meaning of the derivative at a point is the slope of the tangent line. Suppose there is no tangent line. Then we would not expect the derivative to exist. As an example, consider the absolute value function, $F(s) = |s|$. The graph of the function should convince you that there is no tangent line at the point $s = 0$. Try to compute the derivative of this function at $s = 0$. In the expression for the limit of the difference quotient

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

the value for h can be positive or negative (close to 0). Try $h = .1$ and $h = -.1$ to see why the limit will not exist.

You might wonder how useful this procedure for finding the derivative is since forming the difference quotient and computing the limit would be very complicated except for the simplest functions. Fortunately, this process can be broken down into a sequence of separate steps, and some general rules developed which makes the computation quite simple in many cases. For example, one can write down the derivative of a polynomial with essentially no computation. We shall see how to do this below.

Perhaps the most important and useful property of differentiation is *linearity* (don't confuse this linearity with linear function - linearity used here describes a property of the differentiation process). Suppose $f(s)$ and $g(s)$ are two functions and A and B are two constants. Define a new function by

$$h(s) = Af(s) + Bg(s)$$

Then

$$h'(s) = Af'(s) + Bg'(s)$$

provided the derivatives exist. Thus the *derivative of a sum is the sum* *of the derivatives* and the *derivative of a constant times a function is the constant times the derivative of the function*. These properties together with the rule given above for the derivative of a power allows us to differentiate polynomials very easily. Suppose

$$P(s) = 5s^4 + 7s^2 + 3s + 8$$

Then

$$\begin{aligned} P'(s) &= 5(4)s^3 + 7(2)s + 3 \\ &= 20s^3 + 14s + 3 \end{aligned}$$

As a second (non polynomial) example let

$$Q(s) = \frac{2}{s^2} + \frac{3}{s} + s + 4$$

$$Q(s) = 2s^{-2} + 3s^{-1} + s + 4$$

Then

$$\begin{aligned} Q'(s) &= 2(-2)s^{-3} + 3(-1)s^{-2} + 1 \\ &= \frac{-4}{s^3} + \frac{-3}{s^2} + 1 \end{aligned}$$

Several different notations are commonly used to denote the derivative of a function. We present some of them below. Suppose $f(x)$ is a function and the derivative exists at all values of x. Two ways of indicating the derivative are $f'(x)$ and df/dx. If we write the equation

$$y = f(x),$$

setting the dependent variable y equal to the value of the function at the point x, we may also write $y'(x)$ or dy/dx to denote the derivative. If we wish to evaluate the derivative the value $x = a$, then we write this as $f'(a)$, $\frac{df}{dx}|_{x=a}$, $y'(a)$, or $\frac{dy}{dx}|_{x=a}$

Higher Derivatives

Note that the derivative of a function is a function itself. For example, if $f(x) = x^2$ then $f'(x) = 2x$. Hence it makes sense to talk about the derivative of the derivative. Since this would make rather awkward terminology, we refer to the derivative of the derivative as the *second derivative*. We write this as $f''(x)$. Alternate notation is

$$f''(x)$$

$$\frac{d^2f}{dx^2}$$

$$d^2y/dx^2$$

We also have similar notation for the second derivative evaluated at a point. We can also take more than two derivatives, In fact we can take as many as we like.

Example. Suppose $f(x) = x^3 + 7x^2 + 2x + 3$. Then we have

$$\begin{aligned} f(x) &= x^3 + 7x^2 + 2x + 3 \\ f'(x) &= 3x^2 + 14x + 2 \\ f''(x) &= 6x + 14 \\ f'''(x) &= 6 \\ f''''(x) &= 0 \end{aligned}$$

All additional (higher) derivatives of $f(x)$ equal the 0 function.

7. Product, Quotient, and Chain Rules for Differentiation

In the last section, we pointed out that computing derivatives by forming the difference quotient and taking limits would be very tedious at best. We also mentioned that the difference quotient method could be broken into steps and some general rules developed. In this section we present three of these general rules that will allow us to take the derivative of a large class of functions. However, even with these rules we will still be able to deal only with quotients of polynomials, and compositions of sums of polynomials. Later, we will apply these rules to more general functions. We give the rules and then a few examples. In each case we assume that when we write, for example, $f'(x)$, that this derivative actually exists.

Product Rule

The product rule allows us to compute the derivative of a product of two functions by computing the derivative of each separately. If $f(x)$ and $g(x)$ are two functions and we define

$$h(x) = f(x)g(x)$$

then

$$h'(x) = f'(x)g(x) + f(x)g'(x)$$

Example.

$$f(x) = x^2 + 2x + 3 \text{ and } g(x) = 4x + 3$$

$$h(x) = f(x)g(x) = (x^2 + 2x + 3)(4x + 3)$$

Computing the derivatives of f and g,

$$f'(x) = 2x + 2 \text{ and } g'(x) = 4$$

Using the product rule gives

$$\begin{aligned} h'(x) &= (2x + 2)(4x + 3) + (x^2 + 2x + 3)4 \\ &= (8x^2 + 14x + 6) + (4x^2 + 8x + 12) \\ &= 12x^2 + 22x + 18 \end{aligned}$$

We could have obtained the same result by writing out $h(x)$ as a polynomial and then taking the derivative.

Quotient Rule

The quotient rule allows us to compute the derivative of a quotient of two functions by computing the derivative of each separately. If $f(x)$ and $g(x)$ are two functions and we define

$$h(x) = \frac{f(x)}{g(x)}$$

then

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

Example.

$$f(x) = x + 3 \text{ and } g(x) = x^2 + 1$$

$$h(x) = \frac{f(x)}{g(x)} = \frac{x + 3}{x^2 + 1}$$

Then

$$f('x) = 1 \text{ and } g'(x) = 2x$$

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} = \frac{(1)(x^2 + 1) - (x + 3)(2x)}{(x^2 + 1)^2}$$

$$= \frac{x^2 + 1 - 2x^2 - 6x}{(x^2 + 1)^2} = \frac{-x^2 - 6x + 1}{(x^2 + 1)^2}$$

Example. Here is another example that can be checked by a previous method.

$$f(x) = 1 \text{ and } g(x) = x$$

$$h(x) = \frac{f(x)}{g(x)} = \frac{1}{x}$$

Then

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} = \frac{0 - 1}{x^2} = \frac{-1}{x^2}$$

Chain Rule

The chain rule allows us to compute the derivative of the composition of two functions by computing the derivative of each separately. If $f(x)$ and $g(x)$ are two functions and we define

$$h(x) = f(g(x)) \quad = f \circ g(x)$$

then

$$h'(x) = f'(g(x))g'(x)$$

It is worth writing this out in words.
The derivative of the composition of $f(x)$ with $g(x)$ is the derivative of $f(x)$ evaluated at $g(x)$ times the derivative of $g(x)$.
Example. In this example we could obtain the same result by writing the composition as a polynomial and then differentiating the polynomial. Convince yourself that this would be a poor way to compute the derivative!

$$f(x) = x^{10} \text{ and } g(x) = x^3 + x^2 + x + 1$$

$$h(x) = f(g(x)) \quad = (x^3 + x^2 + x + 1)^{10}$$

Then

$$f'(x) = 10x^9 \text{ and } g'(x) = 3x^2 + 2x + 1$$

$$h'(x) = f'(g(x))g'(x) = \{10(x^3 + x^2 + x + 1)^9\}\{3x^2 + 2x + 1\}$$

Finally, let's use this last computation, and find the linear approximation to

$$h(x) = (x^3 + x^2 + x + 1)^{10}$$

at $x = 0$. First we need the value of $h'(0)$. We get this immediately from the above computation.

$$h'(0) = 10(1)^9(1) = 10$$

Hence the linear approximation to $h(x)$ at $x = 0$ is

$$L(x) = h(0) + h'(0)(x - 0)$$

$$= 1 + 10x$$

How good is this linear approximation of $h(x)$ for values of x close to 0?

$$h(.05) = 1.670078187 \text{ (approximately)}$$

$$L(.05) = 1.5$$

Note that the computation of $L(.05)$ is immediate, but the calculation of $h(.05)$ requires a calculator or a great deal of paper!

