The Mathematics of Symmetry

Beth Kirby and Carl Lee

University of Kentucky
MA 111

Fall 2009

Course Information

Text: Peter Tannenbaum, *Excursions in Modern Mathematics*, second custom edition for the University of Kentucky, Pearson.

Course Website:

http://www.ms.uky.edu/~lee/ma111fa09/ma111fa09.html

11.0 Introduction to Symmetry

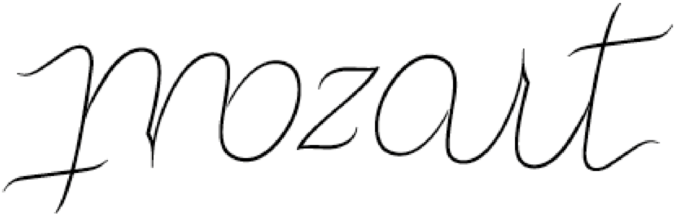
Is This Symmetrical?

Is This Symmetrical?

Is This Symmetrical?

Is This Symmetrical?

Is This Symmetrical?



Is This Symmetrical?

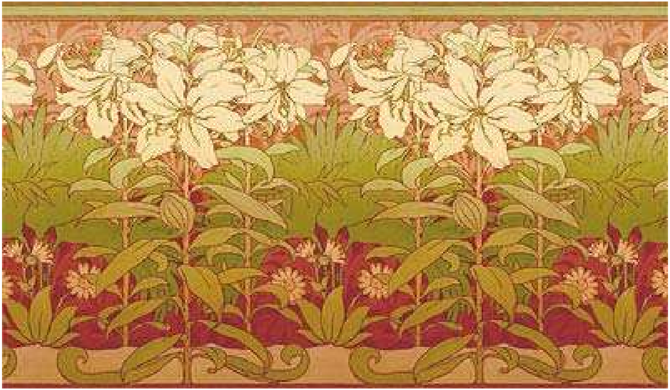
Is This Symmetrical?

Is This Symmetrical?

Are These Symmetrical?

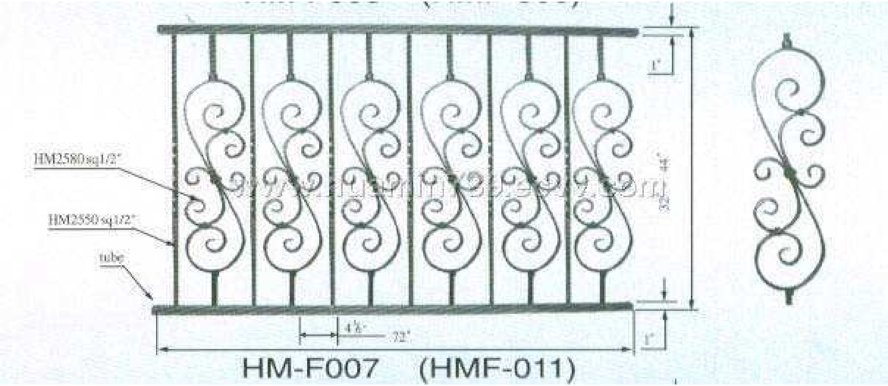
Is This Symmetrical?

Is This Symmetrical?



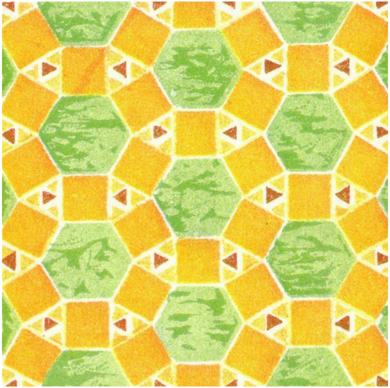
Assume this extends forever to the left and right

Is This Symmetrical?



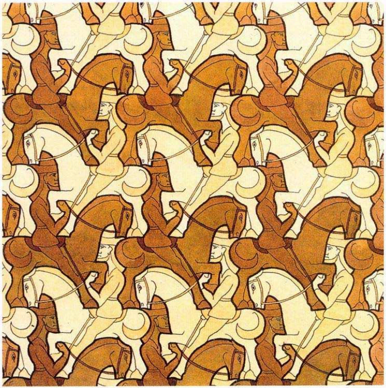
Assume this extends forever to the left and right

Is This Symmetrical?



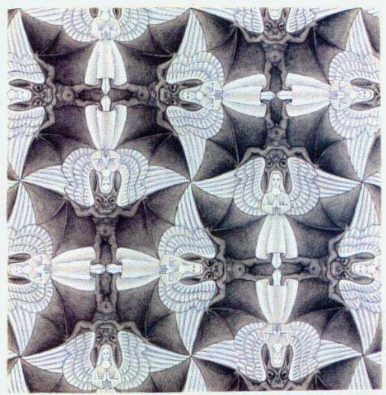
Assume this extends forever in all directions

Is This Symmetrical?



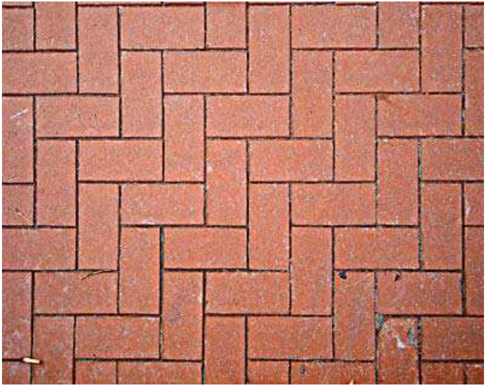
Assume this extends forever in all directions

Is This Symmetrical?



Assume this extends forever in all directions

Is This Symmetrical?



Assume this extends forever in all directions

Is This Symmetrical?

Assume this extends forever in all directions

Is This Symmetrical?

Is the Image of the Sun Symmetrical?

Is This Symmetrical?

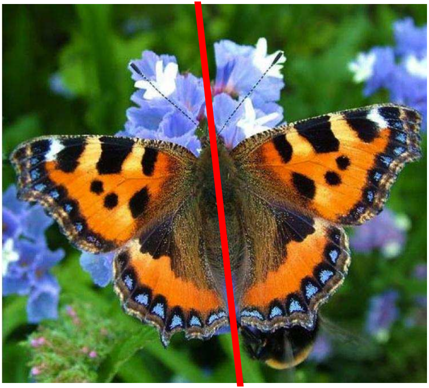
11.6 Symmetry of Finite Shapes

Symmetries of Finite Shapes

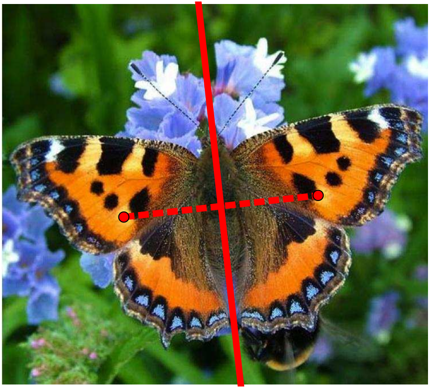
Let's look at the symmetries of some finite shapes — shapes that do not extend forever in any direction, but are confined to a bounded region of the plane.

