The Mathematics of Money

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Simple Interest

Compound Interest

Geometric Sequences

Deferred Annuities

Installment Loans

10.2 Simple Interest

The Time Value of Money

When you deposit $1000 into a savings account at the bank, you expect that amount to gain interest over time.
A year from now, you would have more than $1000.

In return for having access to the present value of your money, the bank increases the future value of the money by adding interest.

The Time Value of Money

If you take out a car loan for $10,000, you expect to pay it back with interest.
Suppose the total amount you repay over time is $12,000.

The present value is $P = \$10,000$.
The future value is $F = \$12,000$.

What determines the future value?

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Since the amount of interest should depend on the amount of the loan, we consider an interest rate.

The standard way to describe an interest rate is the annual percentage rate or APR.

Simple Interest

With simple interest, only the principal (the original money invested or borrowed) generates interest over time.

The amount of interest generated each year will be the same throughout the life of the loan/investment.

Example

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How much interest will be earned in one year?

$$\$1000 \cdot \frac{5}{100} = \$1000(0.05) = \$50.$$

Example

After one year, the bond will be worth

$$\$1000 + \$50 = \$1050.$$

After two years, the bond will be worth

$$\$1000 + \$50 + \$50 = \$1100.$$

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How much will the bond be worth after t years?

$$\$1000 + \$50t.$$

Simple Interest Formula

Remember that the annual interest was found by multiplying $\$1000 \times \frac{5}{100}$.

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In general, if the principal is P dollars and the interest rate is $R\%$, the amount of annual interest is

$$P\left(\frac{R}{100}\right)$$

$$\text{or } P \cdot r \text{ where } r = \frac{R}{100}.$$

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$$\text{or } P \cdot r \text{ where } r = \frac{R}{100}.$$

Over t years, the amount of interest accrued is

$$P \cdot r \cdot t.$$

Simple Interest Formula

Thus, the total future value will be

$$P + P \cdot r \cdot t$$

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$$P + P \cdot r \cdot t$$

If P dollars is invested under simple interest for t years at an APR of $R\%$, then the future value is:

$$\boxed{F = P\left(1 + r \cdot t\right)}$$

where r is the decimal form of $R\%$.

Using the Simple Interest Formula

Suppose you want to buy a government bond that will be worth $2500 in 8 years. If there is 5.75% annual simple interest on the bond, how much do you need to pay now?

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We know the future value $F = \$2500$ and we want to find the present value or principal P.

Solve for P:

$$\begin{aligned} 2500 &= P\,(1 + (0.0575)(8)) \\ 2500 &= P(1.46) \\ P &= \frac{2500}{1.46} \\ P &= \$1712.33. \end{aligned}$$

Using the Simple Interest Formula

Page 393, #27: A loan of $5400 collects simple interest each year for eight years. At the end of that time, a total of $8316 is paid back. Find the APR for the loan.

Using the Simple Interest Formula

Solution: \$5400 is the present value P, and \$8316 is the future value F. Solve for r:

$$8316 = 5400(1 + 8r)$$

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The APR is 6.75%.

10.3 Compound Interest

Compound Interest

With compound interest, the interest rate applies to the principal *and* the previously accumulated interest.

Money collecting compound interest will grow faster than that collecting simple interest. Over time, the difference between compound and simple interest becomes greater and greater.

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After three years?

