The Mathematics of Voting

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MA 111

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Course Information

Text: Peter Tannenbaum, *Excursions in Modern Mathematics*, second custom edition for the University of Kentucky, Pearson.

Course Website:

`http://www.ms.uky.edu/~lee/ma111fa09/ma111fa09.html`

1.1 Preference Ballots and Preference Schedules

Who Wins the Election?

Below is a preference schedule giving the voter preferences in an election.

Number of Voters	8	4	3	2
1st choice	A	B	B	D
2nd choice	C	D	C	C
3rd choice	B	C	D	B
4th choice	D	A	A	A

Who should win the election? Why?

Who Wins the Election?

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- If everyone votes for their first choice, who gets the most votes? (This is the Plurality Method.)

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- If everyone votes for their first choice, who gets the most votes? (This is the Plurality Method.) A.
- If you do repeated voting for top choices, eliminating last place candidates until someone receives the majority of the votes, who wins? (This is the Plurality with Elimination Method.)

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- If you compare candidates head to head, which candidate does the majority of voters prefer to each of the others? (Such a candidate is called a Condorcet candidate.)

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Preference Ballots

Each voter prepares a preference ballot, in which the candidates are ranked in order of preference.

Example of a filled-out preference ballot:

Ballot	
1st	A
2nd	C
3rd	B
4th	D

Preference Ballots

Each voter prepares a preference ballot, in which the candidates are ranked in order of preference.

Example of a filled-out preference ballot:

Ballot	
1st	A
2nd	C
3rd	B
4th	D

We will assume that every voter can prepare a preference ballot with no ties. Such a ballot is called a linear ballot.

Preference Schedules

The preference ballots from an election are collected and tallied in a preference schedule.

Example of a preference schedule:

Number of Voters	8	4	3	2
1st choice	A	B	B	D
2nd choice	C	D	C	C
3rd choice	B	C	D	B
4th choice	D	A	A	A

Transitivity

It is natural to assume that a voter's preferences are transitive. That is to say, if a voter prefers candidate A over candidate B, and prefers candidate B over candidate C, then the voter prefers candidate A over candidate C.

Elimination of Candidate

It is also natural to assume that the relative preferences of a voter are not affected by the elimination of one or more of the candidates—the voter would just move the lower-ranked remaining candidates higher on the preference ballot.

Some Questions

- In an election with three candidates, what is the maximum number of columns possible in the preference schedule?

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- In an election with three candidates, what is the maximum number of columns possible in the preference schedule?
- In an election with four candidates, what is the maximum number of columns possible in the preference schedule?
- In an election with N candidates, what is the maximum number of columns possible in the preference schedule?

Some Questions

Answers:

- If there are three candidates, then there are 3 options for the first choice, then 2 options for the second choice, then only 1 option for the third choice, yielding $3 \times 2 \times 1 = 3! = 6$ possible ballot types.

Some Questions

Answers:

- If there are three candidates, then there are 3 options for the first choice, then 2 options for the second choice, then only 1 option for the third choice, yielding $3 \times 2 \times 1 = 3! = 6$ possible ballot types.
- If there are four candidates, then there are 4 options for the first choice, then 3 options for the second choice, then 2 options for the third choice, then only 1 option for the fourth choice, yielding $4 \times 3 \times 2 \times 1 = 4! = 24$ possible ballot types.

Some Questions

- If there are N candidates, then there are N options for the first choice, then $N-1$ options for the second choice, then $N-2$ options for the third choice, and so on, yielding $N \times (N-1) \times \cdots \times 3 \times 2 \times 1 = N!$ possible ballot types.

1.2 The Plurality Method

The Plurality Method

Plurality Method: The candidate with the most first-place votes wins.

Example

Using the Plurality Method, who wins the election with the following preference schedule?

Number of Voters	7	5	4	4
1st choice	D	B	B	C
2nd choice	A	D	A	D
3rd choice	C	C	D	B
4th choice	B	A	C	A

Why might some candidates object to this election result?