Symmetries of Finite Shapes

Symmetries of Finite Shapes



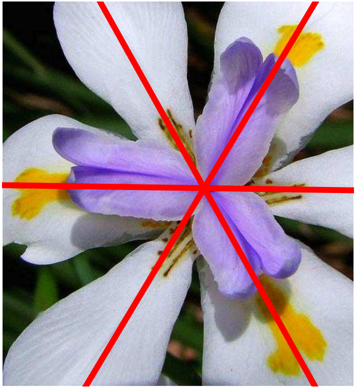
Symmetries of Finite Shapes



This shape has 1 line or axis of reflectional symmetry. It has symmetry of type D_1.

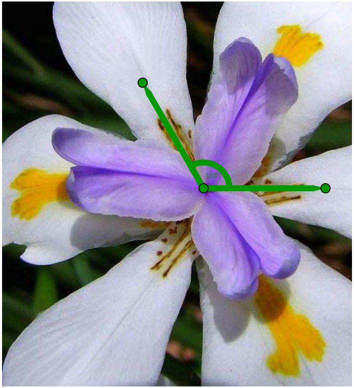
Symmetries of Finite Shapes

Symmetries of Finite Shapes



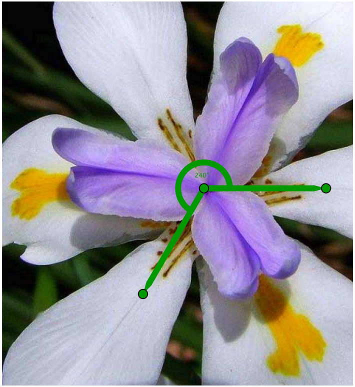
This shape has 3 axes of reflectional symmetry.

Symmetries of Finite Shapes



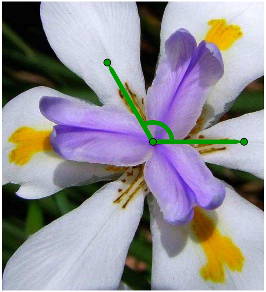
It has a 120 degree angle of rotational symmetry. (Rotate counterclockwise for positive angles.)

Symmetries of Finite Shapes



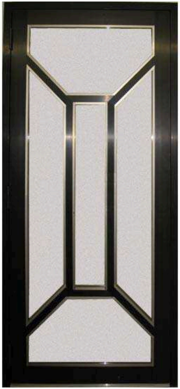
By performing this rotation again we have a 240 degree angle of rotational symmetry.

Symmetries of Finite Shapes

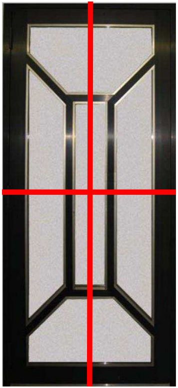


If we perform the basic 120 degree rotation 3 times, we bring the shape back to its starting position. We say that this shape has 3-fold rotational symmetry. With 3 reflections and 3-fold rotational symmetry, this shape has symmetry type D_3.

Symmetries of Finite Shapes

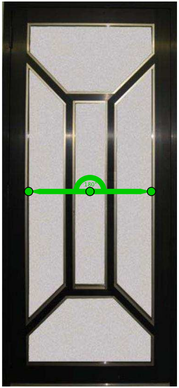


Symmetries of Finite Shapes



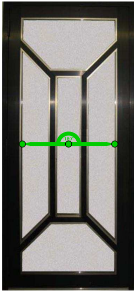
This shape has 2 axes of reflectional symmetry.

Symmetries of Finite Shapes



It has a 180 degree angle of rotational symmetry.

Symmetries of Finite Shapes



If we perform the basic 180 degree rotation 2 times, we bring the shape back to its starting position. We say that this shape has 2-fold rotational symmetry. With 2 reflections and 2-fold rotational symmetry, this shape has symmetry type D_2.

Symmetries of Finite Shapes

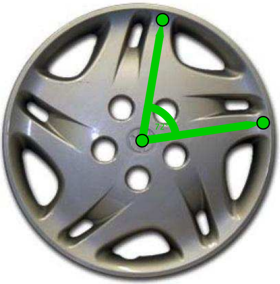
Can you think of some common objects that have symmetry type:

- D_4?
- D_5?
- D_6?
- D_8?

Symmetries of Finite Shapes

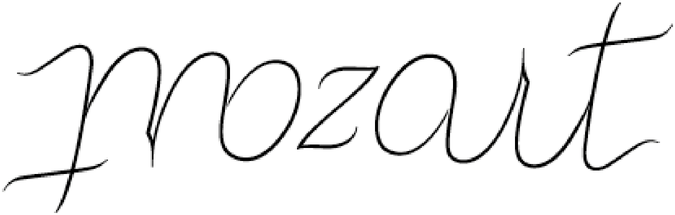
This shape has no axes of reflectional symmetry.

Symmetries of Finite Shapes



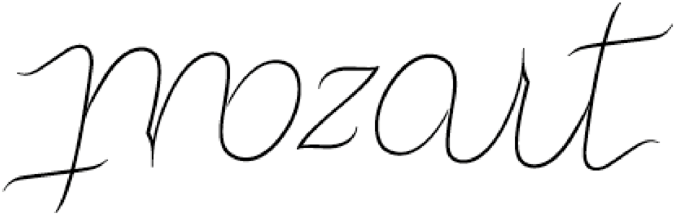
It has a 72 degree angle of rotational symmetry. If we perform the basic 72 degree rotation 5 times, we bring the shape back to its starting position. We say that this shape has 5-fold rotational symmetry. With 0 reflections and 5-fold rotational symmetry, this shape has symmetry type Z_5.

Symmetries of Finite Shapes



What is the symmetry type of this shape?

Symmetries of Finite Shapes



What is the symmetry type of this shape? Type Z_2.

Symmetries of Finite Shapes

Can you think of some common objects that have symmetry type:

- Z_2?
- Z_3?
- Z_4?

Symmetries of Finite Shapes

What is the symmetry type of this shape?

Symmetries of Finite Shapes

What is the symmetry type of this shape? Type D_9.

Symmetries of Finite Shapes

What is the symmetry type of this shape?

Symmetries of Finite Shapes

What is the symmetry type of this shape? Type D_4.

Symmetries of Finite Shapes

What is the symmetry type of this shape?

Symmetries of Finite Shapes

What is the symmetry type of this shape? Type D_1.

Symmetries of Finite Shapes

What is the symmetry type of this image of the sun?

Symmetries of Finite Shapes

Because it has infinitely many axes of reflectional symmetry and infinitely many angles of rotational symmetry, this symmetry type is D_∞.

Symmetries of Finite Shapes

What are the symmetry types of these various names?

Symmetries of Finite Shapes

What is the symmetry type of this shape?

Symmetries of Finite Shapes

Because this has no symmetry other than the one trivial one (don't move it at all, or rotate it by an angle of 0 degrees), it has symmetry type Z_1.