$$2000(1.06)^2(1.06) = 2000(1.06)^3.$$

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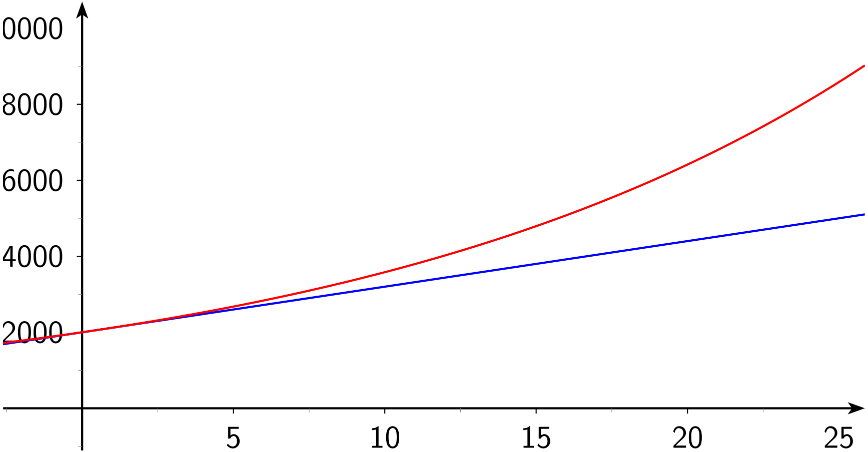
$$2000(1.06)^{15} = 2000(2.3966) = \$4793.20.$$

Compound vs. Simple Interest



Blue line: 6% annual simple interest

Red line: 6% annual compound interest

Compound Interest Formula

If P dollars is compounded annually for t years at an APR of $R\%$, then the future value is

$$\boxed{F = P\left(1 + r\right)^{t}}$$

where r is the decimal form of $R\%$.

Example

Suppose you invest \$2000 in a fund with a 6% APR that is *compounded monthly*. That is, interest is applied at the end of each month (instead of just the end of each year).

Since the interest rate is 6% annually (APR), it must be

$$\frac{6\%}{12} = 0.5\% \text{ per month.}$$

After one month, you'll have $\$2000(1.005) = \$2010$.
After one year (twelve months), you'll have $\$2000(1.005)^{12} = \$2123.36$.

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$$2000\,(1.005)^{15\cdot 12} = 2000(1.005)^{180} = \$4908.19.$$

Compound Interest Formula

If P is invested at an APR of $R\%$ compounded n times per year, for t years, then the future value F is:

$$\boxed{F = P\left(1 + \frac{r}{n}\right)^{nt}}$$

where r is the decimal form of $R\%$.

Using the Compound Interest Formula

You put \$800 in a bank account that offers a 4.5% APR compounded weekly. How much is in the account in 5 years?

Since there are 52 weeks in a year, the interest rate is
$\frac{r}{n} = \frac{4.5\%}{52} = 0.086538\%$ or 0.00086538.
In 5 years, interest will be compounded $nt = 52 \cdot 5 = 260$ times.

$$\begin{aligned} 800(1+0.00086538)^{260} &= 800(1.00086538)^{260} \\ &= 800(1.252076) \\ &= 1001.66. \end{aligned}$$

Using the Compound Interest Formula

You want to save up $1500. If you can buy a 3 year CD (certificate of deposit) from the bank that pays an APR of 5% compounded biannually, how much should you invest now?

Note: Biannually means two times per year.

Using the Compound Interest Formula

You want to save up \$1500. If you can buy a 3 year CD (certificate of deposit) from the bank that pays an APR of 5% compounded biannually, how much should you invest now?

Note: Biannually means two times per year.
Solve for P:

$$\begin{aligned} 1500 &= P\left(1+\frac{.05}{2}\right)^{2\cdot 3} \\ 1500 &= P(1.025)^6 \\ 1500 &= P(1.159693) \\ P &= \$1293.45. \end{aligned}$$

Annual Percentage Yield

The annual percentage yield or APY of an investment is the percentage of profit that is generated in a one-year period.

The APY is essentially the same as the *percent increase* in the investment over one year.

Example: APY

If an investment of \$575 is worth \$630 after one year, what is the APY?

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The profit made is $\$630 - \$575 = \$55$. Thus the annual percentage yield is:

$$\frac{\$630 - \$575}{\$575} = \frac{\$55}{\$575} = 0.096 = 9.6\%.$$

Comparing Investments

The APY allows us to compare different investments.

