History of Mathematics #2

1. Before Tuesday, January 24

   (a) Read Dunham Chapter 2, MTA Sketches 12, 14, 19, and Boyer Chapters 6–7, 23–24.

   (b) Go to the Forum "Axiomatics" and make at least two substantive contributions and at least two substantive responses to others' postings. First reread the following extract from my musings on my axiomatic journey that I presented in the summer geometry course. Write about the following: On which level do you place yourself? On what level should/can students be in various points in the K–16 timeline? How does this relate to the paragraphs near the end of Sketch 14 of MTA: "In the 20th century. . . pressed for time."? What does this have to do with the van Hiele levels?

   Extract:

   I believe that there are several different levels at which people (and mathematicians) may think about axiomatic systems. Let me elaborate a bit with respect to views of geometry.

   Levels:

   1. We have an image in our minds of geometrical objects, and we regard geometry as a (large) collection of facts and properties, not necessarily organized in any particular way.
   2. We have an image in our minds of geometrical objects, and we organize the facts from simplest to more complicated, with later facts provable from earlier facts. The simplest facts are regarded as "self-evident" and therefore exempt from proof.
   3. We have an image in our minds of geometrical objects, and we organize facts as in (2), referring to the simplest, unproven facts, as the axioms. We recognize that despite our mental image, we cannot use any properties in our proofs that are not derivable from the axioms.
   4. We have an image in our minds of geometrical objects, and we organize facts as in (3). We further recognize that despite our mental image, objects and relations specified in the axioms (such as "point", "line", "incidence", "between") are truly undefined, and that therefore in any other model in which we attach an interpretation to the undefined objects and relations for which the axioms hold, all subsequent theorems will hold also.

5. We have an image in our minds of geometrical objects, and we organize facts as in (4). But we further become familiar with and work with alternative models, and models of alternative axiom systems.
6. We have an image in our minds of geometrical objects, and we organize facts as in (5). But we fully recognize that all proofs in an axiom system are completely independent of any image in anyone's mind. (If we receive a set of axioms from an alien race about its version of geometry, we realize that we can prove the theorems without knowing what is in the minds of the aliens.)
7. We regard the formal system of axioms and theorems as all that there is—there is "nothing more out there" in terms of mathematical reality. (The aliens may in fact have nothing in their heads but operate formally with the symbols and procedures of formal logic.)

I distinctly remember the struggle I had in high school of trying to understand the teacher's explanation of levels (3) and (4), but I don't believe I really understood levels (4) and (5) until college. I believe that I presently operate in practice in levels (5) and (6). Computer automated proof systems (but not necessary those who use them) operate at level (7).

(c) Think about the following questions for discussion at the Centra session:

i. Why mention I Kings 7:23 on page 30 of Dunham?

ii. Study some of the statements of Euclid's *Elements* on the website http://aleph0.clarku.edu/~djoyce/java/elements/elements.html, thinking about the wording and meaning, and studying the diagram. Does having a "dynamic" (draggable) diagram help in your understanding?

iii. Look at some of the proofs of the propositions not covered in Dunham.

iv. Illustrate some of the statements of hyperbolic geometry using the web resource NonEuclid, http://cs.unm.edu/~joel/NonEuclid. For example, create a triangle and measure its angles.

v. Find a list of Hilbert's axioms for geometry. Find a list of the SMSG axioms for geometry (these are the ones I used in my 10th grade geometry course). Look at the list of axioms given in the geometry book by Kay that we used in the summer course. Compare these various lists with with those of Euclid.

vi. Find a list of axioms for:

A. Set theory (Zermelo-Fraenkel)

B. The natural numbers (Peano)

C. The real numbers (and why don't the rational numbers satisfy all of them?)

D. Groups

E. Vector spaces

Name some college level mathematics courses that typically begin with a set of axioms for some collection of objects.

vii. Look at the material related to the Pythagorean Theorem on the website http://nlvm.usu.edu/en/nav/vlibrary.html.