11.7 Symmetries of Patterns

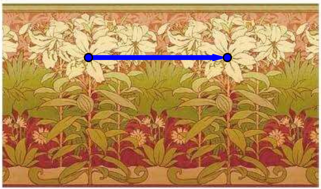
Symmetries of Border Patterns

Now let's look at symmetries of border patterns — these are patterns in which a basic motif repeats itself indefinitely (forever) in a single direction (say, horizontally), as in an architectural frieze, a ribbon, or the border design of a ceramic pot.

Symmetries of Border Patterns

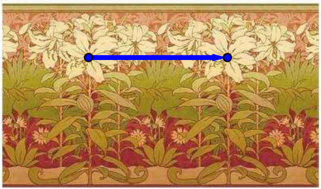
What symmetries does this pattern have?

Symmetries of Border Patterns



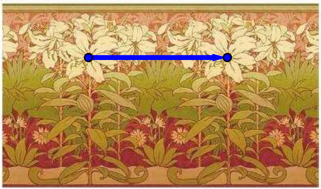
You can slide, or translate, this pattern by the basic translation shown above.

Symmetries of Border Patterns



You can slide, or translate, this pattern by the basic translation shown above. This translation is the smallest translation possible; all others are multiples of this one, to the right and to the left.

Symmetries of Border Patterns



You can slide, or translate, this pattern by the basic translation shown above. This translation is the smallest translation possible; all others are multiples of this one, to the right and to the left. So this border pattern only has translational symmetry.

Symmetries of Border Patterns

What symmetries does this pattern have?

Symmetries of Border Patterns

You can translate this pattern by the basic translation shown above.

Symmetries of Border Patterns

You can translate this pattern by the basic translation shown above. This translation is the smallest translation possible; all others are multiples of this one, forward and backward.

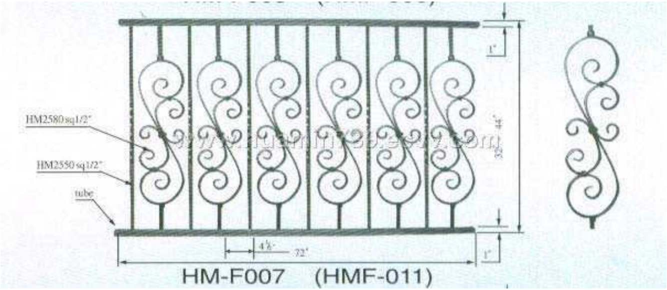
Symmetries of Border Patterns

You can also match the pattern up with itself by a combination of a reflection followed by a translation parallel to the reflection. This is called a glide reflection.

Symmetries of Border Patterns

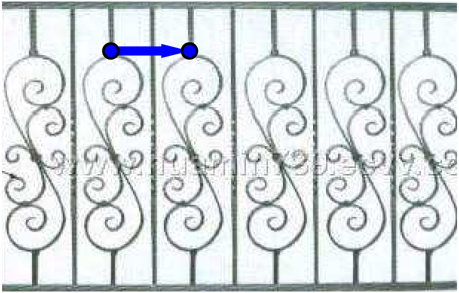
You can also match the pattern up with itself by a combination of a reflection followed by a translation parallel to the reflection. This is called a glide reflection. So this border pattern has both translational symmetry and glide reflectional symmetry.

Symmetries of Border Patterns



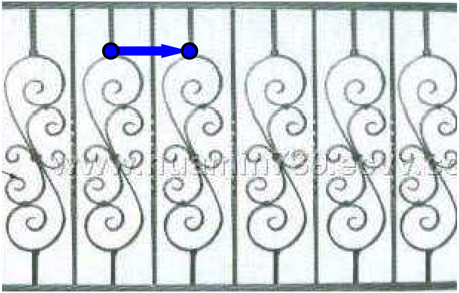
What symmetries does this pattern have?

Symmetries of Border Patterns



You can slide, or translate, this pattern by the basic translation shown above.

Symmetries of Border Patterns



You can slide, or translate, this pattern by the basic translation shown above. This translation is the smallest translation possible; all others are multiples of this one, to the right and to the left.

Symmetries of Border Patterns

There are infinitely many centers of 180 degree rotational symmetry. Here is one type of location of a rotocenter.

Symmetries of Border Patterns

And here is another type of location of a rotocenter.

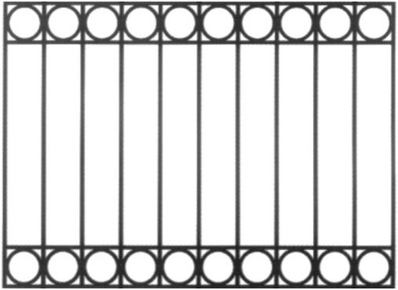
Symmetries of Border Patterns

And here is another type of location of a rotocenter. But this pattern has no reflectional symmetry or glide reflectional symmetry.

Symmetries of Border Patterns

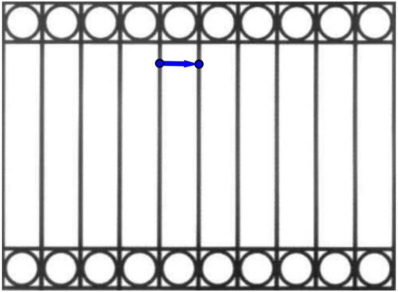
And here is another type of location of a rotocenter. But this pattern has no reflectional symmetry or glide reflectional symmetry. So this border pattern only has translational symmetry and 2-fold (or half-turn) rotational symmetry.

Symmetries of Border Patterns



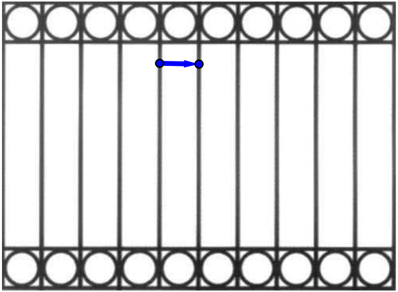
What symmetries does this pattern have?

Symmetries of Border Patterns



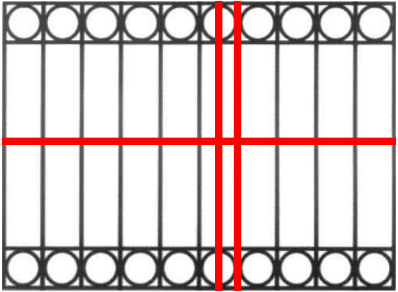
You can translate this pattern by the basic translation shown above.

Symmetries of Border Patterns



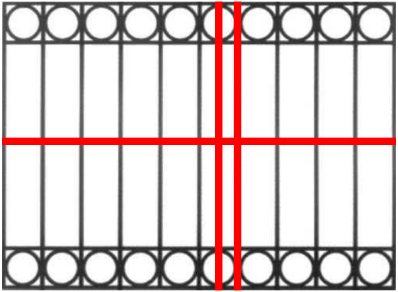
You can translate this pattern by the basic translation shown above. This translation is the smallest translation possible; all others are multiples of this one, to the right and to the left.