8. Derivatives and Rates

One common practical application of the derivative is as a measurement of *rate of change*. So for a quantity that is increasing we want to measure how fast it is increasing. For a quantity that is decreasing we want to measure how fast it is decreasing. An investment that is growing quickly is better than one that is growing slowly (or decreasing!) and we would like a way to quantify this. We frequently use time as the independent variable when talking about rates, so in this section we will use the symbol t for our variable.

Let $A(t)$ denote a quantity whose value is changing with time. Perhaps the most straightforward way to measure the rate of growth (assume $A(t)$ is increasing), at some particular time, T, is to compute $A(T)$, compute $A(T+1)$, and take the difference, $A(T+1)-A(T)$. This gives us the *average rate of growth over one time unit* beginning at T. More generally, we could compute

$$\frac{A(T+\Delta T)-A(T)}{(T+\Delta T)-T} = \frac{A(T+\Delta T)-A(T)}{\Delta T}$$

This gives us the average rate of growth over the time interval $[T, T+\Delta T]$ per unit time since we have divided by the length of the time interval. You should recognize the above as a difference quotient very similar to what we used in computing the derivative. The *instantaneous rate of growth at time T is the limit of the average rate of growth over the interval* $[T, T+\Delta T]$ *as* ΔT *tends to zero*. Of course this is just the derivative! Thus the instantaneous rate of growth of $A(t)$ at $t = T$ is $A'(T)$.

One particular quantity whose rate of change is physically important is "distance from a reference point." The rate of change in this case is called *velocity*. For example, if $x(t)$ represents the position of a point on the x axis at time t, then $x'(t)$ represents the velocity of the point at time t.

The chain rule is often useful in computing rates. For example suppose the radius of a circle, $r(t)$, is increasing at a constant rate. This means that $r'(t)$ is a constant. Suppose this constant is 3, so $r'(t) = 3$ for all values of t How fast is the area of the circle changing when the radius is 5? The relationship between area and radius is given by

$$A(t) = \pi r^2(t)$$

Applying the chain rule we have

$$A'(t) = 2\pi r(t) r'(t)$$

Using the given values for r and r' we have

$$A' = 2\pi(5)(3) = 30\pi$$

You might note that we omitted writing the t dependence of the left side of the equation. The point here is that the rate at which the area is increasing when $r = 5$ depends only on the value of r and its derivative. It does not depend on t directly. We do not know the t value that corresponds to $r = 5$ and we don't need to know this t value.

Higher Derivatives and Rates

There is a nice geometric or physical interpretation of the second derivative of the distance function. It is *acceleration*. So if $x(t)$ represents position (or distance from 0) then

$$x(t) \text{ distance}$$
$$x'(t) = v(t) \text{ velocity}$$
$$x''(t) = v'(t) = a(t) \text{ acceleration}$$

Note that if velocity is negative then $x(t)$ is decreasing so the point is moving to the left. If acceleration is negative then the velocity is decreasing so the point is slowing down. The term *acceleration* is also frequently used in economics. For example, we can talk about "accelerating growth" to mean that the rate of increase of growth is increasing. The term "inflation is increasing" or "accelerating inflation" is a statement about the second derivative of prices.

Example. Suppose sales of a small business are increasing at a constantly increasing rate of 50 units per year per year. Current sales ($t = 0$) are 8,000 units, and sales are currently increasing at a rate of 100 units per year. When will sales reach 10,000 units?

Let S(t) represent sales at time t measured in years from right now (t=0). We don't have an expression for S(t) but we know $S''(t) = 50$. (Read the statement very carefully to understand where this comes from.) First let's find an expression for $S'(t)$. Since $S''(t)$ is constant, we must have $S'(t) = A + 50t$. We know $S'(0) = 100$ so $A = 100$ and $S'(t) = 100 + 50t$. Now what about $S(t)$? From the expression for $S'(t)$ we conclude $S(t) = k + 100t + 50(t^2/2)$ where k is a constant. Convince yourself this is true by taking the derivative. Now, since current sales are 800 units, we have k=8000. Hence the expression for $S(t)$ is

$$S(t) = 8000 + 100t + 50(t^2/2)$$

We want to know when sales will reach 10000 units.

$$S(t) = 8000 + 100t + 50(t^2/2)$$

$$10000 = 8000 + 100t + 50(t^2/2)$$

solve for t to get

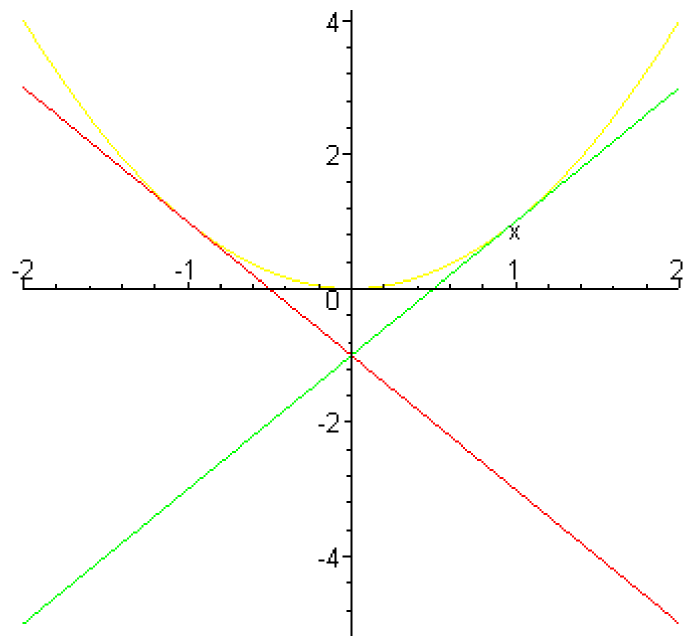
$t = 7.165$ (approximately)

Hence sales will reach 10000 approximately 7.165 years form now.

This example required you to realize that a function whose derivative is constant must be linear and a function whose derivative is linear must be quadratic. We will come to more complicated inferences of this type when we consider antiderivatives and integration. For now, you should at least be able to verify the conclusions obtained above.

9. Increasing and Decreasing Functions

The derivative can be a very useful tool in graphing functions. In this section we will use the derivative as an aid in graphing by determining regions for which the function is increasing or decreasing. As a start, recall that the derivative of a function $f(x)$ at a point corresponds to the slope of the tangent line to the graph of the function at that point. Thus, if the derivative is positive (negative), the slope of the tangent is positive (negative). If the derivative is zero, the slope of the tangent is zero and therefore the tangent is horizontal. Here is a plot of $f(x) = x^2$ showing the two tangents at $x = 1$ and $x = -1$.



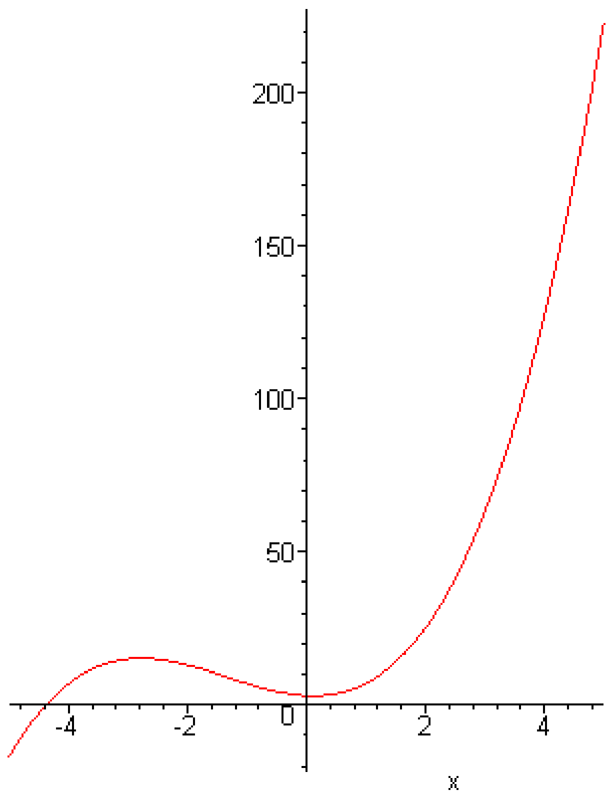
For which value of x will the tangent be horizontal?

