The Banach-Tarski Paradox: How to Disassemble a Ball the Size of a Pea and Reassemble it into a Ball the Size of the Sun

Based on notes taken at a talk at Yale in the early 1970's (I do not remember who gave the lecture)

Carl W. Lee

February 26, 1992

1 Introduction

The following is taken from the foreword by Jan Mycielski of the book by Stan Wagon [3].

> This book is motivated by the following theorem of Hausdorff, Banach, and Tarski: Given any two bounded sets A and B in three-dimensional space $\mathbf{R}^3$, each having nonempty interior, one can partition A into finitely many disjoint parts and rearrange them by rigid motions to form B. This, I believe, is the most surprising result of theoretical mathematics. It shows the imaginary character of the unrestricted idea of a set in $\mathbf{R}^3$. It precludes the existence of finitely-additive, congruence-invariant measures over all bounded subsets of $\mathbf{R}^3$ and it shows the necessity of more restricted constructions such as Lebesgue's measure.

The following is taken from the preface of this book.

> While many properties of infinite sets and their subsets were considered to be paradoxical when they were discovered, the development of paradoxical decompositions really began with the formalization of measure theory at the beginning of the twentieth century. The now classic example (due to Vitali in 1905) of a non-Lebesgue measurable set was the first instance of the use of a paradoxical decomposition to show the nonexistence of a certain type of measure. Ten years later, Hausdorff constructed a truly surprising paradox on the surface of

the sphere (again, to show the nonexistence of a measure), and this inspired some important work in the 1920's. Namely, there was Banach's construction of invariant measures on the line and in the plane (which required the discovery of the main ideas of the Hahn-Banach Theorem) and the famous Banach-Tarski Paradox on duplicating, or enlarging, spheres and balls. This latter result, which at first seems patently impossible, is often stated as: It is possible to cut up a pea into finitely many pieces that can be rearranged to form a ball the size of the sun!

Their construction has turned out to be much more than a curiosity. Ideas arising from the Banach-Tarski Paradox have become the foundation of a theory of finitely additive measures, a theory that involves much interplay between analysis (measure theory and linear functionals), algebra (combinatorial group theory), geometry (isometry groups), and topology (locally compact topological groups). Moreover, the Banach-Tarski Paradox itself has been useful in recent work on the uniqueness of Lebesgue measure: It shows that certain measures necessarily vanish on the sets of Lebesgue measure zero.

2 The Banach-Tarski Paradox

Definition 1 Subsets A, B of Euclidean space are *congruent* if A and B can be made to coincide by rigid motions (combinations of translations and rotations). Notation: $A \cong B$.

$A \overset{n}{\cong} B$ if A can be partitioned into disjoint sets $A = A_1 \cup \cdots \cup A_n$, B can be partitioned into disjoint sets $B = B_1 \cup \cdots \cup B_n$, with $A_i \cong B_i$, $i = 1, \ldots, n$. In this case we say that A and B are *equidecomposable.*

$A \overset{n}{\preceq} B$ if $A \overset{n}{\cong} B'$ for some subset B' of B.

Lemma 1 *If A and C are disjoint, B and D are disjoint, $A \overset{m}{\cong} B$, and $C \overset{n}{\cong} D$, then $A \cup C \overset{m+n}{\cong} B \cup D$. Similarly with $\overset{m}{\preceq}$ and $\overset{n}{\preceq}$.*

Lemma 2 *If $A \overset{n}{\preceq} B \overset{m}{\preceq} C$ then $A \overset{mn}{\preceq} C$. Similarly with $\overset{m}{\cong}$ and $\overset{n}{\cong}$.*

Proposition 1 *If $A \overset{m}{\preceq} B \overset{n}{\preceq} A$ then $A \overset{m+n}{\cong} B$.*

Proof. This is a simple modification of the Schröder-Bernstein Theorem. Let $\phi : A \to B$ be the injective map realizing $A \overset{m}{\preceq} B$, and let $\psi : B \to A$ be the injective map realizing

$B \overset{n}{\preceq} A$. Given $a \in A$, consider the chain $\psi^{-1}(a), \phi^{-1}\psi^{-1}(a), \ldots$ as far as you can go. Call this the *ancestral chain* of a. Then $A = A_e \cup A_o \cup A_\infty$, where $A_e = \{a \in A : a$ has an even length ancestral chain$\}$, $A_o = \{a \in A : a$ has an odd length ancestral chain$\}$, and $A_\infty = \{a \in A : a$ has an infinite length ancestral chain$\}$. Similarly $B = B_e \cup B_o \cup B_\infty$. Then $\phi : A_e \cup A_\infty \overset{\text{onto}}{\to} B_o \cup B_\infty$ and $\psi^{-1} : A_o \overset{\text{onto}}{\to} B_e$. Partition $A_e \cup A_\infty$ into m pieces via ϕ and A_o into n pieces via ψ. This yields $A \overset{m+n}{\cong} B$. $\square$