Symmetries of Border Patterns



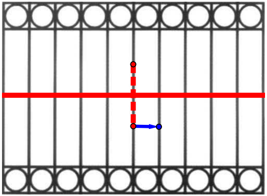
This pattern has one horizontal axis of reflectional symmetry

Symmetries of Border Patterns



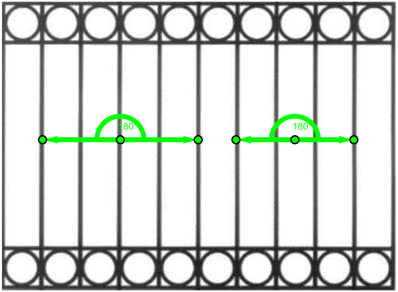
This pattern has one horizontal axis of reflectional symmetry but infinitely many vertical axes of reflectional symmetry, that have two types of locations.

Symmetries of Border Patterns



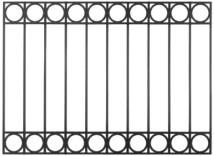
Because there are translations and horizontal reflections, we can combine them to get glide reflections. Here is one type of glide reflection. Others use the same reflection axis but multiples of this translation, to the right and to the left.

Symmetries of Border Patterns



There are infinitely many centers of 180 degree rotational symmetry. Here the two types of locations of rotocenters.

Symmetries of Border Patterns



So this border pattern has translational, horizontal reflectional, vertical reflectional, and 2-fold rotational symmetry. Even though glide reflections also work, our text states that we don't say this pattern has "glide reflectional symmetry" because the glide reflections are in this case just a consequence of translational and the horizontal reflectional symmetry.

11.2 Reflections

What is a Reflection?

A reflection is a motion that moves an object to a mirror image of itself.

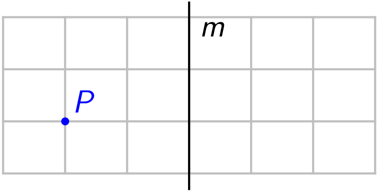
The “mirror” is called the axis of reflection, and is given by a line m in the plane.

What is a Reflection?

To find the image of a point P under a reflection, draw the line through P that is perpendicular to the axis of reflection m. The image P' will be the point on this line whose distance from m is the same as that between P and m.

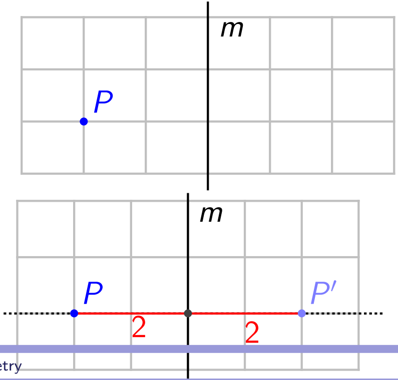
What is a Reflection?

To find the image of a point P under a reflection, draw the line through P that is perpendicular to the axis of reflection m. The image P' will be the point on this line whose distance from m is the same as that between P and m.

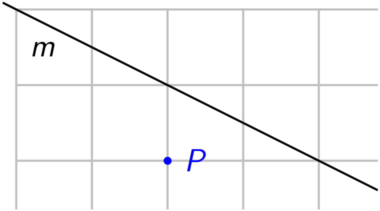


What is a Reflection?

To find the image of a point P under a reflection, draw the line through P that is perpendicular to the axis of reflection m. The image P' will be the point on this line whose distance from m is the same as that between P and m.

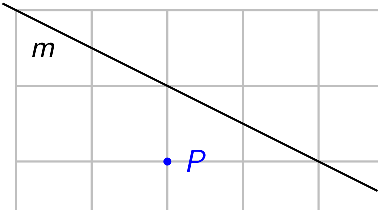


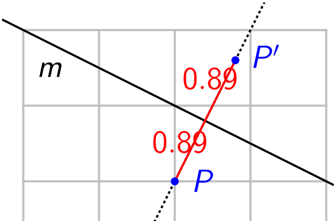
Examples

1. 

Examples

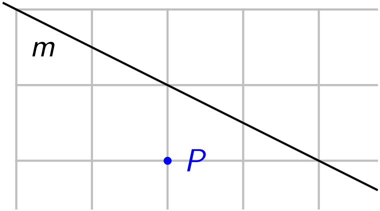
1.

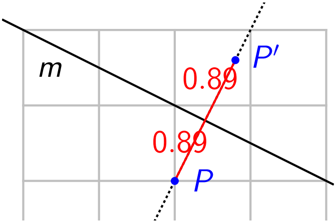




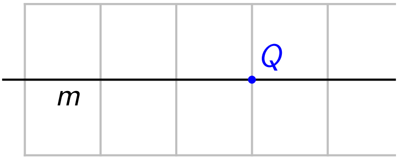
Examples

1\.



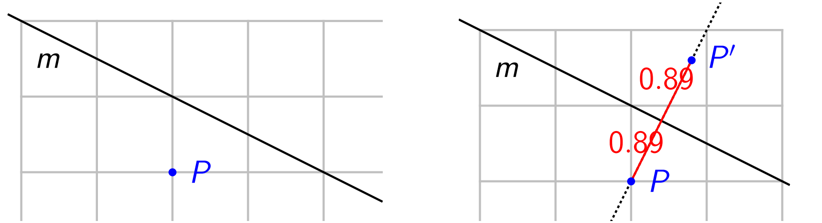


2\.

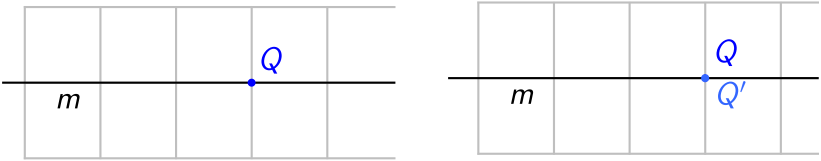


Examples

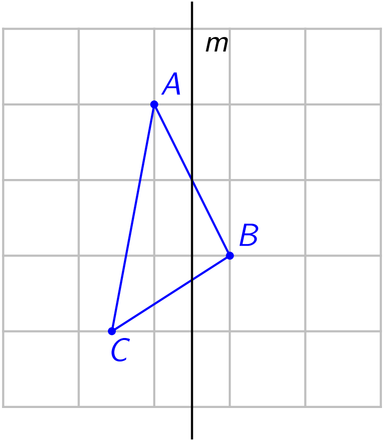
1.



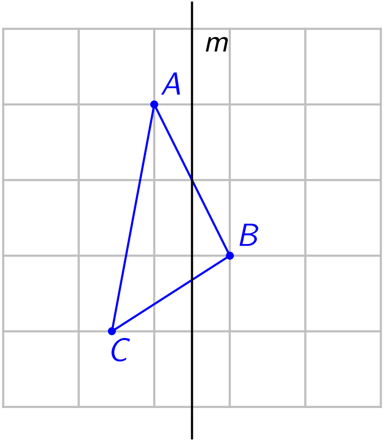
2.