How Do Voters Really Feel?

Number of Voters	7	5	4	4
1st choice	D	B	B	C
2nd choice	A	D	A	D
3rd choice	C	C	D	B
4th choice	B	A	C	A

- How many voters prefer D to A?

How Do Voters Really Feel?

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1st choice	D	B	B	C
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- How many voters prefer D to A? 16 to 4

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- How many voters prefer D to B?

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- How many voters prefer D to A? 16 to 4
- How many voters prefer D to B? 11 to 9

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- How many voters prefer D to B? 11 to 9
- How many voters prefer D to C?

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- How many voters prefer D to A? 16 to 4
- How many voters prefer D to B? 11 to 9
- How many voters prefer D to C? 16 to 4

Terminology

- D is a Condorcet candidate because she is preferred by a majority of voters when compared head-to-head to each other candidate.

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- D is a Condorcet candidate because she is preferred by a majority of voters when compared head-to-head to each other candidate.
- B is a Plurality candidate because he received the most first-place votes.

Fairness Criteria

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Fairness Criteria

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- A voting method satisfies a fairness criterion if *every* election using that method conforms to the criterion.
  A voting method violates a fairness criterion if *at least one* election using that method does not conform to the criterion.

- Our example shows that, using the Plurality Method, it is possible for a Condorcet candidate to lose, so the Plurality Method violates the Condorcet Criterion.

Fairness Criteria

Method	Majority	Condorcet	Monotonicity	Ind. of Irr. Alt.
Plurality		Violates		
Borda Count				
Plurality with Elim.				
Pairwise Comparison				

Who wins using the Plurality Method?

Number of voters	25	11	9
1st choice	A	C	B
2nd choice	C	A	C
3rd choice	B	B	A

Who wins using the Plurality Method?

Number of voters	25	11	9
1st choice	A	C	B
2nd choice	C	A	C
3rd choice	B	B	A

A wins.

Who wins using the Plurality Method?

Number of voters	25	11	9
1st choice	A	C	B
2nd choice	C	A	C
3rd choice	B	B	A

A wins.

A is a majority candidate because he received a majority (more than half) of the first-place votes.

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- The Plurality Method satisfies the Majority Criterion.
- Why?

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Plurality	Satisfies	Violates		
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Some Questions

1. In an election with 64 voters, how many first-place votes does B need to be a majority candidate?

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2. In an election with 65 voters, how many first-place votes does B need to be a majority candidate?

3. In an election with 65 voters and 4 candidates, at least ________ first-place votes are needed to have a plurality.

Some Questions

Answers:

1. To have more than half of the 64 first-place votes, the candidate must receive more than $\frac{64}{2} = 32$ votes. That is, candidate B needs **33 votes** to be a majority candidate.

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Answers:

1. To have more than half of the 64 first-place votes, the candidate must receive more than $\frac{64}{2} = 32$ votes. That is, candidate B needs **33 votes** to be a majority candidate.

2. The candidate needs more than $\frac{65}{2} = 32.5$ first-place votes. Candidate B needs **33 votes** to be a majority candidate.

Some Questions

Answers:

1. To have more than half of the 64 first-place votes, the candidate must receive more than $\frac{64}{2} = 32$ votes. That is, candidate B needs **33 votes** to be a majority candidate.

2. The candidate needs more than $\frac{65}{2} = 32.5$ first-place votes. Candidate B needs **33 votes** to be a majority candidate.

3. The case to consider is if all 65 voters spread their first-place votes evenly among the 4 candidates. Since $\frac{65}{4} = 16.25$, a plurality candidate needs **at least 17 votes**.

Manipulating Elections

Example: The Marching Band Election

Number of voters	49	48	3
1st choice	R	H	F
2nd choice	H	S	H
3rd choice	F	O	S
4th choice	O	F	O
5th choice	S	R	R

Winner by the Plurality Method: R.