Now suppose we do not have an explicit expression for a function, but we do know something about the derivative. For example, we might know where the derivative is positive and where it is negative. We can use this information to get a rough idea of the graph of the function. If we have specific values for the derivative at several points we can plot tangent lines at those points and sketch in the graph.

Example. Suppose $f(x)$ is a function with the following values of the derivative at five points.

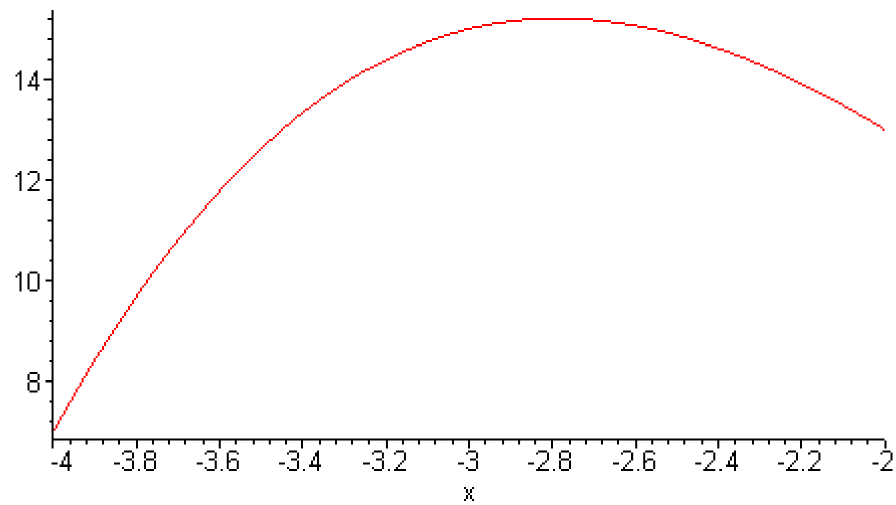
x	$f'(x)$
-3	2
-1	-6
0	-1
1	10
2	27

From this information we see that the graph will have a horizontal tangent between $x = -3$ and $x = -1$. We also see that the tangent gets very steep as we go from $x = 1$ to $x = 2$. Here is a sketch of a curve that has the specified values for the derivatives.



Keep in mind that positive slope of the tangent corresponds to an increasing function, negative slope of tangent corresponds to a decreasing function, and slope equal to zero corresponds to a horizontal tangent.

A point (x value) at which $f'x) = 0$ is called a *critical point* of a function. The corresponding value of the function at a critical point is called a *critical value*. **CAUTION**. The term critical point is used in two different ways. It may refer to the value of the argument of the function at which the derivative is zero or undefined (the x value) or it may refer to the value of the argument and the value of the function at that point (the (x, y) values. The context should make it clear if we are talking about the x value alone or the (x, y) values. These points are important because they indicate horizontal tangents and a possible maximum or minimum of the function. You should see that the function corresponding to the above graph has two critical points. In this case, these points correspond to a *local* maximum and a *local* minimum. We use the term local maximum (minimum) because if we look only at x values close to the critical points, then f will have a maximum (minimum) at the critical point. A graph concentrated around the critical point should make this clear.



In this case, it is clear from the graph that these local maximum and local minimum values are not the maximum and minimum values that the function attains. Note that the function is increasing for x values slightly to the left of the critical point and it is decreasing slightly to the right of the critical point. Hence the derivative will go from positive to negative (left to right) near the local maximum.

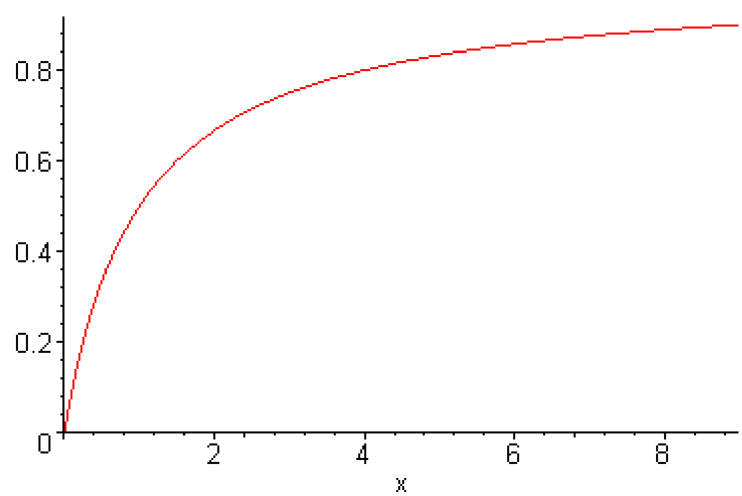
Note that a function may go from (for example) decreasing to increasing with the derivative never equal to zero. The absolute value function is an example ; the derivative (if it exists) is either -1 or $+1$. The derivative does not exist at zero.

At this point you should have a good understanding of the relationship between the sign of the derivative and the increasing/decreasing behavior of a function. In particular, you should be able to give an example (by a graph) of a function with specified signs of the derivative.

10. Concavity

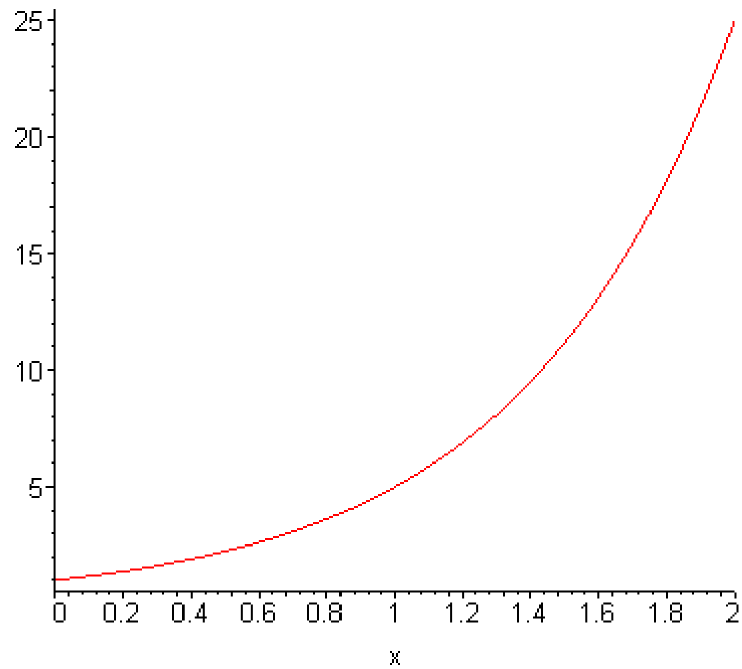
Concavity is an important concept in applications. Here are two examples from economics.

Decreasing Marginal Return. Suppose we invest resources in production of a product. A common phenomenon is decreasing return on additional investment. In other words, when we first begin investing a resource, we see a large return for each unit of resource invested. But, as we invest more and more of the resource, each additional unit of resource does not result in as much additional production as the previous unit of resource. How does this show up mathematically? Suppose $P(r)$ denotes production as a function of the amount of resource, r. As r increases, P also increases. This means that $P'(r) > 0$. Recall that $P'(r)$ measures the marginal return. In other words, it measures how much production increases per unit increase in resource, r. Hence decreasing marginal return means that $P'(r)$ is decreasing. This means that the *derivative* of $P'(r)$ is negative. But the derivative of $P'(r)$ is just $P''(r)$. Hence *decreasing marginal return* corresponds to $P''(r) < 0$. Here is the graph of a function with this property.



A function with this property is said to be *concave*. The term *concave down* is also sometimes used instead of simply concave.

Increasing Marginal Return Now we consider a situation with marginal return increasing. This occurs, for example, with deposits earning interest. The larger the amount in the account, the faster the account grows. Suppose $A(t)$ denotes the amount in an account at time t. The account earns interest so $A(t)$ is increasing with t. Hence, $A'(t) > 0$. If the rate of increase also increases as A and t increase, then $A'(t)$ is an increasing function, Hence the derivative of $A'(t)$ is positive, or in other words, $A''(t) > 0$. Here is a graph of a function of this type.