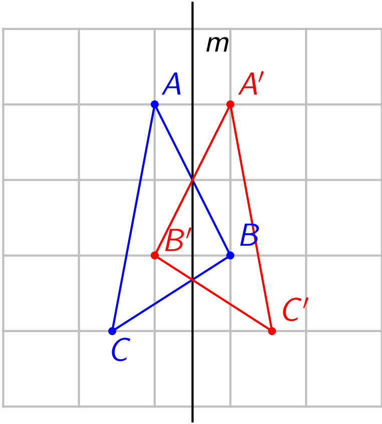


Examples



Examples





Properties of Reflections

1. A reflection is completely determined by its axis of reflection.

Properties of Reflections

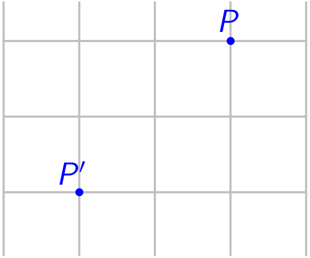
1. A reflection is completely determined by its axis of reflection.

...or...

2. A reflection is completely determined by a single point-image pair P and P' (if $P \neq P'$).

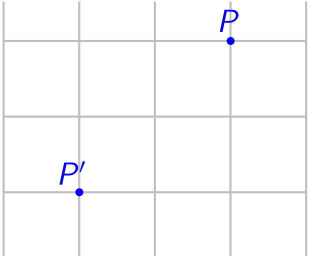
Properties of Reflections

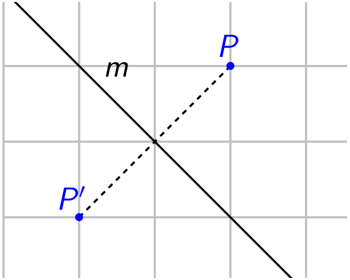
Given a point P and its image P', the axis of reflection is the perpendicular bisector of the line segment PP'.



Properties of Reflections

Given a point P and its image P', the axis of reflection is the perpendicular bisector of the line segment PP'.





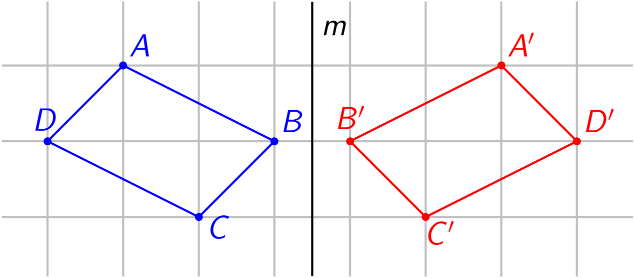
Properties of Reflections

A fixed point of a motion is a point that is moved onto itself.

For a reflection, any point on the axis of reflection is a fixed point.

3. Therefore, a reflection has *infinitely many* fixed points (all points on the line m).

Properties of Reflections



The orientation of the original object is clockwise: read $ABCDA$ going in the clockwise direction.

The orientation of the image under the reflection is counterclockwise: $A'B'C'D'A'$ is read in the counterclockwise direction

Properties of Reflections

4. A reflection is an improper motion because it reverses the orientation of objects.

5. Applying the same reflection twice is equivalent to not moving the object at all. So applying a reflection twice results in the identity motion.

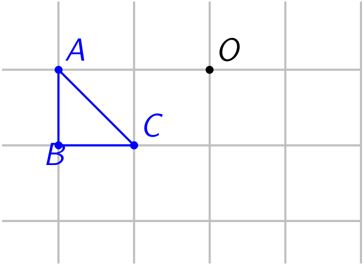
11.3 Rotations

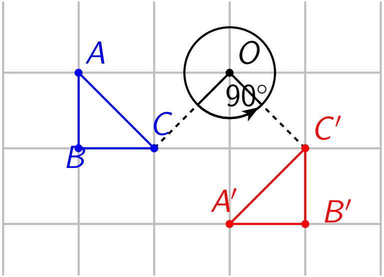
What is a Rotation?

A rotation is a motion that swings an object around a fixed point.

The fixed center point of the rotation is called the rotocenter.

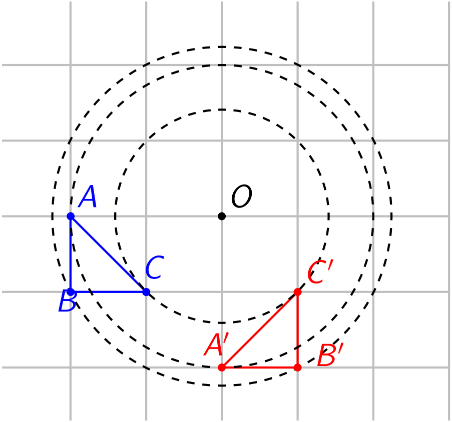
The amount of swing is given by the angle of rotation.





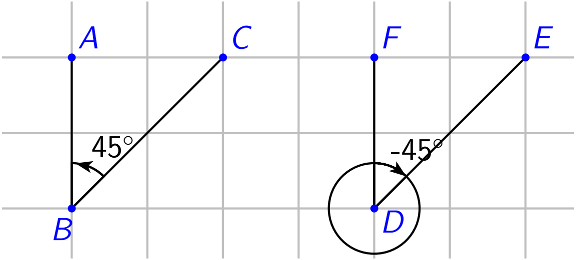
What is a Rotation?

Notice that the distance of each point from the rotocenter O does not change under the rotation:



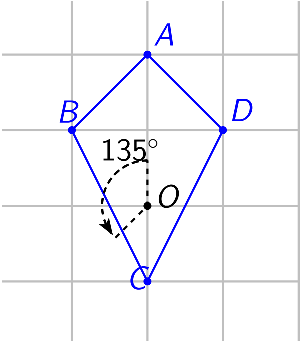
The Angle of Rotation

As a convention, any angle in the counterclockwise direction has a positive angle measure.
Any angle in the clockwise direction has a negative angle measure.



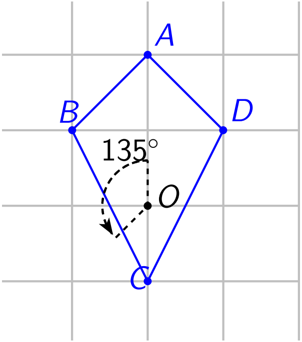
Examples

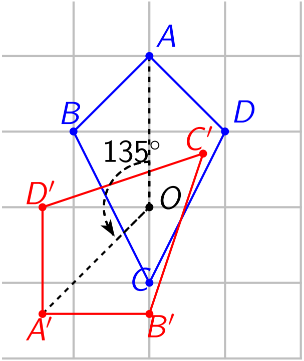
The rotation with rotocenter O and angle of rotation $135°$:



Examples

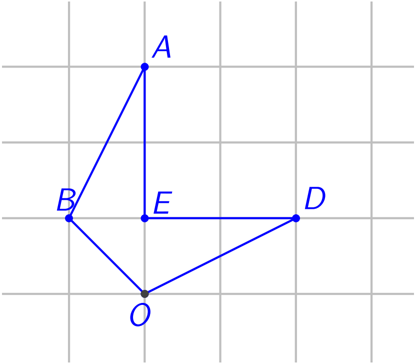
The rotation with rotocenter O and angle of rotation 135°:





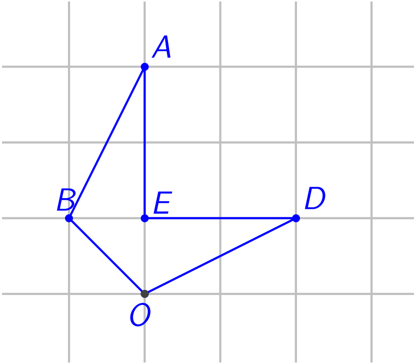
Examples

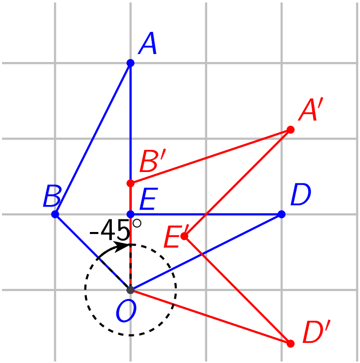
The rotation with rotocenter O and angle of rotation -45°:



Examples

The rotation with rotocenter O and angle of rotation -45°:





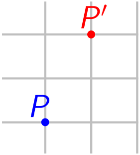
Properties of Rotations

A rotation is completely determined by ______________.

Properties of Rotations

A rotation is completely determined by ________________.

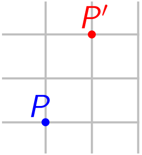
If we know one point-image pair...



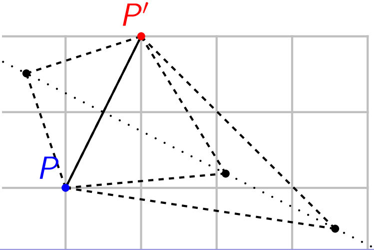
Properties of Rotations

A rotation is completely determined by ______________.

If we know one point-image pair...



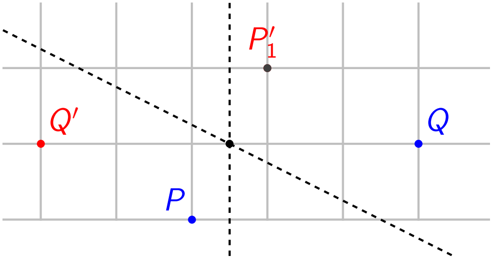
there are infinitely many possible rotocenters:



Properties of Rotations

1. A rotation is completely determined by two point-image pairs.

The rotocenter is at the intersection of the two perpendicular bisectors:

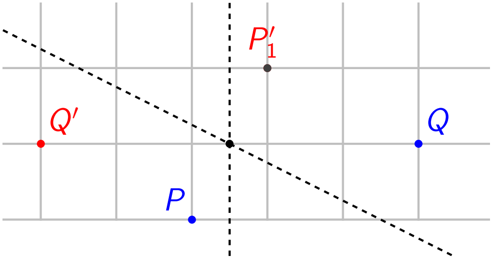


What is the angle of rotation of this rotation?

Properties of Rotations

1. A rotation is completely determined by two point-image pairs.

The rotocenter is at the intersection of the two perpendicular bisectors:



What is the angle of rotation of this rotation? 180°.

Properties of Rotations

Once we know the rotocenter, the angle of rotation is the measure of angle $\angle POP'$.

This will be the same as the measure of angle $\angle QOQ'$.

Properties of Rotations

What are the fixed points of a rotation?

Properties of Rotations

What are the fixed points of a rotation?

2. A rotation has one fixed point, the rotocenter.

Is a rotation a proper or improper motion?

Properties of Rotations

What are the fixed points of a rotation?

2. A rotation has one fixed point, the rotocenter.

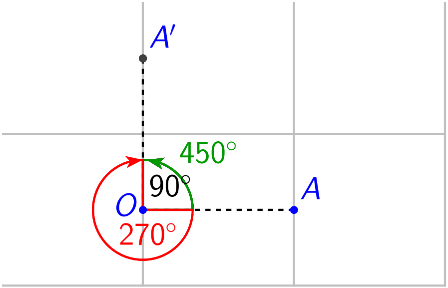
Is a rotation a proper or improper motion?

3. A rotation is a proper motion. The orientation of the object is maintained.

4. A 360° rotation around any rotocenter is equivalent to the identity motion.

Properties of Rotations

Notice that any rotation is equivalent to one with angle of rotation between 0° and 360°:

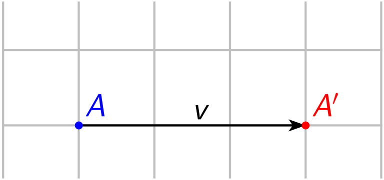


11.4 Translations

What is a Translation?

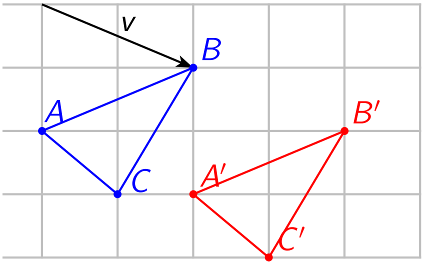
A translation is a motion which drags an object in a specified direction for a specified length.

The direction and length of the translation are given by the vector of translation, usually denoted by v.



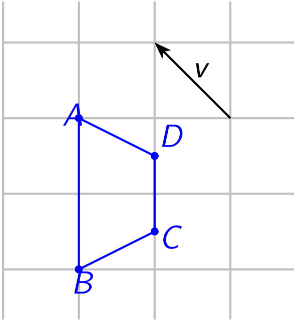
What is a Translation?

The placement of the vector of translation on the plane does not matter. We only need to see how long it is and in what direction it's pointing to know the image of an object.

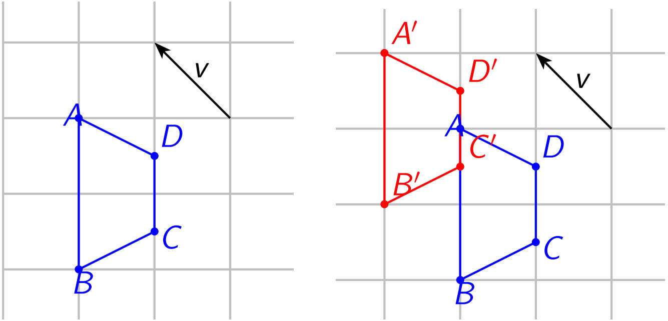


The vector v indicates that each point moves down one unit and to the right two units.