Manipulating Elections

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1st choice	R	H	F
2nd choice	H	S	H
3rd choice	F	O	S
4th choice	O	F	O
5th choice	S	R	R

Winner by the Plurality Method: R.
The three voters in the third column are disappointed that their last choice was chosen. But they can change the outcome if they do not vote according to their real preferences, and vote for H instead. This is called insincere voting.

Manipulating Elections

Can you think of real life examples where voters may not have voted sincerely?

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Can you think of real life examples where voters may not have voted sincerely?

One example was during the 2000 and 2004 presidential elections, during which Ralph Nader lost many votes from voters who actually favored him. In general, this kind of behavior can happen often when there are small parties and fringe candidates. The Plurality Method often encourages an entrenched two-party system.

Who Uses the Plurality Method?

Can you think of some real-life elections in which the Plurality Method is employed?

1.3 The Borda Count Method

The Borda Count Method

To determine the winner of an election using the Borda Count Method in an election with N candidates: Give 1 point for *last* place, 2 points for *second from last* place, and so on. Tally the points for each candidate. The candidate with the highest total is the winner, and is called the Borda winner.

The Borda Count Method

Example: The Math Club Election

Number of Voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

The Borda Count Method

Example: The Math Club Election

Number of Voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

Number of Voters	14	10	8	4	1
1st choice: 4 points	A	C	D	B	C
2nd choice: 3 points	B	B	C	D	D
3rd choice: 2 points	C	D	B	C	B
4th choice: 1 point	D	A	A	A	A

The Borda Count Method

Number of Voters	14	10	8	4	1
1st choice: 4 points	A:56 pts	C:40 pts	D:32 pts	B:16 pts	C:4 pts
2nd choice: 3 points	B:42 pts	B:30 pts	C:24 pts	D:12 pts	D:3 pts
3rd choice: 2 points	C:28 pts	D:20 pts	B:16 pts	C:8 pts	B:2 pts
4th choice: 1 point	D:14 pts	A:10 pts	A:8 pts	A:4 pts	A:1 pt

The Borda Count Method

Number of Voters	14	10	8	4	1
1st choice: 4 points	A:56 pts	C:40 pts	D:32 pts	B:16 pts	C:4 pts
2nd choice: 3 points	B:42 pts	B:30 pts	C:24 pts	D:12 pts	D:3 pts
3rd choice: 2 points	C:28 pts	D:20 pts	B:16 pts	C:8 pts	B:2 pts
4th choice: 1 point	D:14 pts	A:10 pts	A:8 pts	A:4 pts	A:1 pt

Tally the points:

A: 79 points

B: 106 points

C: 104 points

D: 81 points

Candidate B is the Borda winner—the winner by the Borda Count Method.

The Borda Count Method

Number of Voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

Who is the winner by the Plurality Method?

The Borda Count Method

Number of Voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

Who is the winner by the Plurality Method?

Answer: A.

The Borda Count Method

Another Example: The School Principal Election

Number of Voters	6	2	3
1st choice	A	B	C
2nd choice	B	C	D
3rd choice	C	D	B
4th choice	D	A	A

Who wins according to the Borda Count Method?

The Borda Count Method

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Who wins according to the Borda Count Method? Answer: B.
Does this bother you?

The Borda Count Method

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Who wins according to the Borda Count Method? Answer: B.

Does this bother you?

Who is the majority candidate? Answer: A.

The Borda Count Method

Another Example: The School Principal Election

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1st choice	A	B	C
2nd choice	B	C	D
3rd choice	C	D	B
4th choice	D	A	A

Who wins according to the Borda Count Method? Answer: B.
Does this bother you?
Who is the majority candidate? Answer: A.
Who is the Condorcet candidate? Answer: A.

The Borda Count Method

This example shows that the Borda Count Method violates both the Majority Criterion and the Condorcet Criterion.

Method	Majority	Condorcet	Monotonicity	Ind. of Irr. Alt.
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Borda Count	Violates	Violates		
Plurality with Elim.				
Pairwise Comparison				

The Borda Count Method

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- Although violations of the majority criterion can happen, they do not happen very often, and when there are many candidates such violations are rare.