viii. MTA Sketch 12 comments on lyrics from the *Pirates of Penzance* mentioning the Pythagorean Theorem. Find these lyrics.

ix. When the Scarecrow discovered he had a brain at the end of the *Wizard of Oz*, he quoted the Pythagorean Theorem. Or did he? Yes, even this can be found on the web!

x. Study the generalization of the Pythagorean Theorem involving similar figures, in Sketch 12 of MTA. (This is one of my favorite proofs.) This is Euclid's proposition VI.31.

xi. At the end of Sketch 12 of MTA, the Pythagorean Theorem is related to the distance formula. Is this a standard connection typically made in High School?

xii. Which basic trigonometric identity is actually a statement of the Pythagorean Theorem? (From MTA, Expanded Edition.)

xiii. The compasses commonly used in school hold a fixed opening, allowing lengths to be transferred from place to place. A Euclidean compass did not do that; as soon as it was picked up, the size of the span was lost. However, Propositions 2 and 3 of Book I prove that lengths can be transferred, thereby legitimizing modern compasses. Study these propositions to see how Euclid does this. (From MTA, Expanded Edition.)

xiv. Try to carry out some of Euclid's constructions with Wingeom. Then try dragging around some of the points.

xv. Lewis Carroll, the author of *Alice's Adventures in Wonderland* and *Through the Looking-Glass* also wrote a book called *Euclid and His Modern Rivals*. What is this book about, and why did Lewis Carroll write it? (From MTA, Expanded Edition.)

2. Tuesday, January 24, 7–9 pm. Attend the Centra session to discuss the readings, forum, and comments and questions on the assigned homework due on Friday.

3. Homework problems due Friday, January 27, 11 pm, uploaded on the moodle site or submitted to the email address mathhist@ms.uky.edu.

(a) Make a directed graph to illustrate the logical connection among the postulates and early propositions of Book I in the following way: Make a vertex for each of the five postulates of Book I and for each of the first 16 propositions of Book I. Draw a directed arrow from vertex A to vertex B if A is used in the proof of B—these relations are explicitly given in the right hand margin of the proofs of each of the propositions on the website http://aleph0.clarku.edu/~djoyce/java/elements/Euclid.html. Does this help in your visualization? I believe it is possible to do this within powerpoint in such a way that if you drag around the vertices, the arrows will remain connecting the various pairs.

(b) Here is a way to generate some Pythagorean triples: Take any number a. If a is odd, square it, subtract 1, and divide by 2 to get b. Add 1 to b to get c. If a is even, square it, subtract 4, and divide by 4 to get b. Add 2 to b to get c. Why does this method generate Pythagorean triples? Find a Pythagorean triple that cannot be generated by this method.

(c) Take a standard deck of 52 cards. Imagine they are numbered 1 to 52 as follows: A through K of clubs, A through K of diamonds, A through K of hearts, and A through K of spades. Shuffle the deck thoroughly and pick a card. Determine its number. Now go to the website http://www.cut-the-knot.org/pythagoras/index.shtml and describe, in your own words, the proof of the Pythagorean theorem corresponding to your selected number. (Yes, I realize that there are actually 54 proofs on that website, so I guess you could include the two jokers as well.)

(d) In MTA Sketch 19 it is mentioned that the ratio of the circumference of a circle to its diameter is not π in non-Euclidean geometry. In fact, it is not even a constant. Illustrate this by considering a sphere of radius 1, marked with lines of latitude and longitude as on a standard globe.

    i. Look at the circle A of 45 degrees north latitude (45 degrees up from the equator). Regard the north pole N as its center, and a great circle arc from N to the circumference of the circle A as its radius. Calculate the ratio of the circumference to the diameter, giving an exact answer, and then evaluating it to several decimal places.

    ii. Repeat with the circle B that is 30 degrees north latitude.

    iii. Repeat with the circle C that is 0 degrees north latitude (i.e., the equator).

    iv. Try to develop a formula for the ratio of the circumference to the diameter of a circle of x degrees N latitude. How does this number compare to π?