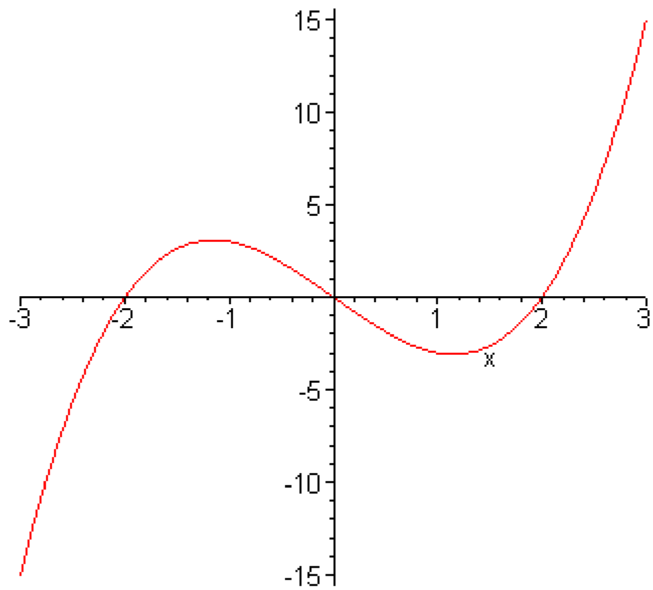
A function with this property is said to be *convex*. The term *concave up* is sometimes used instead of convex.

We summarize the relationship between the second derivative of a function, $f(x)$, and the shape of the graph.

Second Derivative	Shape
$f''(x) > 0$	convex
$f''(x) < 0$	concave

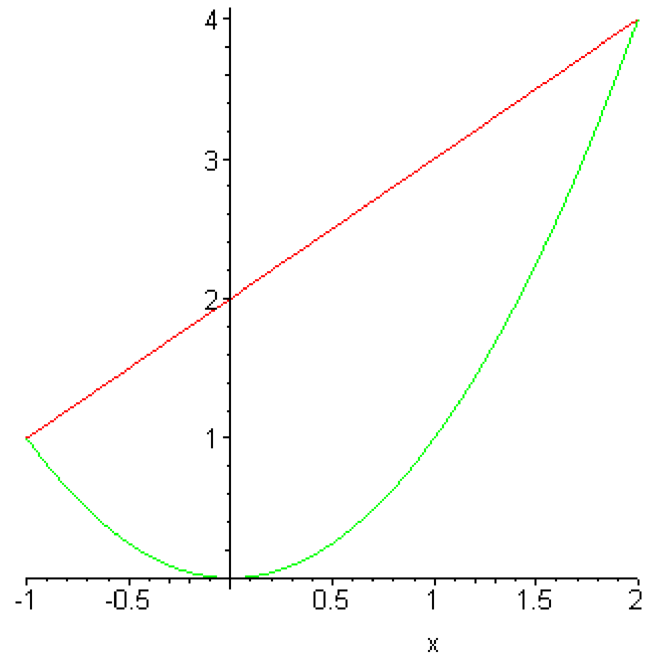
One must keep in mind that the relationship between the sign of the second derivative and the convexity/concavity of the graph is a local property. In other words, the second derivative of a function may be negative in some regions of the domain and positive in other regions. An example is

$f(x) = x^3 - 4x$



A computation gives $f''(x) = 6x$. Hence for $x < 0$, $f''(x) < 0$, and for $x > 0$, $f''(x) > 0$. Thus for $x < 0$ the graph is concave, and for $x > 0$ the graph is convex.

For a continuous function it is possible to give a characterization of convexity and concavity that does not depend upon the second derivative (which may not exist in many cases). Note that if we select two points on the graph of a function which is convex, and connect the two points by a line segment, then the line segment lies on or above the graph of the function.



This property can be used to define convexity and concavity. It has the advantage that is applies even in case the second derivative does not exist. For example, using this concept of convexity, the absolute value function, $|x|$, is seen to be convex. The first two examples of this section introduced convexity and concavity through the example of marginal return. If we have only marginal return data at a discrete set of points, and we do not have an explicit expression for the return or the marginal return, we can still get an idea of the convexity/concavity properties of the return by using this concept of convexity. For example, suppose we have the following set of data for the quantity $S(t)$ as a function of t.

t	$S(t)$
1	1.3
2	3.8
3	9.2

Do you expect $S(t)$ to be concave or convex for t values from 1 to 3?

The first and second derivatives together (or equivalently the properties of increasing/decreasing and convex/concave) give two powerful tools that help us graph a function. We compute the first and second derivatives, and determine regions for which the derivatives are positive and negative. We can then use this information, together with the values of the function at a few points to get a rough idea of the graph. Here is a summary for the function $f(x)$.

f'	f''	Shape	Example
$+$	$+$	increasing and convex	$f(x) = x^2$ for $x > 0$
$+$	$-$	increasing and concave	$f(x) = \sqrt{x}$ for $x > 0$
$-$	$+$	decreasing and convex	$f(x) = 1/x$ for $x > 0$
$-$	$-$	decreasing and concave	$f(x) = -x^2$ for $x > 0$

We suggest you graph each of the examples to be sure you understand them.

11. Optimization

One of the fundamental and most common applications of basic calculus is optimization. In other words, one would like to find the value of a controllable quantity to give an optimum (maximum or minimum) result. For example, suppose we can vary the number of employees at a particular facility. We might be interested in determining the number of employees that will maximize our profit or the number of employees that will minimize our cost. Note that the answer to the second problem is not necessarily zero as zero employees may result in a lost opportunity or lost profit cost.

The previous sections on increasing/decreasing functions and convexity/concavity provide the necessary tools to solve many optimization problems. In many cases the most difficult part of an optimization problem is the translation from a written description of the problem to a mathematical description of the problem. It requires a great deal of practice and experience to be able to perform this translation quickly and effectively. Hence, the only way to learn this material is to work lots of examples. Here is one.

Inventory Control Problem. This example is used frequently in finding an optimal ordering policy for a product. Suppose a product is sold at a steady rate. The seller has three costs associated with stocking the product. The costs are the purchase cost, a storage cost, and an order cost. Suppose the product is sold at a steady rate of 100 units per month, and each unit costs \$50. Each time the seller orders, a fixed cost of \$20 is incurred. The storage cost is \$10 per unit per month. Suppose the seller places an order for Q units each time an order is placed. The units are then sold at the constant rate until they run out. A new order for Q units is then placed. We want to find the optimal order policy. In other words, determine the quantity Q that should be ordered each time an order is placed to minimize costs. We would also like to know the time between the placement of orders.

The first step is to write down an expression for the total cost per unit time (in this case one month). Let's call this cost C. We have

$$C = O + S + P$$

O is the ordering cost

S is the storage cost

P is the purchasing cost

First we determine O , the ordering cost. Let N be the number of orders we place per month. Since we order a quantity Q each time we order, and we need 100 units per month we must have

$$NQ = 100$$

or

$$N = \frac{100}{Q}$$

Now since we have an ordering cost of $20 each time we order, we have a total ordering cost of

$$O = \$20 \times \frac{100}{Q} \text{ dollars per month}$$

Next we determine the storage cost S. Since we order a quantity Q and we sell the product at a steady rate until we run out, we store an average of $Q/2$ units per order cycle. The same process repeats itself. At the beginning of a cycle we have Q units in stock and at the end of the cycle we have no units in stock. Hence on the average we have $Q/2$ units in stock. Hence we finally get as the storage cost per month