Use the APY to compare an investment at 3.5% compounded monthly with an investment at 3.8% compounded annually.

The amount of principal is unimportant. Pick $P = \$1$ to make our lives easier.

Comparing Investments

For the first loan, after one year we have

$$1\left(1+\frac{.035}{12}\right)^{12}=1.035571.$$

So the APY is $\frac{1.035571-1}{1}=0.035571\approx 3.56\%$.

For the second loan, after one year we have

$$1(1+.038)=1.038.$$

So the APY is $\frac{1.038-1}{1}=0.038=3.8\%$.
The second loan is better because it has a higher APY.

Geometric Sequences

Suppose \$5000 is invested with an annual interest rate of 6% compounded annually. Let G_N represent the amount of money you have at the end of N years. Then:

$$\begin{aligned} G_0 &= 5000 \\ G_1 &= (1.06)5000 = 5300 \\ G_2 &= (1.06)^2 5000 = 5618 \\ G_3 &= (1.06)^3 5000 = 5955.08 \\ &\vdots \\ G_N &= (1.06)^N 5000 \end{aligned}$$

Note that each term is obtained from the previous one by multiplying by 1.06. This is called the common ratio. The number 5000 is the initial term.

Geometric Sequences

A geometric sequence starts with an initial term P, and from then on every term in the sequence is obtained by multiplying the preceding term by the same constant c, called the common ratio.

Examples:

- $5, 10, 20, 40, 80, \ldots$
- $27, 9, 3, 1, \frac{1}{3}, \frac{1}{9}, \ldots$
- $27, -9, 3, -1, \frac{1}{3}, -\frac{1}{9}, \ldots$

In each case, what is the initial term P and what is the common ratio c?

Geometric Sequences

Here is the general form of a geometric sequence:

$$P, cP, c^2P, c^3P, c^4P, \ldots, c^NP, \ldots$$

We write G_N to label the terms of a geometric sequence:

$$G_0 = P, G_1 = cP, G_2 = c^2P, G_3 = c^3P, \ldots, G_N = c^NP, \ldots$$

- $G_0 = P$ and $G_N = cG_{N-1}$. This expresses the statement that the initial term is a number P and each term is obtained by multiplying the preceding term by c. This is called a recursive formula.
- $G_N = c^NP$. This expresses the term G_N directly in terms of P and c. this is called an explicit formula.

Examples

A principal amount of P is invested with annual compound interest rate r (expressed as a decimal). Express the yearly amounts of money as a geometric sequence.

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$P, P(1+r), P(1+r)^2, P(1+r)^3, \ldots, \ldots$
What is the initial term and what is the common ratio?
The initial term is P and the common ratio is $(1+r)$.

Examples

In 2008 there were 1 million reported cases of the gamma virus. The number of cases has been dropping each year by 70% since then. If the present rate continues, how many reported cases can we predict by the year 2014?

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To decrease a number by 70%, remember that we multiply by $(1 - \frac{70}{100})$ or 0.30. So we have a geometric sequence with initial term 1,000,000 and common ratio 0.30. We are interested in the term after the elapse of 6 years. So we need to calculate $G_6 = c^6 P$. In this case,

$$G_6 = (0.30)^6 (1000000) = 729.$$

The Geometric Sum Formula

Here is a formula for the sum of the first N terms in a geometric sequence (in the case that $c \neq 1$):

$$P + cP + cP^2 + \cdots + c^{N-1}P = P\left(\frac{c^N - 1}{c - 1}\right).$$

Note that we are adding the first N terms, up to $c^{N-1}P$. Note also that the power of c on the right-hand side (N) is one more than the power of c on the left-hand side ($N - 1$). We are going to need this formula soon!

The Geometric Sum Formula

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Now subtract the upper expression from the lower expression. There is lots of cancellation!