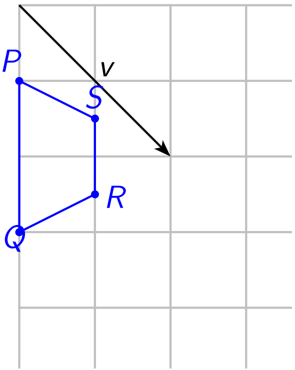
Examples



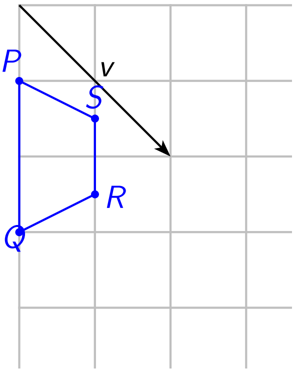
Examples

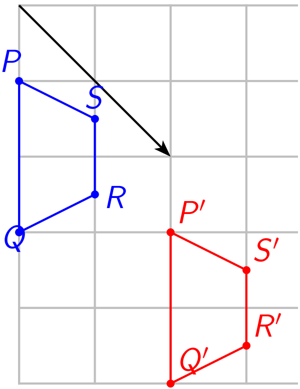


Examples



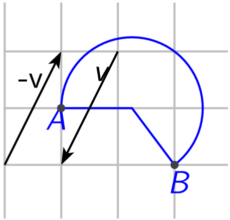
Examples





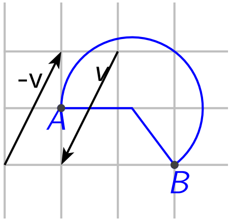
Examples

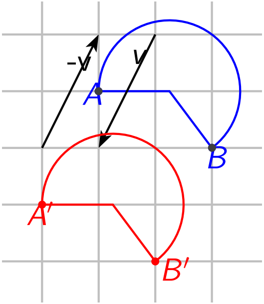
Given a vector v, the vector $-v$ has the same length but the opposite direction.
Apply the translation with vector v and then the translation with vector $-v$.



Examples

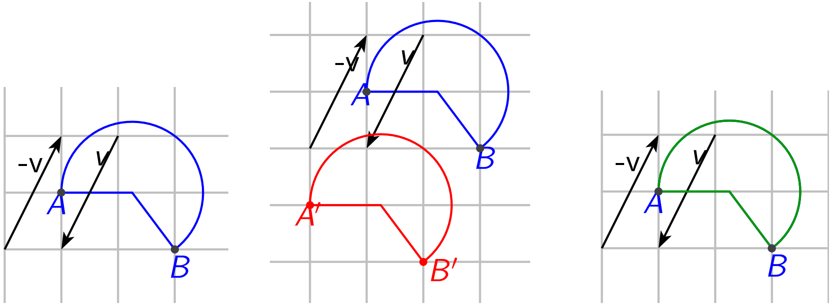
Given a vector v, the vector $-v$ has the same length but the opposite direction.
Apply the translation with vector v and then the translation with vector $-v$.





Examples

Given a vector v, the vector $-v$ has the same length but the opposite direction.
Apply the translation with vector v and then the translation with vector $-v$.



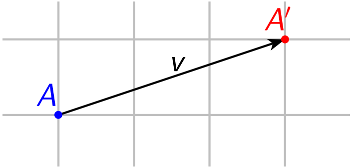
Properties of Translations

How many point-image pairs do we need to determine the translation?

Properties of Translations

How many point-image pairs do we need to determine the translation?

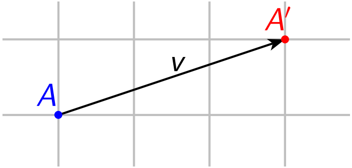
Given one point-image pair, the vector of translation is the arrow that connects the point to its image.



Properties of Translations

How many point-image pairs do we need to determine the translation?

Given one point-image pair, the vector of translation is the arrow that connects the point to its image.



1. A translation is completely determined by one point-image pair.

Properties of Translations

2. A translation has no fixed points. Why?

3. A translation is a proper motion because the orientation of the object is preserved.

4. By applying the translation with vector $-v$ after the translation with vector v, we obtain a motion that is equivalent to the identity motion.

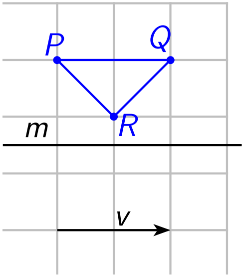
11.5 Glide Reflections

What is a Glide Reflection?

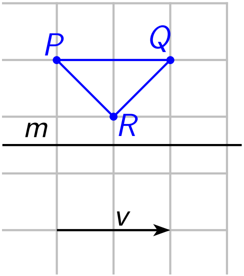
A glide reflection is a combination of a translation and a reflection.

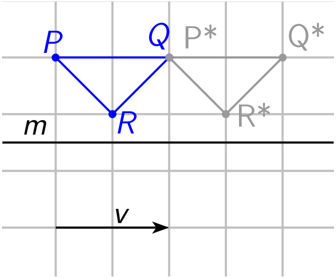
The vector of translation v and the axis of reflection m must be parallel to each other.

Examples

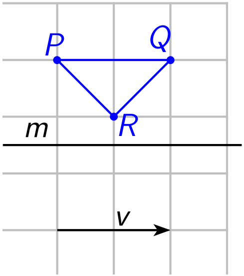


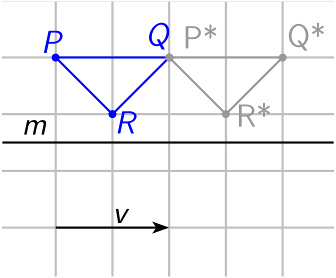
Examples

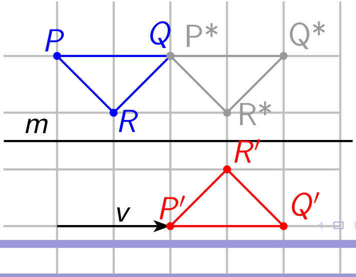




Examples

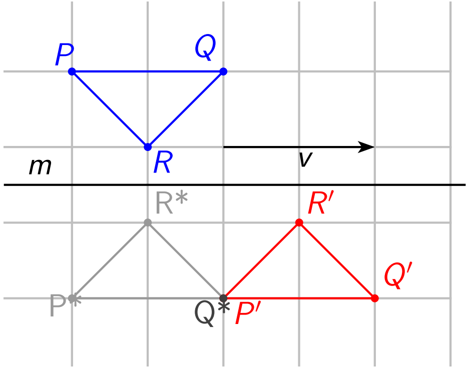






Examples

Since the vector of translation and the axis of reflection are parallel, it does not matter which motion is done first in the glide reflection.

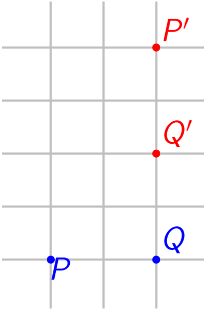


Properties of Glide Reflections

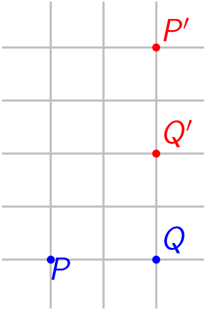
1. A glide reflection is completely determined by two point-image pairs.

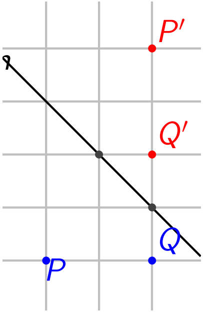
- The axis of reflection is the line passing through the two midpoints of the segments PP' and QQ'.
- Use the axis of reflection to find an intermediate point. For example, the image of P under the reflection is the intermediate point $P*$.
- Finally, the vector of translation is the vector connecting P to $P*$.