$$S = \$10 \times \frac{Q}{2}.$$

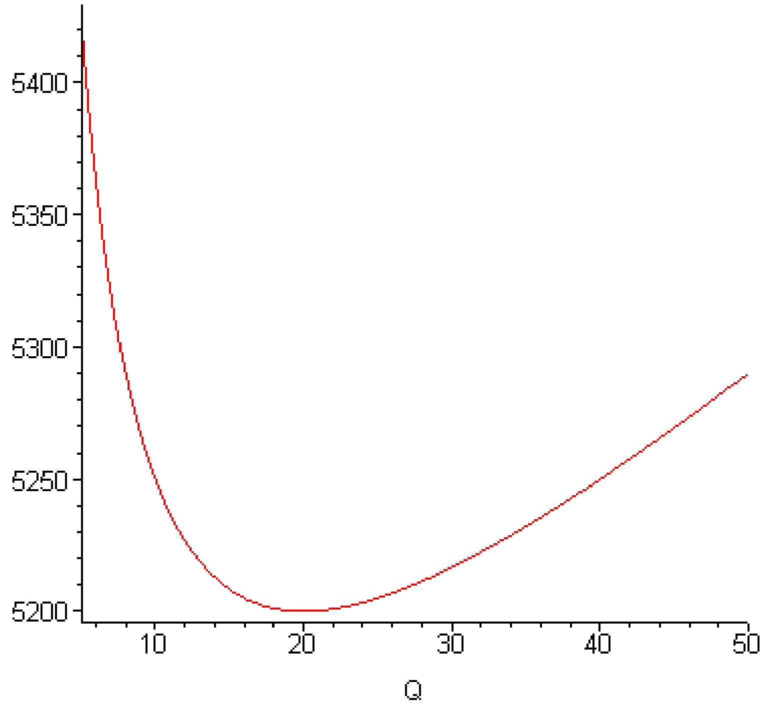
Finally we compute the purchase cost. Since each unit costs $50 and we need 100 units per month, our total purchase cost per month is

$$P = \$50 \times 100$$

We can now write down the total cost C per month as a function of Q.

$$C = (20 \times \frac{100}{Q}) + (10 \times \frac{Q}{2}) + (50 \times 100)$$

where we have omitted the $ signs. At this point, we might be tempted to simplify the arithmetic. However, there is an advantage in leaving the expression for C in this form since we can easily see where each of the constants comes from. Here is a plot of C as a function of Q.



The next step is to find the value of Q that minimizes cost. Of course we are looking for a Q value greater than 0. The above plot gives us a pretty good idea of the value we expect as a minimizer. To find the minimizing value of Q we differentiate the function $C(Q)$ and set the derivative equal to 0.

$$C'(Q) = (-20 \times \frac{100}{Q^2}) + (10 \times \frac{1}{2})$$

$$C'(Q) = (-20 \times \frac{100}{Q^2}) + (10 \times \frac{1}{2}) = 0$$

$$10 \times \frac{1}{2} = 20 \times \frac{100}{Q^2}$$

$$Q^2 = \frac{20 \times 100}{10 \times 1/2} = \frac{2 \times 20 \times 100}{10}$$

$$Q = \sqrt{\frac{2 \times 20 \times 100}{10}}$$

$$Q = 20$$

So the **Economic Order Quantity** or **EOQ** (as the quantity that minimizes cost is called) is 20 units. In other words, we should order 20 units each time we order and we will order 5 times per month. You should be able to provide an argument that this value actually corresponds to a minimum (and not a maximum). Notice from the cost function, that for very small values of Q the cost is very large.

This example was somewhat complicated, but it illustrates what is involved in solving *real world* business problems. Note that we solved the problem by breaking it up into several steps. It seems pretty clear that setting up the problem in mathematical form is the most difficult part. It should also be clear at this point how we can use the above calculations to develop a general expression for the EOQ that holds for arbitrary values of the costs. Let s denote the storage cost per unit time, K the ordering cost per order, and d the demand or number sold per unit time. Write down an expression for the EOQ (call it Q^*) in terms of $s, K,$ and d. Note that the price per unit plays no role in determining the EOQ.

12. Exponential and Logarithmic Functions

Exponential and logarithmic functions occur frequently in mathematical models for economics, finance, other social sciences as well as in the natural sciences. For example, the exponential function is used to model the growth of stock and bonds, radioactive decay, and growth of bacteria, and the logarithmic function is used to model the utility of wealth. A good understanding of these functions is essential for all fields that use mathematical models.

First we review some of the basic rules and notation for exponentials and logarithms. For a positive number a and a positive integer n, we have

$$a^n = a \times a \times a \times \bullet \bullet \bullet \times a$$

where the a on the right side of the equation appears n times. Similarly for n a positive integer

$$a^{1/n} = b$$

means

$$b \times b \times \bullet \bullet \bullet \times b = a$$

where the b appears n times on the left side of the equation. Putting these two notations together we write, for example,

$$a^{3/2} = [a^{1/2}]^3$$

We also write

$$a^{-1} = \frac{1}{a}$$

Now with the notational rule for composition for exponentials,

$$(a^p)^q = a^{pq}$$

and the first three notational rules, we can define exponentials for rational numbers (quotients of integers). When the exponent is not a rational number, a precise definition of the exponential is much more difficult and involves a limit in one form or another. At any rate the following notational rules hold even when the exponent is not a rational number.

$$
\begin{aligned}
(a^p)^q &= a^{pq} \\
a^p a^q &= a^{p+q} \\
\frac{1}{a^p} &= a^{-p} \\
\frac{a^p}{a^q} &= a^{p-q} \\
a^0 &= 1
\end{aligned}
$$

The logarithm is the inverse of the exponential. Hence we must have some way to indicate the base used in the exponential when we write a logarithm. For example

$$
\begin{aligned}
2^3 &= 8 \\
\log_2 8 &= 3
\end{aligned}
$$

express a relationship between the numbers 2,3 and 8 in two different but equivalent ways. The second equation requires a subscript on the $\log$ to indicate the base. If the base is 10 we frequently omit the subscript. Hence

$$
\begin{aligned}
10^2 &= 100 \\
\log_{10}(100) &= \log(100) = 2
\end{aligned}
$$

There are notational rules for logarithms that correspond to the rules for exponents. We give examples using the base 10 logarithm.

$$
\begin{aligned}
\log(a^p) &= p\log(a) \\
\log(ab) &= \log(a) + \log(b) \\
\log(\frac{1}{a}) &= -\log(a) \\
\log(\frac{a}{b}) &= \log(a) - \log(b) \\
\log(1) &= 0
\end{aligned}
$$

There is one base for exponentials that has very nice properties of differentiation that we will see later. This base is the real number e which is approximately 2.71828. The number e is not a rational number; it can not be written as a fraction. (The number π is another example of an irrational). It is so important for applications that most calculators have a function that enables them to compute powers of e.

Perhaps the easiest way to define the number e is as a limit.

$$e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$$

The numbers n in the limit are taken to be positive integers. For example

$$(1 + \frac{1}{10000})^{10000} \text{ is approximately } 2.718145927$$

The logarithm associated with the base e is called the *natural logarithm* and frequently written as $\ln$. We have for example

If $a = e^5$ then

$$\ln(a) = 5$$

Since the exponential and logarithm are inverses of one another, we have

$$e^{\ln(a)} = a \text{ and}$$

$$\ln(e^b) = b$$

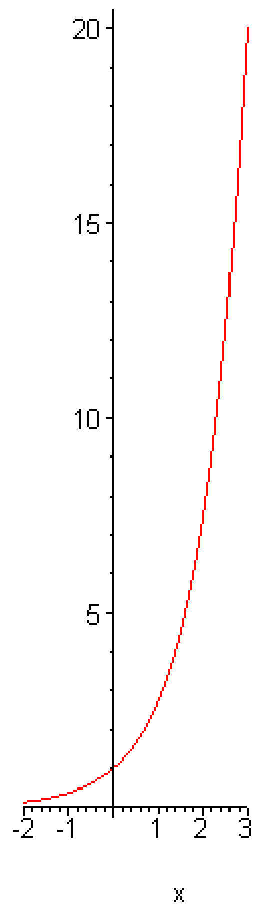
Now we are ready to discuss the exponential and logarithmic functions. We consider the function defined by exponentiating the number e.

$$f(x) = e^x$$

This is often written as

$$\exp(x) = e^x$$

Here is a plot of this function.



Because the function increases so rapidly, it might have been more convenient to have chosen different scales on the horizontal and vertical axes. However, we use the same scale to help explain the nice property of the exponential function when e is used as base. First note that

$$e^0 = 1$$

as one sees from the plot. Note also the slopes of tangents to the graph are always positive and increase as we move left to right on the graph. What about the slope of the tangent through the point $(0, 1)$? You should be able to see that the slope of the tangent through this point would be fairly close to 1. In fact it is 1. The important property of the exponential function with respect to differentiation can almost be seen from the graph. That property is

$$\text{If } f(x) = \exp(x) = e^x \text{ then}$$
$$f'(x) = \exp(x) \text{ or}$$
$$\frac{d}{dx}[\exp(x)] = \exp(x)$$

In other words the exponential function is its own derivative!