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$$cS - S = c^NP - P.$$

Finally solve for S:

$$S(c-1) = P(c^N - 1)$$

$$S = P\left(\frac{c^N - 1}{c - 1}\right).$$

The Geometric Sum Formula

Calculate $1 + 2 + 4 + 8 + 16 + \cdots + 2^{63}$.
(This is the "rice on the chessboard" problem.)

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Here $P = 1$, $c = 2$, and $N - 1 = 63$.

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$$\begin{aligned}&1 + 2(1) + 2^2(1) + 2^3(1) + \cdots + 2^{63}(1)\\&= (1)\left(\frac{2^{64}-1}{2-1}\right)\\&= 18,446,744,073,709,551,615.\end{aligned}$$

The Geometric Sum Formula

Calculate $\frac{1}{2} + \frac{1}{6} + \frac{1}{18} + \cdots + \frac{1}{2}(\frac{1}{3})^{10}$.

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Here $P = \frac{1}{2}$, $c = \frac{1}{3}$, and $N - 1 = 10$. The sum is

$$\left(\frac{1}{2}\right)\left(\frac{\left(\frac{1}{3}\right)^{11} - 1}{\frac{1}{3} - 1}\right) \approx 0.75.$$

The Geometric Sum Formula

Example: Suppose we want to count the total number of cases of a particular disease. Suppose in 2008 there were 5000 cases, and that for each after that the number of *new* cases was 40% more than the year before. How many total cases occurred in the 10-year period from 2008 to 2017?

The Geometric Sum Formula

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Number of cases in 2008: 5000,
Number of new cases in 2009: $5000(1.40)$,
Number of new cases in 2010: $5000(1.40)^2$,
etc.
Number of new cases in 2017: $5000(1.40)^9$.

The Geometric Sum Formula

Total number of cases:

$$5000 + 5000(1.40) + 5000(1.40)^2 + \cdots + 5000(1.40)^9$$
$$= 5000\frac{(1.40)^{10}-1}{1.40-1}$$
$$\approx 1,932,101.$$

Fixed Annuities

A fixed annuity is a sequence of equal payments made or received over regular time intervals.

Examples:

- making regular payments on a car or home loan
- making regular deposits into a college fund
- receiving regular payments from an retirement fund or inheritance

Two Types of Fixed Annuities

A deferred annuity is one in which regular payments are made first, so as to produce a lump-sum payment at a later date.

- Example: Making regular payments to save up for college.

An installment loan is an annuity in which a lump sum is paid first, and then regular payments are made against it later.

- Example: Receiving a car loan, and paying it back with monthly payments.

Deferred Annuities

Example: A newborn's parents set up a college fund. They plan to invest \$100 each month. If the fund pays 6% annual interest, compounded monthly, what is the future value of the fund in 18 years?

Notice that each monthly installment has a different "lifespan":

- The first installment will generate interest for all $18 \cdot 12 = 216$ months.
- The second installment will generate interest for 215 months.

  $\vdots$
- The last installment will generate interest for only one month.

Deferred Annuities: Example

The first installment will generate interest for all $18 \cdot 12 = 216$ months. Using the compound interest formula, after 18 years the installment is worth:

$$100\left(1+\frac{.06}{12}\right)^{18\cdot 12} = 100(1.005)^{216}.$$

The future value of the second installment is:

$$100\left(1+\frac{.06}{12}\right)^{215} = 100(1.005)^{215}.$$

$$\vdots$$

The future value of the final installment is:

$$100(1.005)^{1}.$$

Deferred Annuities: Example

The total future value is the sum of all of these future values:

$$\begin{aligned} F &= 100(1.005) + 100(1.005)^2 + \cdots + 100(1.005)^{215} + 100(1.005)^{216} \\ &= 100(1.005)\left[1 + 1.005 + \cdots 1.005^{214} + 1.005^{215}\right]. \end{aligned}$$

Deferred Annuities: Example

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Inside the brackets, we have a geometric sum with initial term $P = 1$ and common ratio $c = 1.005$. Use the geometric sum formula:

$$F = 100(1.005) \times 1 \left[\frac{1.005^{216} - 1}{1.005 - 1}\right] = \$38,929.00$$

Notice that the exponent is 216 because there are 216 terms in the sum, and because the last exponent in the sum is 215.