Proposition 2 *Let $D \subset \mathbf{R}^2$ be the closed unit disk centered at the origin. Then $D \overset{n+2}{\cong} D \cup (n$ copies of $(0,1])$.*

Proof. Let A be the set of segments of the form $re^{i\theta}$, $0 < r \leq 1$, $\theta = 1, 2, 3, \ldots$, and let B be the complement of A with respect to D. Then $A \overset{n+1}{\cong} A \cup (n$ copies of $(0,1])$ (rotate A backwards through n radians), and $B \overset{1}{\cong} B$. $\square$

Proposition 3 *Let $S \subset \mathbf{R}^3$ be the unit sphere centered at the origin, D be a countable subset of S, and D' be the complement of D with respect to S. Then $S \overset{2}{\cong} D'$.*

Proof. Pick an axis which misses D and choose a rotation α about that axis such that $D, \alpha D, \alpha^2 D, \ldots$ are all disjoint. Let $A = D \cup \alpha D \cup \alpha^2 D \cup \cdots$ and B be the complement of A with respect to S. Then $A \overset{1}{\cong} \alpha D \cup \alpha^2 D \cup \cdots = \alpha A$ (rotate via α) and $B \overset{1}{\cong} B$. Then $S \overset{2}{\cong} D' = \alpha A \cup B$. $\square$

Theorem 1 *If $\overline{S}$ and $\overline{S}_1$ are disjoint unit balls, then $\overline{S} \overset{9}{\cong} \overline{S} \cup \overline{S}_1$; i.e., a unit ball can be disassembled into nine pieces and reassembled to make two unit balls.*

Proof. Let $S \subset \mathbf{R}^3$ be the unit sphere centered at the origin. Let α be the rotation by $180°$ about the z-axis and let β be the rotation by $120°$ about another axis such that $\alpha^2 = e$ (where e is the identity), $\beta^3 = e$, and no other relations hold between α and β. Why can we do this? Suppose β is a rotation about an axis in the xz-plane. Any relation between α and β is an algebraic equation involving the cosine of the angle between the two axes. Only a countable number of words formed from α and β is possible, so there are only a countable number of bad choices for the axis of β.

Let G be the group generated by α and β. Besides e, a typical element of G is $\alpha\beta^{\epsilon_1}\alpha\beta^{\epsilon_2}\cdots$ or $\beta^{\epsilon_1}\alpha\beta^{\epsilon_2}\alpha\cdots$ where $\epsilon_i = 1$ or 2. Let γ be an element of G other than e. Then γ is a rotation about some axis. Let D be the collection of all points of S on these axes. D is countable, and $\delta D = D$ if $\delta \in G$ (for if x is on the axis of γ, then δx is on the axis of $\delta\gamma\delta^{-1}$). On D', the complement of D with respect to S, no element is fixed by any element of G (except by e).

Given $x \in D'$, let $S_x = \{\gamma x : \gamma \in G\}$ (the orbit of x). For two elements $x, y \in D'$, either $S_x = S_y$ or $S_x \cap S_y = \emptyset$, so we have an equivalence relation on the points of D'. Pick one element from each equivalence class (invoking the Axiom of Choice) to form the set T. Then every element of D' is of the form γt for some $t \in T$, $\gamma \in G$, with γ and t uniquely determined by that element of D'.

Let
$$\begin{aligned} A &= \{\gamma t : t \in T,\ \gamma = e \text{ or } \gamma = \alpha\beta^{\epsilon_1}\cdots\}, \\ B &= \{\gamma t : t \in T,\ \gamma = \beta\alpha\beta^{\epsilon_2}\cdots\}, \\ C &= \{\gamma t : t \in T,\ \gamma = \beta^2\alpha\beta^{\epsilon_2}\cdots\}. \end{aligned}$$

Then $A \cup B \cup C = D'$, $\beta A = B$, $\beta B = C$, and $\beta C = A$. But $\alpha B \cup \alpha C \subset A$ (since you don't get e). So $B \cup C \overset{1}{\preceq} A$, $A \overset{1}{\cong} B$, $B \overset{1}{\cong} C$, and $C \overset{1}{\cong} A$. Then

$$\begin{aligned} A \cup (B \cup C) &\overset{2}{\preceq} B \cup (C) \\ &\overset{1}{\preceq} A. \end{aligned}$$

Similarly $A \cup B \cup C \overset{2}{\preceq} B$. So $D' \overset{2}{\preceq} A$ and $D' \overset{2}{\preceq} B$. Therefore $S = D' \cup D \overset{2}{\cong} D' \overset{2}{\preceq} A$, where the first congruence comes from Proposition 3. So $S \overset{4}{\preceq} A$.