Two Point-Image Pairs Determine a Glide Reflection

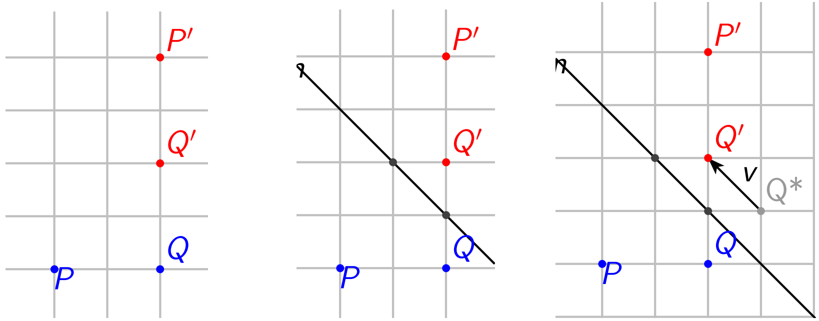


Two Point-Image Pairs Determine a Glide Reflection





Two Point-Image Pairs Determine a Glide Reflection



Properties of Glide Reflections

Does a glide reflection have any fixed points?

Properties of Glide Reflections

Does a glide reflection have any fixed points? No.

2. Since a translation has no fixed points, a glide reflection has no fixed points.

Properties of Glide Reflections

Does a glide reflection have any fixed points? No.

2. Since a translation has no fixed points, a glide reflection has no fixed points.

Is a glide reflection a proper or improper motion?

Properties of Glide Reflections

Does a glide reflection have any fixed points? No.

2. Since a translation has no fixed points, a glide reflection has no fixed points.

Is a glide reflection a proper or improper motion? Improper. Why?

3. A glide reflection is an improper motion.

Properties of Glide Reflections

Given a glide reflection with translation vector v and axis of reflection m, how can we "undo" the motion?

Properties of Glide Reflections

Given a glide reflection with translation vector v and axis of reflection m, how can we "undo" the motion?

To undo the translation, we must apply the translation with vector $-v$.

To undo the reflection, we must apply the reflection with the same axis of reflection m.

Properties of Glide Reflections

Given a glide reflection with translation vector v and axis of reflection m, how can we "undo" the motion?

To undo the translation, we must apply the translation with vector $-v$.

To undo the reflection, we must apply the reflection with the same axis of reflection m.

4. When a glide reflection with vector v and axis of reflection m is followed by a glide reflection with vector $-v$ and axis of reflection m, we obtain the identity motion.

Summary of Motions

Rigid Motion	Specified by	Proper/Improper	Fixed Points	Point-Image Pairs Needed
Reflection	Axis of reflection ℓ	Improper	All points on ℓ	One
Rotation*	Rotocenter O and angle α	Proper	O only	Two
Translation	Vector of translation v	Proper	None	One
Glide Reflection	Vector of translation v and axis of reflection ℓ	Improper	None	Two
*Identity (0° rotation)		Proper	All points	

11.7 Classifying Symmetries of Border Patterns

Classifying Symmetries of Border Patterns

By definition, border patterns always have translational symmetry—there is always a basic, smallest, translation that can be repeated to the right and to the left as many times as desired. There is a basic design or motif that repeats indefinitely in one direction (e.g., to the right and the left).

Classifying Symmetries of Border Patterns

We have seen some border patterns that have no other symmetry, some that have glide reflectional symmetry, some that have rotational (half-turn) symmetry, and even some that have horizontal reflectional symmetry, vertical reflectional symmetry, and half-turn (2-fold rotational) symmetry.

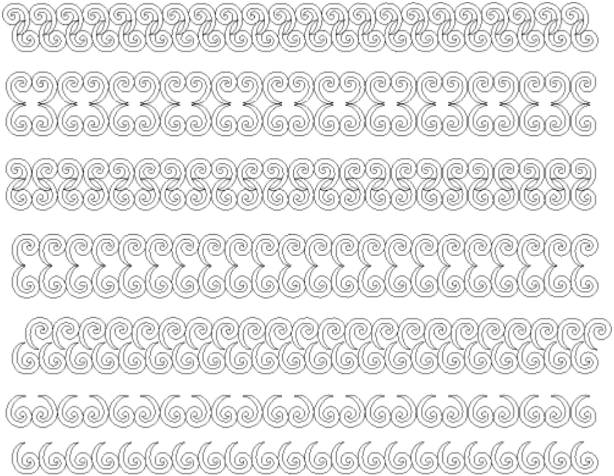
Classifying Symmetries of Border Patterns

We can classify border patterns according to the combinations of symmetries that they possess. It turns out that there are only seven different possibilities:

Symmetry Type	Translation	Horizontal Reflection	Vertical Reflection	Half-Turn	Glide Reflection
11	Yes	No	No	No	No
1m	Yes	Yes	No	No	No*
m1	Yes	No	Yes	No	No
mm	Yes	Yes	Yes	Yes	No*
12	Yes	No	No	Yes	No
1g	Yes	No	No	No	Yes
mg	Yes	No	Yes	Yes	Yes

*These patterns *do* have glide reflections, but only as a result of having translational and horizontal reflectional symmetry.

Classify These Seven Patterns



Classifying Symmetries of Border Patterns

Why are there only seven types?
The net result of a horizontal reflection followed by a vertical reflection is?

Classifying Symmetries of Border Patterns

Why are there only seven types?
The net result of a horizontal reflection followed by a vertical reflection is?
A half turn.

Classifying Symmetries of Border Patterns

Why are there only seven types?

The net result of a horizontal reflection followed by a vertical reflection is?

A half turn.

So if a border pattern admits a horizontal reflection and a vertical reflection, it *must* also have a half turn. You cannot have a border pattern with a horizontal reflection, a vertical reflection, and no half turn.

Classifying Symmetries of Border Patterns

The net result of a horizontal reflection followed by a half turn is?

Classifying Symmetries of Border Patterns

The net result of a horizontal reflection followed by a half turn is?

A vertical reflection.

Classifying Symmetries of Border Patterns

The net result of a horizontal reflection followed by a half turn is?

A vertical reflection.

So if a border pattern admits a horizontal reflection and a half turn, it *must* also have a vertical reflection. You cannot have a border pattern with a horizontal reflection, a half turn, and no vertical reflection.

Classifying Symmetries of Border Patterns

Other combinations of possible symmetries can be considered to rule out other cases, leaving only seven remaining.

Classifying Wallpaper Patterns

Border patterns have a repeated basic design or motif that repeats indefinitely in a *single* direction (e.g., to the right and left).

Wallpaper patterns have a basic design or motif that repeats indefinitely in at least *two* different directions. It turns out that there are 17 different types of symmetry for wallpaper patterns. A flowchart for classifying wallpaper patterns appears on page 627 of the text. Try classifying the wallpaper patterns in the file "More Patterns" on the course website.