The Fixed Deferred Annuity Formula

The future value F of a fixed deferred annuity consisting of T payments of $\$P$ each, having a periodic interest rate p (in decimal form) is:

$$\boxed{F = L\left(\frac{(1+p)^T - 1}{p}\right)}$$

where L denotes the future value of the last payment.

Note that the periodic interest rate $p = \frac{r}{n}$ where r is the APR in decimal form and the interest is compounded n times per year. *Pay attention to what quantities the various variables stand for!*

Example

Page 395, #63:
Starting at age 25, Markus invests $2000 at the beginning of each year in an IRA (individual retirement account) with an APR of 7.5% compounded annually. How much money will there be in Markus's retirement account when he retires at age 65?

Example

Page 395, #63:
Starting at age 25, Markus invests \$2000 at the beginning of each year in an IRA (individual retirement account) with an APR of 7.5% compounded annually. How much money will there be in Markus's retirement account when he retires at age 65?

Notice that the periodic interest rate p is $p = 0.075$ and $T = 65 - 25 = 40$.
The future value of the last payment is $L = 2000(1.075)$ because the final payment will accumulate interest for one year.

Example

So the future value is:

$$
\begin{aligned}
F &= 2000(1.075)\left(\frac{1.075^{40}-1}{0.075}\right)\\
&= 2000(1.075)\left(\frac{18.044239-1}{0.075}\right)\\
&= 2000(1.075)\left(\frac{17.044239}{0.075}\right)\\
&= 2000(1.075)(227.25652)\\
&= 488601.52.
\end{aligned}
$$

Markus will have $488,601.52 in his account when he retires.

Example

Same example as before, except that Markus invests the money at the *end* of each year, *after* the interest for that year has been added to the account.

Example

This time, $L = 2000$. So the future value is:

$$\begin{aligned} F &= 2000\left(\frac{1.075^{40}-1}{0.075}\right) \\ &= 2000\left(\frac{18.044239-1}{0.075}\right) \\ &= 2000\left(\frac{17.044239}{0.075}\right) \\ &= 2000(227.25652) \\ &= 454513.04. \end{aligned}$$

Markus will have $454,513.04 in his account when he retires.

Example

Suppose you want to set up an 18-year annuity at an APR of 6% compounded monthly, if your goal is to have $50,000 at the end of 18 years. How much should the monthly payments be?

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Remember

$$F = L\left(\frac{(1+p)^T - 1}{p}\right).$$

We know $F = 50,000$, $p = \frac{.06}{12} = 0.005$, and $T = 18 \times 12 = 216$. Let P be the unknown monthly payment. Then we know $L = P(1.005)$.

Example

Substitute:

$$50,000 = P(1.005)\left(\frac{(1.005)^{216} - 1}{0.005}\right) = P(389.29).$$

So

$$P = \frac{50,000}{389.29} = \$128.44.$$

Example

Suppose you want to have \$2000 at the end of 7.5 years. You already have \$875 in the bank, invested at a 6.75% APR compounded monthly. You want to put more money each month into the bank to end up with the \$2000 goal. What should your monthly deposit be?

Example

Suppose you want to have \$2000 at the end of 7.5 years. You already have \$875 in the bank, invested at a 6.75% APR compounded monthly. You want to put more money each month into the bank to end up with the \$2000 goal. What should your monthly deposit be?

First, the \$875 in the bank will grow to $875(1+\frac{0.0675}{12})^{7.5(12)} = \$1449.62$. So you only need $\$2000 - 1449.62 = \$550.38$ more.