Now let S_1 be another sphere congruent to but disjoint from S. Then $S_1 \overset{4}{\preceq} B$ and $S \cup S_1 \overset{8}{\preceq} A \cup B$. Let $\overline{S}$ and $\overline{S}_1$ be the solid balls bounded by S and S_1, respectively. Let $\overline{A}$ and $\overline{B}$ be the sets obtained by "filling in" the radii from A and B, respectively, into but not including the center of S; e.g, $\overline{A} = \{x \in \overline{S} : 0 < \|x\| \leq 1 \text{ and } x/\|x\| \in A\}$. Then $(\overline{S} \setminus \{O\}) \cup (\overline{S}_1 \setminus \{O'\}) \overset{8}{\preceq} \overline{A} \cup \overline{B}$, where O' is the center of $\overline{S}_1$.

Mapping O to O and O' to anywhere else in $\overline{S} \setminus (\overline{A} \cup \overline{B})$, we have that $\overline{S} \cup \overline{S}_1 \overset{9}{\preceq} \overline{S}$. But $\overline{S} \overset{1}{\preceq} \overline{S} \cup \overline{S}_1$. Hence $\overline{S} \cup \overline{S}_1 \overset{9}{\cong} \overline{S}$. Therefore one unit ball can be disassembled into nine pieces and reassembled into two unit balls. □

Corollary 1 *A ball the size of a pea can be disassembled into a finite number of pieces and reassembled into a ball the size of the sun.*

Proof. Let B be a ball the size of a pea and B' be a ball the size of the sun. Clearly $B \overset{1}{\preceq} B'$. On the other hand, by repeated application of the previous theorem, we can dissemble B

into a finite number of pieces and reassemble it into a large number of balls of the same size whose union contains a ball the size of the sun. So $B' \overset{n}{\preceq} B$ for some n. Therefore $B \overset{n+1}{\cong} B'$. $\square$

3 The Role of the Axiom of Choice

The following is taken from the book by Wagon [3].

> In their original paper..., Banach and Tarski anticipated the controversy that their counterintuitive result would spawn, and they analyzed their use of the Axiom of Choice as follows. "It seems to us that the role played by the axiom of choice in our reasoning deserves attention. Indeed, consider the following two theorems, which are consequences of our research:
>
> **I.** Any two polyhedra are equivalent by finite decomposition.
>
> **II.** Two different polygons, with one contained in the other, are never equivalent by finite decomposition.
>
> Now, it is not known how to prove either of these theorems without appealing to the Axiom of Choice: neither the first, which seems perhaps paradoxical, nor the second, which agrees fully with intuition. Moreover, upon analysing their proofs, one could state that the Axiom of Choice occurs in the proof of the first theorem in a more limited way than in the proof of the second."
>
> Thus Banach and Tarski pointed out that if AC is discarded, then not only would their paradox be lost, but also the result that such paradoxes do not exist in the plane. The last sentence of the excerpt refers to the fact that statement II uses choices from a larger family of sets than does statement I. But Corollary 13.9, which was proved by A. P. Morse...in 1949, shows that statement II is proved in ZF, and is therefore not relevant to a discussion of AC. The exact role of AC can therefore be loosely summarized as follows. It is necessary to disprove the existence of various invariant measures on $\mathcal{P}(\mathbf{R}^n)$ and to construct such measures, but it is not necessary to disprove the existence of paradoxes.

4 Solution to Tarski's Circle-Squaring Problem

The following review by the paper of Laczkovich is by Stan Wagon and appears as MR 91b#51034.

This remarkable paper [2] provides a surprising solution to a 60-year-old open problem of A. Tarski. The Banach-Tarski paradox does not exist in the plane because of Banach's proof that Lebesgue measure extends to an isometry-invariant, finitely additive measure on all subsets of $\mathbf{R}^2$. This result depends heavily on the amenability of the group of plane isometries. Banach's result implies that equidecomposable measurable sets must have the same measure. The question whether the circle (with interior) is equidecomposable with a square of the same area—Tarski's circle-squaring problem—had seen very little progress since its formulation in 1925.

The paper under review provides a most surprising solution to the problem. The author proves that the circle and square are equidecomposable using translations only. The paper begins with a detailed study of the notion of translation-equidecomposability for subsets of $\mathbf{R}$, and then moves to the situation in $\mathbf{R}^2$. En route to the main theorem, a new result for polygons is obtained, namely that any polygon is translation-equidecomposable to the square of the same area. Even for the case of an isosceles right triangle this is a new and difficult result.

The methods of proof are diverse, using deep ideas from number theory, in particular, the theory of uniform distribution of sequences.

References

[1] A. K. Dewdney, A matter fabricator provides matter for thought, *Scientific American*, **260** (1989), 116–119.

[2] M. Laczkovich, Equidecomposability and discrepancy; a solution of Tarski's circle-squaring problem, *J. Reine Angew. Math.*, **404** (1990), 77–117.

[3] S. Wagon, *The Banach-Tarski Paradox*, vol. 24 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge-New York, 1985.