Because of this property of the exponential function, it is very important that we be able to express other exponential functions in terms of the natural exponential. For example

$$2^x = (2)^x = (e^{\ln 2})^x = e^{(\ln 2)x}$$

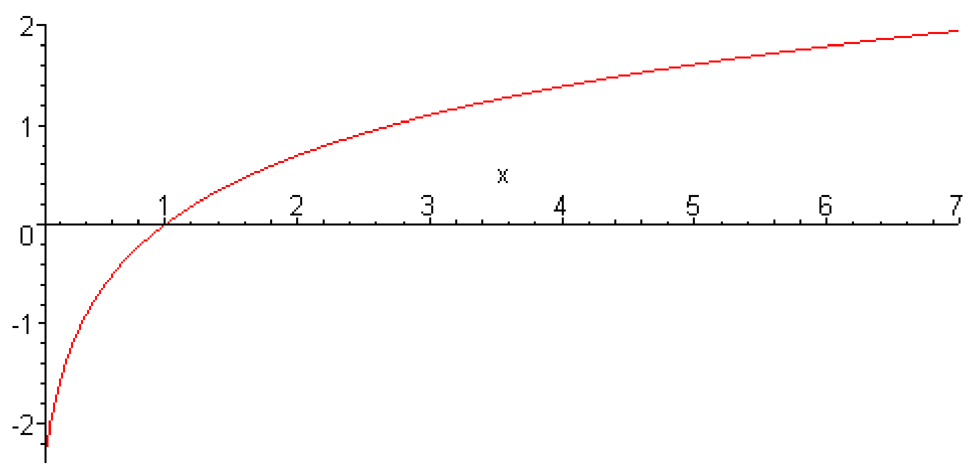
Note how we used the () in the first step to help us rewrite the exponential base 2 in terms of the natural exponential. You should be able to do this for any base. Now we can combine this with the chain rule to compute some more complicated derivatives. For example

$$\frac{d}{dx}[\exp(kx)] = k\exp(x)$$
$$\frac{d}{dx}[\exp(x^2)] = 2x\exp(x^2)$$

By applying the natural logarithm to a variable, we get the natural logarithm function

$$\ln(x)$$

Here is a plot.



Note that this function is defined only for positive values of x. Also note that the graph passes through the point $(1, 0)$ since $\ln(1) = 0$. Again notice that the slope of the tangent to the graph through the point $(1, 0)$

seems to be close to the value 1. In fact it is 1. Using the definition of the derivative as a limit, it is possible to compute the derivative of the function $\ln(x)$.

$$\frac{d}{dx}[\ln(x)] = \frac{1}{x}$$

You should at least be able to see that this is consistent with the graph. The derivative is positive for positive values of x and the derivative becomes a large positive number as x gets close to 0. Again, using the chain rule we can compute some more complicated derivatives involving the natural logarithm. for example

$$\frac{d}{dx}[\ln(3x+1)] = \frac{3}{3x+1}$$

Now let's turn to some applications. First we go back to the problem of compound interest. Recall that if we earn interest at a rate r compounded n times per year and we deposit \$1 at the beginning of the year, then at the end of the year we have an amount A with

$$A = (1+\frac{r}{n})^n$$

If we let

$$\frac{1}{k} = \frac{r}{n}$$

then this can be written as

$$A = (1+\frac{1}{k})^{kr} = [(1+\frac{1}{k})^k]^r$$

Now when n is very large, k is also very large (for a fixed value of the interest rate r). But for large values of k, the expression inside the square brackets,

$$(1+\frac{1}{k})^k$$

is very close to the number e. In fact,

$$\lim_{k\to\infty}(1+\frac{1}{k})^k = e.$$

Hence the amount in the account is approximately

$$A \approx e^r$$

For example if the interest rate is 7% or .07 then now matter how many times per year we compound the interest we will never have more than

$$e^{.07} \approx 1.072508181$$

with an initial deposit of \$1.

Continuously Compounded Interest. From the above example we see that the amount in an account that pays interest at rate r with an initial deposit of $1 will contain no more than the amount e^r at the end of one year. If the amount, $A(t)$, at time t in an account that pays interest at a rate r, with an initial deposit of K is given by

$$A(t) = Ke^{rt}$$

then we say that interest is *compounded continuously*. In fact, it is much easier to work with this model for growth of money in a bond than it is to work with the model of growth with interest compounded at discrete time intervals. Hence, for pricing and modeling many financial instruments, it is the model of choice.

13. Antiderivatives

In the previous sections we have seen how the derivative of a function can be used to give us information about the function itself. For example, the first derivative tells us when a function is increasing or decreasing, and the second derivative tells us if the function is convex or concave. It is natural to ask if the derivative tells us even more about the function. In other words, suppose we know the derivative of a function. How much do we know about the function itself? The answer is - almost everything. This is the idea of the *antiderivative*. We would like to be able to reconstruct the function itself from the derivative. In fact, to accomplish this we need one additional bit of information about the function - we need the value at a single point.

Here is the simplest example. Suppose the function $f(x)$ has the property that its derivative is 0 for all values of x. In other words

$$f'(x) = 0 \text{ for all } x$$

What can we say about $f(x)$? Since the tangent to the graph of $f(x)$ is horizontal everywhere, it should be intuitively clear that $f(x)$ must be constant. We cannot determine the constant without more information since all constant functions have the property that their derivatives equal 0 for all x values. However, if we know the value of $f(x)$ at a single point, then we know the value of this constant. We summarize this as follows.

$$\text{If } f'(x) = 0 \text{ for all } x \text{ and}$$
$$f(a) = A \text{ then } f(x) = A \text{ for all } x.$$

Here is a second example.

$$\text{Suppose } f(x) = k \text{ for all } x$$
$$\text{Then } f(x) = kx + c \text{ where } c \text{ is a constant.}$$

For example suppose

$$f'(x) = 3 \text{ for all } x \text{ and } f(2) = 5$$

$$\text{Then } f(x) = 3x + c \text{ for some constant } c$$

$$\text{Since } f(2) = 5 \text{ we have } f(2) = (3 \times 2) + c = 5$$

$$\text{or } 6 + c = 5 \text{ so } c = -1 \text{ and}$$

$$f(x) = 3x - 1$$

It should be clear at this point that finding the antiderivative of a function (in other words finding a function whose derivative equals some given expression) involves working backwards from the differentiation process. Hence the more differentiation examples you have worked, the easier it will be to find antiderivatives. It might help to recall the rules for differentiation since these will play a role in finding the antiderivatives. First,

$$\frac{d}{dx}(f(x) + g(x)) \ = f'(x) + g'(x)$$

Hence to find the antiderivative of a sum we simply find the antiderivative of each term in the sum and then add the antiderivatives. For example

$$\text{If } F'(x) = 3x + 7 \text{ and } F(1) = 2 \text{ then}$$

$$F(x) = 3[\frac{x^2}{2}] + 7x + c$$

We get this by noting that an antiderivative of x is $x^2/2$ and an antiderivative of 7 is $7x$. If we differentiate the above expression for $F(x)$ we do get $3x + 7$. Now we determine the constant c.

$$F(x) = 3[\frac{x^2}{2}] + 7x + c \text{ so}$$

$$F(1) = 3[\frac{1^2}{2}] + (7 \times 1) + c = 2$$

$$c = 2 - 3[\frac{1^2}{2}] - 7 = 1/2 - 7 = \frac{-13}{2} \text{ and}$$

$$F(x) = 3[\frac{x^2}{2}] + 7x + \frac{-13}{2}$$

A more complicated example involves the chain rule. Recall

$$\frac{d}{dx}\{F(u(x)\} \ = F'(u(x)) \ \times u'(x) = u'(x)F'(u(x))$$

Suppose we are given

$$H'(x) = (2x)(x^2 + 8)$$

If we let

$$u(x) = x^2 + 8 \text{ then } u'(x) = 2x$$

Therefore we can write

$$H'(x) = u'(x)F'(u(x)) \text{ if}$$

$$F'(u) = u \text{ since then } F'(x^2 + 8) = x^2 + 8$$

Now we must find $F(u)$. Since

$$F'(u) = u$$

$$F(u) = \frac{u^2}{2} + c$$

This gives

$$H(x) = F(u(x)) + c = \frac{[u(x)]^2}{2} + c$$

$$= \frac{[x^2 + 8]^2}{2} + c$$

You should check this by differentiating the expression for $H(x)$ to be sure you get $(2x)(x^2 + 8)$.