Example

So we have $F = 550.38$, $p = \frac{0.0675}{12} = 0.005625$, $T = 7.5(12) = 90$, and $L = P(1.005625)$. Substituting,

$$550.38 = P(1.005625)\left(\frac{(1.005625)^{90} - 1}{0.005625}\right) = P(117.404).$$

Therefore

$$P = \frac{550.38}{117.404} = \$4.69.$$

Installment Loans

We already know that the future value F of a payment P made today is

$$F = P(1+p)^T,$$

where p is the periodic interest rate $p = \frac{r}{n}$, and T is the total number of time periods.

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where p is the periodic interest rate $p = \frac{r}{n}$, and T is the total number of time periods.

For example, if we have an annual interest rate of 6% compounded monthly, \$200 today is worth $200(1+\frac{.06}{12})^{36} = 200(1.005)^{36} = \$239.34$ in 36 months.

Installment Loans

Thinking backwards in time and solving for P, we also know that if we want a future value of F in T months, then we must invest

$$P = \frac{F}{(1+p)^T}$$

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For example, if we have an annual interest rate of 6% compounded monthly, \$300 in 36 months is worth $\frac{300}{(1.005)^{36}} = \$250.69$ today.

Installment Loans

This is how we can figure out the payments to pay back an installment loan. You will receive a certain loan amount today (in the present), and make periodic payments on into the future. The *present* values of all of these future payments must add up to the *present* amount of the loan.

Installment Loans

Example. You receive a loan of $25,000, at an annual interest rate of 6% compounded monthly. You will repay the loan by making monthly payments over 36 months. How much will each payment be? Here, $p = \frac{.06}{12} = 0.005$.

Installment Loans

Example. You receive a loan of \$25,000, at an annual interest rate of 6% compounded monthly. You will repay the loan by making monthly payments over 36 months. How much will each payment be? Here, $p = \frac{.06}{12} = 0.005$.

Since the payments occur in the future, we denote the amount of each payment by F.
Present value of payment 1: $\frac{F}{(1.005)^1}$,
Present value of payment 2: $\frac{F}{(1.005)^2}$,
Present value of payment 3: $\frac{F}{(1.005)^3}, \ldots$
Present value of payment 36: $\frac{F}{(1.005)^{36}}$.

Installment Loans

These must all sum up to the present value of the loan, \$25,000.

$$25,000 = \frac{F}{(1.005)^1} + \frac{F}{(1.005)^2} + \frac{F}{(1.005)^3} + \cdots + \frac{F}{(1.005)^{36}}$$

Using the geometric sum formula,

$$\begin{aligned} & \frac{F}{(1.005)^1} + \frac{F}{(1.005)^2} + \frac{F}{(1.005)^3} + \cdots + \frac{F}{(1.005)^{36}} \\ = \; & \frac{F}{1.005}\left[1 + \frac{1}{(1.005)^1} + \frac{1}{(1.005)^2} + \cdots + \frac{1}{(1.005)^{35}}\right] \\ = \; & \frac{F}{1.005} \cdot 1 \left[\frac{\left(\frac{1}{1.005}\right)^{36} - 1}{\left(\frac{1}{1.005}\right) - 1}\right] \\ = \; & (32.871)F. \end{aligned}$$

Installment Loans

Finishing up by solving for F:

$$25,000 = (32.871)F$$

$$F = \frac{25,000}{32.871} = \$760.55.$$

So you will pay \$760.55 each month. Over 36 months, this amounts to a total of \$27,379.80.

Amortization Formula

This procedure leads to the general Amortization Formula: If an installment loan of P dollars is paid off in T payments of F dollars at a periodic interest of p (written in decimal form), then

$$P = Fq\left[\frac{q^T - 1}{q - 1}\right],$$

where $q = \frac{1}{1+p}$.

Example: Car Loan

You want to buy a car for \$23,995 for which you have \$5000 for a down payment, and the dealer offers you two choices:

1. Cash rebate of \$2000, and financing for 6.48% annual interest for 60 months.
2. Financing for 0% APR for 60 months.

Which is better?