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The Borda Count Method

So why use this method?

- Although violations of the majority criterion can happen, they do not happen very often, and when there are many candidates such violations are rare.
- Violations of the Condorcet criterion automatically follow violations of the majority criterion, since a majority candidate is automatically a Condorcet candidate. Why?

The Borda Count Method

Who uses this method (or some variation)?

The Borda Count Method

Who uses this method (or some variation)?

- Heisman Trophy winner
- NBA Rookie of the Year
- NFL MVP

Voting Methods Website

Here is a useful website for setting upon preference schedules and carrying out computations:
`http://www.cut-the-knot.org/Curriculum/SocialScience/SocialChoice.shtml`

Some Questions

In an election with 100 voters and 4 candidates:

- What is the minimum number of points a candidate can receive?
- What is the maximum number of points a candidate can receive?
- How many points are given out by one ballot?
- What is the total number of points given out to all four candidates?

Some Questions

Answers.

Some Questions

Answers. Note that the answer is not just a *calculation*, but includes *explanation and justification*.

Some Questions

- If a candidate were rated last by all 100 voters, he/she would receive 1 point from each voter, for a total of 100 points.

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- If a candidate were rated first by all 100 voters, he/she would receive 4 points from each voter (since there are four candidates), for a total of 400 points.

Some Questions

- If a candidate were rated last by all 100 voters, he/she would receive 1 point from each voter, for a total of 100 points.
- If a candidate were rated first by all 100 voters, he/she would receive 4 points from each voter (since there are four candidates), for a total of 400 points.
- Each voter effectively distributes 4 points for the first choice, 3 points for the second, 2 points for the third, and 1 point for the fourth, for a total of 10 points for the entire ballot.

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- If a candidate were rated last by all 100 voters, he/she would receive 1 point from each voter, for a total of 100 points.
- If a candidate were rated first by all 100 voters, he/she would receive 4 points from each voter (since there are four candidates), for a total of 400 points.
- Each voter effectively distributes 4 points for the first choice, 3 points for the second, 2 points for the third, and 1 point for the fourth, for a total of 10 points for the entire ballot.
- There are 100 ballots, with 10 points associated with each ballot, for a total of 1000 points.

Some Questions

- In an election with 29 voters and 3 candidates A,B,C, suppose A gets a total of 51 points and B gets a total of 62 points. Who wins the election?

Some Questions

- In an election with 29 voters and 3 candidates A,B,C, suppose A gets a total of 51 points and B gets a total of 62 points. Who wins the election?
- Answer: Each ballot distributes $1+2+3=6$ points. There are 29 ballots for a total of $29\times 6=174$ points. Candidates A and B together receive $51+62=113$ of these points, leaving the remaining points, $174-113=61$, for C. Therefore B wins the election.

1.4 The Plurality-with-Elimination Method

The Plurality-with-Elimination Method

To decide a winner by the Plurality-with-Elimination Method:

- Count first-place votes, as in the Plurality Method. If there is a *majority candidate*, he wins.
- If not, eliminate the candidate with the *fewest* first-place votes. Redistribute the eliminated candidate's first-place votes according to the rankings.
- Recount the first-place votes. If there is a majority candidate, she wins. If not, repeat the process of eliminating and redistributing votes until there is a winner.

The Plurality-with-Elimination Method

Example:

Number of Voters	5	3	5	2	1	4
1st choice	A	A	C	D	D	B
2nd choice	B	D	D	C	A	D
3rd choice	C	B	A	B	C	C
4th choice	D	C	B	A	B	A

The Plurality-with-Elimination Method

Example:

Number of Voters	5	3	5	2	1	4
1st choice	A	A	C	D	D	B
2nd choice	B	D	D	C	A	D
3rd choice	C	B	A	B	C	C
4th choice	D	C	B	A	B	A

How many votes are needed for a majority?