The above method of finding the antiderivative is known as *substitution* since we substituted the symbol u for the expression $(x^2 + 8)$. We did this since the derivative of u appeared in the original expression for $H'(x)$; it was $2x$. The method of substitution works when we are given the derivative of a composition.

Notation The following notation is used for the antiderivative of a function $f(x)$,

$$\text{Antiderivative of } f(x) \text{ is written as } \int f(x)dx$$

Note that the antiderivative is determined only up to a constant. If we are given the value of the antiderivative at some point, then this constant can be determined. Another term used (perhaps more commonly) for the antiderivative is the *integral* and the sign, $\int$, is an "integral" sign. Writing the term dx after the function $f(x)$ is customary for historical reasons that we shall explain in the next section. However the dx does serve the useful purpose of reminding us that the function expression is to be considered as a function of x. This may be important when the function contains one or more parameters. For example

$$\int (x + A)dx = ?$$

The symbol A is to be regarded as a constant parameter and the variable is x. A more subtle example is

$$\int (x + t)dt$$

In this case x is to be regarded as a constant parameter and the variable is t. You should evaluate both integrals above.

The dx plays a second useful notational (or heuristic) role with respect to actually evaluating integrals (or finding antiderivatives) by substitution. Let's rework the example with substitution (after making it slightly more complicated).

$$\int (2x)(x^2 + 8)^3 dx$$

Let

$$u = x^2 + 8$$

Again we make this substitution since the derivative of u appears after the integral sign. Now we write

$$du = (2x)dx$$

This "equation" can be given a precise mathematical meaning but you should realize that our discussion of the derivative does not give a precise definition. (We defined

$$\frac{du}{dx} = u'(x)$$

as a limit, but du by itself does not mean anything. At any rate the notational trick above, $du = (2x)dx$ makes the substitution method very algorithmic. Substituting u and du in the integral, integrating (or finding the antiderivative), and substituting back, gives

$$\int (2x)(x^2 + 8)^3 dx = \int (x^2 + 8)^3 (2x)dx = \int u^3 du = \frac{u^4}{4} + c = \frac{(x^2 + 8)}{4} + c$$

Most people find this notational trick very helpful in making substitutions for finding antiderivatives.

Finding antiderivatives is something of an art (or skill with algebra) and many methods have been devised to help. Many calculators are able to find the antiderivatives of a large number of functions, so perfecting this skill is not terribly important. However, you should be able to do simple examples such as those above and the problems from the homework.

It is very important to realize that it may be impossible to write down an explicit expression for the antiderivative of even fairly simple functions. For example,

$$\int \exp(x^2)dx = \int e^{(x^2)}dx$$

cannot be written in terms of functions involving exponentials and polynomials. This antiderivative plays an extremely important role in statistics.

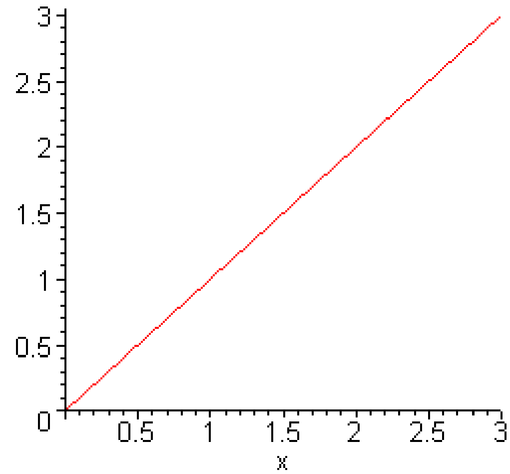
14. Integrals

In the last section we discussed the antiderivative and introduced the “integral sign”, $\int$, as notation for the antiderivative. So, in some sense we have already discussed ”integrals”. However the integral has a second interpretation, a geometric interpretation that was historically developed before the antiderivative. This section concerns the geometric meaning of the integral and will explain the integral notation.

The integral was developed to solve the problem of finding the area enclosed by a curve - a circle for example. We can algebraically compute areas only for very simple curves or figures such as rectangles, trapezoids, and right triangles. How might we find the area enclosed by a circle or a more complicated curve? One approach would be to fill up the area by many small rectangles, compute the area of each rectangle, and then add up all of the areas. Of course, this would give only an approximation since we cannot exactly fill the area inside the circle with small rectangles. However, we could get better and better approximations by making the rectangles smaller and smaller. With some good fortune we would be able to compute the limiting value of the total areas of the rectangles. This is exactly the idea of the integral.

The remarkable result of calculus is the equivalence of the problem of finding area enclosed by a curve and the antiderivative. This result is known as *the fundamental theorem of calculus*. We first consider an example where we can check the result by algebra. Let

$$f(x) = x$$



Suppose we want to find the area under the curve $f(x) = x$, above the x axis, and between the vertical lines, $x = 0$ and $x = 3$. In other words, we just want to find the area of the right triangle with nonhypotenuse side lengths 3 and 3. We know the answer by algebra. The area is

$$A = \frac{1}{2}(3 \times 3) = \frac{9}{2}$$

We could, using the idea of approximation mentioned above, get an approximate value by filling up, as best as possible, the triangle by rectangles. The base of each rectangle will lie on the x –axis, and the height of each rectangle will be as large as possible without going outside the triangle. For example, if the rectangles are placed side by side with base length 1, we get for the approximate area

$$A \approxeq (1 \times 0) + (1 \times 1) + (1 \times 2) = 3$$

If the base length is 1/2 we get for the approximate area

$$A \approxeq (.5 \times 0) + (.5 \times .5) + (.5 \times 1) + (.5 \times 1.5) + (.5 \times 2) + (.5 \times 2.5) = 3.75$$

If the base length is .1, we get for the approximate area

$$A \approxeq (.1 \times 0) + (.1 \times .1) + (.1 \times .2) + \bullet \bullet \bullet + (.1 \times 1.9) = 4.35$$

The approximations are getting closer and closer to the actual value of the area. In order to explain the integral sign notation and to give the precise expression of the area as a limit we introduce one additional bit of notation that is commonly used in many applications - the summation notation. Suppose the length of each base is some small value we call Δx. Then the total number of rectangles will be N with

$$N = \frac{3}{\Delta x}$$

The first rectangle will have height 0 and the last rectangle will have height $3 - \Delta x$. The approximate area will be

$$A \approx (\Delta x \times 0) + (\Delta x \times \Delta x) + (\Delta x \times 2\Delta x) + \bullet \bullet \bullet + (\Delta x \times (N-1)\Delta x)$$

There are N terms in the sum (since the first term is 0). The notation for this is

$$A \approx \sum_{k=0}^{k=N-1} \{k(\Delta x)(\Delta x)\}$$

Note that the value $k\Delta x$ is just the value of the function $f(x) = x$ evaluated at the point $k\Delta x$. Therefore we could write the above sum as

$$A \approx \sum_{k=0}^{k=N-1} f(k\Delta x)(\Delta x)$$

The value $k = 0$ corresponds to $x = 0$ and $k = N - 1$ corresponds to $x = 3 - \Delta x$. Now we can write the actual area as a limit.

$$A = \lim_{\Delta x \to 0} \sum_{k=0}^{k=N-1} f(k\Delta x)(\Delta x) = \sum_{k=0}^{k=N-1} k\Delta x(\Delta x)$$

Note that as $\Delta x \to 0$ $N \to \infty$. In other words as the base length tends to zero, the number of rectangles tends to infinity.
In fact this is precisely the definition of the integral, and we write

$$\int_0^3 f(x)dx = \lim_{\Delta x \to 0} \sum_{k=0}^{k=N-1} f(k\Delta x)(\Delta x)$$

where $N = \dfrac{3}{\Delta x}$

The exact same definition is used when $f(x)$ is an arbitrary function.