Example: Car Loan

First option: Finance $P = \$16,995$. $p = \frac{0.0648}{12} = 0.0054$.

$$16,995 = \left(\frac{F}{1.0054}\right)\left[\frac{\left(\frac{1}{1.0054}\right)^{60} - 1}{\left(\frac{1}{1.0054}\right) - 1}\right].$$

Solve for F to get $F = \$332.37$ for your monthly payment.

Example: Car Loan

First option: Finance $P = \$16,995$. $p = \frac{0.0648}{12} = 0.0054$.

$$16,995 = \left(\frac{F}{1.0054}\right)\left[\frac{\left(\frac{1}{1.0054}\right)^{60} - 1}{\left(\frac{1}{1.0054}\right) - 1}\right].$$

Solve for F to get $F = \$332.37$ for your monthly payment.
Second option: Finance $P = \$18,995$, with monthly payment $\frac{18{,}995}{60} = \$316.59$.
So the second option has a lower monthly payment.

Example: Lottery

Suppose you win \$9 million in the lottery, and that after taxes this amounts to \$6.8 million. You are offered two choices:

1. Annuity option: Receive 25 annual installments of \$272,000 per year.
2. Lump sum option: Receive an immediate lump sum of \$3.75 million.

Which is better?

Example: Lottery

We can calculate the present value P of this sequence of payments, and compare to the value of the lump sum. We should pick an interest rate that is reasonable for the current market.

If we try 5%, then $p = \frac{0.05}{1} = 0.05$ and

$$P = 272,000 \left[\frac{\left(\frac{1}{1.05}\right)^{25} - 1}{\left(\frac{1}{1.05}\right) - 1} \right] = \$4,025,230.$$

Note that the first payment comes immediately, so we do not have to divide F by 1.05.

Example: Lottery

If we try 6%, then $p = \frac{0.06}{1} = 0.06$ and

$$P = 272,000 \left[\frac{\left(\frac{1}{1.06}\right)^{25} - 1}{\left(\frac{1}{1.06}\right) - 1} \right] = \$3,685,700.$$

So if we are conservative about interest rates, the annuity option appears better, but if we are less conservative about interest rates, the lump sum option appears better.

Example: Mortgage

This problem is more complicated, but more realistic!
You take out a mortgage on your home, borrowing $180,000 for 30 years at an annual rate of 6.75% with monthly payments.

1. What is your monthly payment?
2. What is the balance on your mortgage after you have made 30 payments?
3. How much interest will you pay over the life of the loan?

Example: Mortgage

Note that $p = \frac{0.0675}{12} = 0.005625$.

$$180,000 = \left(\frac{F}{1.005625}\right)\left[\frac{\left(\frac{1}{1.005625}\right)^{360} - 1}{\left(\frac{1}{1.005625}\right) - 1}\right].$$

Solve for F to get $F = \$1167.48$.

Example: Mortgage

For the second question, calculate the present value of the loan when only 330 payments remain:

$$P = \left(\frac{1167.48}{1.005625}\right)\left[\frac{\left(\frac{1}{1.005625}\right)^{330} - 1}{\left(\frac{1}{1.005625}\right) - 1}\right] = \$174,951.$$

So even though you have made 30 payments of $1167.48, you have only reduced your loan by $5049!

Example: Mortgage

For the second question, calculate the present value of the loan when only 330 payments remain:

$$P = \left(\frac{1167.48}{1.005625}\right)\left[\frac{\left(\frac{1}{1.005625}\right)^{330} - 1}{\left(\frac{1}{1.005625}\right) - 1}\right] = \$174,951.$$

So even though you have made 30 payments of \$1167.48, you have only reduced your loan by \$5049!
Over the life of the loan you will make 360 payments of \$1147.48, for a total of \$413,093, so your total interest will be \$413,093-180,000=\$233,093.