The Plurality-with-Elimination Method

Example:

Number of Voters	5	3	5	2	1	4
1st choice	A	A	C	D	D	B
2nd choice	B	D	D	C	A	D
3rd choice	C	B	A	B	C	C
4th choice	D	C	B	A	B	A

How many votes are needed for a majority? 11.

The Plurality-with-Elimination Method

Example:

Number of Voters	5	3	5	2	1	4
1st choice	A	A	C	D	D	B
2nd choice	B	D	D	C	A	D
3rd choice	C	B	A	B	C	C
4th choice	D	C	B	A	B	A

How many votes are needed for a majority? 11.

Count the first-place votes:

Candidate	A	B	C	D
First-Place Votes	8	4	5	3

The Plurality-with-Elimination Method

Example:

Number of Voters	5	3	5	2	1	4
1st choice	A	A	C	D	D	B
2nd choice	B	D	D	C	A	D
3rd choice	C	B	A	B	C	C
4th choice	D	C	B	A	B	A

How many votes are needed for a majority? 11.

Count the first-place votes:

Candidate	A	B	C	D
First-Place Votes	8	4	5	3

Candidate D is eliminated.

Round 2

Redistribute D's first-place votes:

Candidate	A	B	C
First-Place Votes	9	4	7

Round 2

Redistribute D's first-place votes:

Candidate	A	B	C
First-Place Votes	9	4	7

There is still no majority candidate, so B is eliminated.

Round 2

Redistribute D's first-place votes:

Candidate	A	B	C
First-Place Votes	9	4	7

There is still no majority candidate, so B is eliminated.
Since D has already been eliminated, B's four votes go to C:

Round 2

Redistribute D's first-place votes:

Candidate	A	B	C
First-Place Votes	9	4	7

There is still no majority candidate, so B is eliminated.
Since D has already been eliminated, B's four votes go to C:

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Round 2

Redistribute D's first-place votes:

Candidate	A	B	C
First-Place Votes	9	4	7

There is still no majority candidate, so B is eliminated.
Since D has already been eliminated, B's four votes go to C:

Candidate	A	C
First-Place Votes	9	11

Candidate C is the winner.

Fairness Criteria

Do you think the Plurality-with-Elimination Method satisfies the Majority Criterion?

Justify your answer: Explain why it does satisfy the criterion, or find a counterexample to show that it violates the criterion.

Fairness Criteria

Method	Majority	Condorcet	Monotonicity	Ind. of Irr. Alt.
Plurality	Satisfies	Violates		
Borda Count	Violates	Violates		
Plurality with Elim.	Satisfies			
Pairwise Comparison				

The Plurality-with-Elimination Method

Example: Which City Will Host the Olympics?

Number of Voters	7	8	10	4
1st choice	A	B	C	A
2nd choice	B	C	A	C
3rd choice	C	A	B	B

Using the Plurality-with-Elimination Method, who wins?

The Plurality-with-Elimination Method

Round 1:

Candidate	A	B	C
First-Place Votes	11	8	10

B is eliminated.

The Plurality-with-Elimination Method

Round 1:

Candidate	A	B	C
First-Place Votes	11	8	10

B is eliminated.

Round 2:

Candidate	A	C
First-Place Votes	11	18

C wins.

The Plurality-with-Elimination Method

Suppose there is a reelection, and the 4 voters in the last column move C up higher on their ballots. No harm in that, right? After all, C is already going to win.

The Reelection:

Number of Voters	7	8	14
1st choice	A	B	C
2nd choice	B	C	A
3rd choice	C	A	B

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Now who wins according to the Plurality-with-Elimination Method?

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The Reelection:

Number of Voters	7	8	14
1st choice	A	B	C
2nd choice	B	C	A
3rd choice	C	A	B

Now who wins according to the Plurality-with-Elimination Method? Answer: B.

The Monotonicity Criterion

The Monotonicity Criterion: Suppose candidate X is the winner of an election. If, in a reelection, the only changes in ballots are changes that favor X (and only X), then X should remain the winner.

The Plurality-with-Elimination Method violates the Monotonicity Criterion.