In general it would be extremely difficult to write down the sum and compute the limit as above (although in the above case we could do this quite explicitly). However, this is often unnecessary since the fundamental theorem of calculus tells us that the integral defined as a limit can be computed from the antiderivative. In the above example we found the area between the x-axis and the graph of the function $f(x) = x$ for x between 0 and 3. In general we can use any values we like for the variable. In this case we write

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{k=1}^{k=N} f(x_k)(\Delta x)$$

Here, the points x_k are points on the x-axis with x_1 corresponding to the point a and x_N corresponding to the point $b - \Delta x$. (We could just as well let x_1 correspond to the point $a + \Delta x$ and x_N corresponding to the point b.) The number N equals the number of rectangles we use in an approximation. $N \times \Delta x = (b - a)$. The above expression for the integral can be modified in various ways. For example, it is not necessary to use rectangles with the length of the bases equal to one another, but for simplicity we will stick to the above form when possible. We incorporate the above expression for the integral and the integral notation for the antiderivative to give a statement of the Fundamental Theorem of Calculus

$$F(b) - F(a) = \int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{k=1}^{k=N} f(x_k)(\Delta x)$$

where

$$F'(x) = f(x)$$

In other words, to evaluate the area between the graph of $f(x)$ and the x-axis for x between a and $b,$ we compute the difference of the antiderivative of $f(x),$ namely $F(x),$ evaluated at a and $b.$

The above example with $f(x) = x,$ $a = 0,$ and $b = 3$ follows.

$$\int_0^3 f(x)dx = \int_0^3 xdx = F(3) - F(0)$$

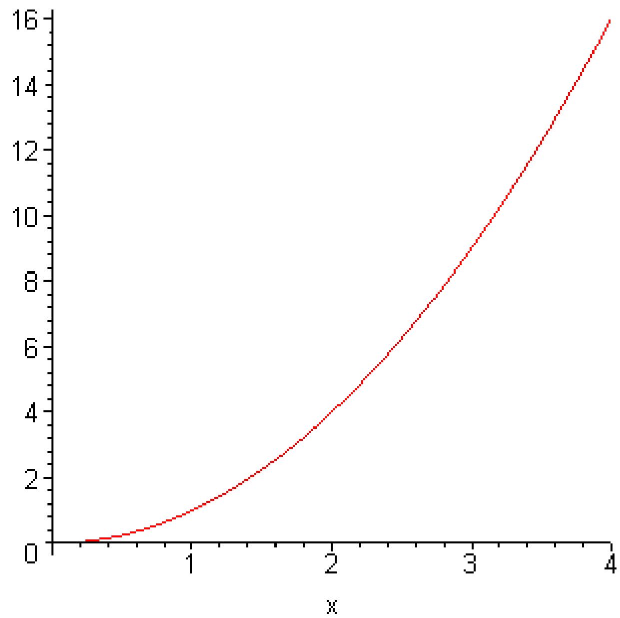
where

$$F(x) = \frac{x^2}{2} \text{ is an antiderivative of } f(x) = x$$

Hence

$$\int_0^3 f(x)dx = \int_0^3 xdx = F(3) - F(0) = \frac{3^2}{2} - \frac{0^2}{2} = \frac{9}{2}$$

Here is a more complicated example. Find the area between the graph of $f(x) = x^2$ and the x –axis for x between 1 and $4.$



$$\int_1^4 x^2 dx = F(4) - F(1) \text{ with } F(x) = \frac{x^3}{3}$$

$$\int_1^4 x^2 dx = F(4) - F(1) = \frac{4^3}{3} - \frac{1^3}{3} = \frac{63}{3} = 21$$

What about functions that take on negative values? For example, consider the function $f(x) = -x^2$ for $x \geq 0$. The antiderivative of $f(x) = -x^2$ is $F(x) = -x^3/3$. Hence if we use the above expression for the integral we have

$$\int_1^4 (-x^2) dx = F(4) - F(1) \text{ with } F(x) = \frac{-x^3}{3}$$

$$\int_1^4 x^2 dx = F(4) - F(1) = \frac{-4^3}{3} - \frac{-1^3}{3} = \frac{-63}{3} = -21$$

In other words we just get the negative of the previous result. For negative functions the integral will be negative. If a function is sometimes positive and sometimes negative on a certain interval, then integrating over that interval will result in some of the negative contribution canceling some of the positive contribution. For example

$$\int_{-3}^3 x dx = F(3) - F(-3) \text{ with } F(x) = \frac{x^2}{2}$$

$$\int_{-3}^3 x dx = F(3) - F(-3) = \frac{3^2}{2} - \frac{(-3)^2}{2} = 0$$

One must be very careful about signs when using the integral to compute area since area must have a positive value.

We observed above that multiplying a function by (-1) and integrating gives the negative of the integral of the function. This property can be extended to multiplication by any constant. The basic properties of derivatives transfer over to corresponding properties for integrals. The two most important properties are *linearity*.

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

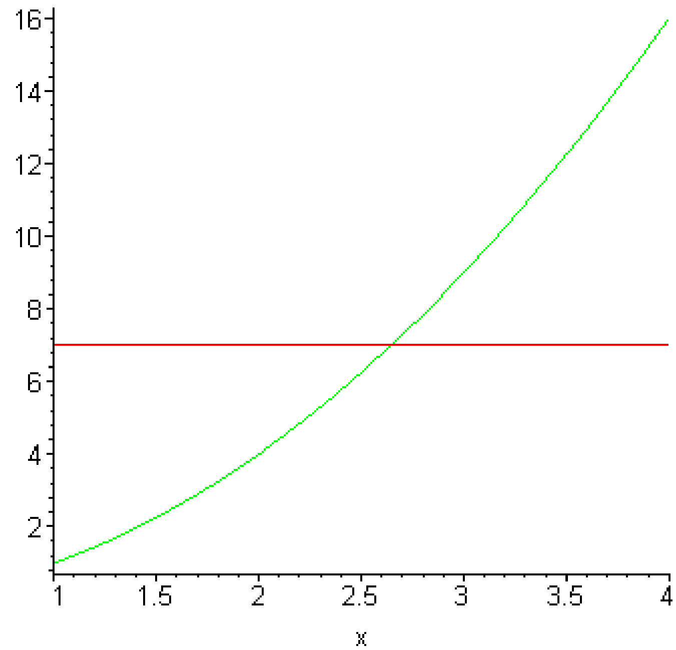
Another important additive property of the integral lets us break up integration over an interval into the sum of integrals over subintervals.

If $a < b < c$ then

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

As we pointed out in the section on antiderivatives, even very simple functions may not have antiderivatives that can be written in terms of elementary functions such as polynomials, exponentials, and so on. In this case it may be only possible to obtain approximations to the integral using the rectangle method.

Another important application of the summation approximation to the integral is the case of an unknown function. We first explain what we mean by the average of a continuous function. What value we would use for the "average" of the function $f(x) = x^2$ for x between 1 and 4? A picture will help.



The horizontal line is at height 7 and the area under the horizontal line equals the area under the graph of $f(x) = x^2$ for $1 \leq x \leq 4$. So the average is just the value of the integral divided by the length of the interval over which we integrate.

$$Avg(f(x)) = \frac{1}{b-a}\int_a^b f(x)dx$$

Now suppose we have collected data that represents the values of some unknown continuous function of time at various points in time. Here is an example of what such data might look like.

Time	0	1	3	4	7	8	10	13
Function Value	1.2	3.4	5.1	6.7	8.2	9.3	14.6	17.0

How would we find the “average value” of this function between times 0 and 13? It would not make sense to simply add all the function values and divide by the total number of values (Why not?). The total length of the interval of x values is 13. Hence if we approximate the integral of the unknown function using rectangles and then divide by 13 we will have an approximation to the average. To approximate the integral we have

$$\text{Approximate Integral} = (1-0)\times 1.2 + (3-1)\times 3.4 + (4-3)\times 5.1 + (7-4)\times 6.7 + (8-7)\times 8.2 + (10-8)\times 9.3 + (13-10)\times 14.6 = 103.8$$

We can therefore compute the approximate average of the function f.

$$Avg(f) \approx \frac{103.8}{13} = 7.98$$