Fairness Criteria

Method	Majority	Condorcet	Monotonicity	Ind. of Irr. Alt.
Plurality	Satisfies	Violates		
Borda Count	Violates	Violates		
Plurality with Elim.	Satisfies		Violates	
Pairwise Comparison				

The Monotonicity Criterion

Explain why the Plurality Method satisfies the Monotonicity Criterion.

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Pairwise Comparison				

Fairness Criteria

Try exercise 33 on page 35 of the text.

Fairness Criteria

Try exercise 33 on page 35 of the text.

The Plurality-with-Elimination Method violates the Condorcet Criterion.

Method	Majority	Condorcet	Monotonicity	Ind. of Irr. Alt.
Plurality	Satisfies	Violates		
Borda Count	Violates	Violates		
Plurality with Elim.	Satisfies	Violates	Violates	
Pairwise Comparison				

Who Uses the Plurality-with-Elimination Method?

This method is often used when the a candidate must have a majority of votes to win the election.

- Local elections
- Choosing the host city of the Olympics
- Amendment 12 of the U.S. Constitution:

`http://www.usconstitution.net/const.html#Am12`

1.5 The Method of Pairwise Comparisons

The Method of Pairwise Comparisons

To carry out the Method of Pairwise Comparisons, every candidate is matched head-to-head against every other candidate. Each of these head-to-head matches is called a pairwise comparison. For each comparison and for each ballot, the vote goes to whichever candidate is listed higher on the ballot. The winner of a particular pairwise comparison is the candidate with the most votes. The winner of that pairwise comparison gets 1 point while the loser gets 0 points; in the case of a tie each candidate gets 1/2 point.

The winner of the entire election is the candidate with the most points after tabulating all pairwise comparisons.

The Method of Pairwise Comparisons

Example: The Math Club Election

Number of Voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

The Method of Pairwise Comparisons

Example: The Math Club Election

Number of Voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

A versus B: 14 votes to 23 votes. B wins and gets 1 point.
A versus C: 14 votes to 23 votes. C wins and gets 1 point.
A versus D: 14 votes to 23 votes. D wins and gets 1 point.
B versus C: 18 votes to 19 votes. C wins and gets 1 point.
B versus D: 28 votes to 9 votes. B wins and gets 1 point.
C versus D: 25 votes to 12 votes. C wins and gets 1 point.

The Method of Pairwise Comparisons

Example: The Math Club Election

Number of Voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

A versus B: 14 votes to 23 votes. B wins and gets 1 point.
A versus C: 14 votes to 23 votes. C wins and gets 1 point.
A versus D: 14 votes to 23 votes. D wins and gets 1 point.
B versus C: 18 votes to 19 votes. C wins and gets 1 point.
B versus D: 28 votes to 9 votes. B wins and gets 1 point.
C versus D: 25 votes to 12 votes. C wins and gets 1 point.

Final tally: A:0, B:2, C:3, D:1, so C is the winner.

Fairness of the Pairwise Comparison Method

- A majority candidate (if there is one) automatically wins every pairwise comparison. If there are N candidates, then the majority candidate gets $N - 1$ points. No other candidate gets $N - 1$ points. So a majority candidate will win the election. Therefore the Pairwise Comparison Method satisfies the Majority Criterion.

Fairness of the Pairwise Comparison Method

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- A Condorcet candidate (if there is one) also wins every pairwise comparison. So by the same reasoning, a Condorcet candidate will win the election. Therefore the Pairwise Comparison Method satisfies the Condorcet Criterion.

Fairness of the Pairwise Comparison Method

- A majority candidate (if there is one) automatically wins every pairwise comparison. If there are N candidates, then the majority candidate gets $N-1$ points. No other candidate gets $N-1$ points. So a majority candidate will win the election. Therefore the Pairwise Comparison Method satisfies the Majority Criterion.
- A Condorcet candidate (if there is one) also wins every pairwise comparison. So by the same reasoning, a Condorcet candidate will win the election. Therefore the Pairwise Comparison Method satisfies the Condorcet Criterion.
- It can also be shown that the Pairwise Comparison Method satisfies the Monotonicity Criterion.

Fairness of the Pairwise Comparison Method

Method	Majority	Condorcet	Monotonicity	Ind. of Irr. Alt.
Plurality	Satisfies	Violates		
Borda Count	Violates	Violates		
Plurality with Elim.	Satisfies	Violates	Violates	
Pairwise Comparison	Satisfies	Satisfies	Satisfies	

Independence-of-Irrelevant-Alternatives Criterion (IIA)

Example: LAXer's Draft Choice Election

Number of Voters	2	6	4	1	1	4	4
1st choice	A	B	B	C	C	D	E
2nd choice	D	A	A	B	D	A	C
3rd choice	C	C	D	A	A	E	D
4th choice	B	D	E	D	B	C	B
5th choice	E	E	C	E	E	B	A

IIA Criterion

Pairwise comparisons:
A versus B: 7 votes to 15 votes. B wins and gets 1 point.
A versus C: 16 votes to 6 votes. A wins and gets 1 point.
A versus D: 13 votes to 9 votes. A wins and gets 1 point.
A versus E: 18 votes to 4 votes. A wins and gets 1 point.
B versus C: 10 votes to 12 votes. C wins and gets 1 point.
B versus D: 11 votes to 11 votes. B and D each get 1/2 point.
B versus E: 14 votes to 8 votes. B wins and gets 1 point.
C versus D: 12 votes to 10 votes. C wins and gets 1 point.
C versus E: 10 votes to 12 votes. E wins and gets 1 point.
D versus E: 18 votes to 4 votes. D wins and gets 1 point.
A:3, B:2.5, C:2, D:1.5, E:1. A would be the winner.

IIA Criterion

Now suppose at the last minute C drops out of the race. Readjust the preference schedule by removing C and moving candidates up as necessary:

Number of Voters	2	6	4	1	1	4	4
1st choice	A	B	B	B	D	D	E
2nd choice	D	A	A	A	A	A	D
3rd choice	B	D	D	D	B	E	B
4th choice	E	E	E	E	E	B	A

IIA Criterion

Pairwise comparisons:
A versus B: 7 votes to 15 votes. B wins and gets 1 point.
A versus D: 13 votes to 9 votes. A wins and gets 1 point.
A versus E: 18 votes to 4 votes. A wins and gets 1 point.
B versus D: 11 votes to 11 votes. B and D each get 1/2 point.
B versus E: 14 votes to 8 votes. B wins and gets 1 point.
D versus E: 18 votes to 4 votes. D wins and gets 1 point.
A:2, B:2.5, D:1.5, E:0. B is now the winner.

IIA Criterion

The Independence-of-Irrelevant-Alternatives Criterion states that if candidate X is a winner of an election, and in a recount one of the *nonwinning* candidates withdraws or is disqualified, then X should still be a winner of the election.
Equivalently, if candidate X is a winner of an election and in a reelection another candidate that has no chance of winning (an "irrelevant alternative") enters the race, then X should still be the winner of the election.

The Method of Pairwise Comparisons violates the IIA Criterion.

IIA Criterion

Method	Majority	Condorcet	Monotonicity	Ind. of Irr. Alt.
Plurality	Satisfies	Violates		
Borda Count	Violates	Violates		
Plurality with Elim.	Satisfies	Violates	Violates	
Pairwise Comparison	Satisfies	Satisfies	Satisfies	Violates

Other Problems with the Pairwise Comparisons Method

Example: The Restaurant Election

Number of Voters	4	2	6	3
1st choice	A	B	C	B
2nd choice	B	C	A	A
3rd choice	C	A	B	C

In pairwise comparisons, A beats B, B beats C, and C beats A, so each candidate receives 1 point and there is a three-way tie.
Using the Plurality Method, C wins.
Using the Borda Count Method, A wins.
Using the Plurality-with-Elimination Method, B wins.

Number of Pairwise Comparisons

- If there are 3 candidates, how many pairwise comparisons must be made?

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AB BC
AC

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A total of $2 + 1 = 3$.

Number of Pairwise Comparisons

- If there are 3 candidates, how many pairwise comparisons must be made?

  AB BC
  AC

  A total of $2 + 1 = 3$.

- If there are 4 candidates, how many pairwise comparisons must be made?

Number of Pairwise Comparisons

- If there are 3 candidates, how many pairwise comparisons must be made?

AB	BC
AC	

A total of $2 + 1 = 3$.

- If there are 4 candidates, how many pairwise comparisons must be made?

AB	BC	CD
AC	BD	
AD		

Number of Pairwise Comparisons

- If there are 3 candidates, how many pairwise comparisons must be made?

  AB BC
  AC

  A total of $2 + 1 = 3$.
- If there are 4 candidates, how many pairwise comparisons must be made?

  AB BC CD
  AC BD
  AD

  A total of $3 + 2 + 1 = 6$.

Number of Pairwise Comparisons

If there are 10 candidates, how many pairwise comparisons must be made?

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$9 + 8 + 7 + \cdots + 3 + 2 + 1.$

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$9 + 8 + 7 + \cdots + 3 + 2 + 1.$

$$S = 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$$

Number of Pairwise Comparisons

If there are 10 candidates, how many pairwise comparisons must be made?

$9 + 8 + 7 + \cdots + 3 + 2 + 1.$

$$\begin{aligned} S &= 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 \\ S &= 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 \end{aligned}$$

Number of Pairwise Comparisons

If there are 10 candidates, how many pairwise comparisons must be made?

$9 + 8 + 7 + \cdots + 3 + 2 + 1.$

$$\begin{aligned} S &= 9+8+7+6+5+4+3+2+1 \\ S &= 1+2+3+4+5+6+7+8+9 \\ 2S &= 10+10+10+10+10+10+10+10+10 \end{aligned}$$

Number of Pairwise Comparisons

If there are 10 candidates, how many pairwise comparisons must be made?

$9 + 8 + 7 + \cdots + 3 + 2 + 1.$

$$
\begin{aligned}
S &= 9+8+7+6+5+4+3+2+1 \\
S &= 1+2+3+4+5+6+7+8+9 \\
2S &= 10+10+10+10+10+10+10+10+10 \\
2S &= 9 \times 10
\end{aligned}
$$

Number of Pairwise Comparisons

If there are 10 candidates, how many pairwise comparisons must be made?

$9 + 8 + 7 + \cdots + 3 + 2 + 1.$

$$
\begin{aligned}
S &= 9+8+7+6+5+4+3+2+1 \\
S &= 1+2+3+4+5+6+7+8+9 \\
2S &= 10+10+10+10+10+10+10+10+10 \\
2S &= 9 \times 10 \\
S &= \tfrac{9\times 10}{2} = 45.
\end{aligned}
$$

Number of Pairwise Comparisons

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$(N-1)+(N-2)+(N-3)+\cdots+3+2+1.$

Number of Pairwise Comparisons

If there are N candidates, how many pairwise comparisons must be made?
$(N-1)+(N-2)+(N-3)+\cdots+3+2+1.$

$$
\begin{aligned}
S &= (N-1)+(N-2)+(N-3)+\cdots+3+2+1 \\
S &= 1+2+3+\cdots+(N-3)+(N-2)+(N-1) \\
2S &= N+N+N+\cdots+N+N+N \\
2S &= (N-1)\times N \\
S &= \tfrac{(N-1)N}{2}
\end{aligned}
$$

Number of Pairwise Comparisons

- The sum of the numbers from 1 to $N-1$ equals $\frac{(N-1)N}{2}$.

Number of Pairwise Comparisons

- The sum of the numbers from 1 to $N-1$ equals $\frac{(N-1)N}{2}$.
- The sum of the numbers from 1 to L equals $\frac{L(L+1)}{2}$.