Chapter 2
Notes for Instructors

Content

The second chapter of Long and DeTemple is geared toward defining the whole numbers and the arithmetic operations on the whole numbers. The first section provides an introduction to set theory. The second section focuses on equivalence, cardinality, and ordering the whole numbers. The third and fourth sections define whole number addition, subtraction, multiplication, and division. These sections also introduce several models for these operations.

Notes and Suggestions

Notes on Section 2.1: *Sets and Operations on Sets*

- Because the definition for addition is based on the union of disjoint sets, it is important for students to have a basic understanding of some set theory. In particular, students will need to understand the following ideas:

  - Union of sets

  - Intersection of sets

  - Disjoint sets

  - Transitivity of inclusion

  - Commutativity of union

  - Associativity of union

  - Properties of the empty set

  - The inclusion-exclusion principle

  - Subset

  - Proper subset

  You can do a lot of other things in this section, but these are the essentials. Students should be able to understand Venn diagrams and perhaps draw a few on their own, but I do not know that this skill is necessary to understand the key ideas in the remainder of the chapter.

- You will notice that I have placed a special emphasis on the union of sets. I have done this because addition on the whole numbers is defined using the union of disjoint sets. Since the main objective of this chapter is to define arithmetic operations on the whole numbers, the union of sets seems to be one of the most important operation on sets. For example, the commutative property of addition follows from the commutativity of union.

- You will need to define *subset* and *proper subset* so that you can order the whole numbers. These ideas can be slightly confusing for students, but I think they are necessary for students to fully understand the relations $\leq, \geq, <$, and $>$. If $A \subset B$, students should be able to clearly explain why this is true. That is, they should argue that each element of A is also an element of B and that there is an element of B that is not an element of A. Moreover, they should understand that $A \subseteq B$ and $B \subseteq A$ implies that $A = B$.

- If time permits, you should also define the *complement* of a set and the *Cartesian product*. The latter is used as a model for multiplication (and, in fact, can be used to provide an alternate definition for multiplication). The former could be used to formally define subtraction by the take-away model. Moreover, students will need to be familiar with the complement of a set when they study the chapter on probability.

- You can make a lot of good True/False questions from the material in section 2.1. These questions can teach students to read definitions carefully. You can also teach students how to construct good counterexamples for those statements which are false. Moreover, you will need to stress that you can show that a statement is false by providing a counterexample, but you (usually) cannot show that a statement is true by providing an example.

- **An Activity Note:** When introducing the operations on sets, I found that it was helpful to involve the students. Specifically, I told the students with brown hair to stand on the right side of the room. Then I told the students with brown eyes to stand on the left side of the room. At this point they should see the need to have an intersection. You can also talk about complements at this point because some students should not be in either set. Moreover, you can address the inclusion exclusion principle, since it is likely that there will be students in the intersection of the sets. You can certainly make this activity more elaborate if you like. I do believe that the activity was, for my class, more interesting and effective than the traditional lecture approach for teaching set theory.

Notes on Section 2.2: *Sets, Counting and the Whole Numbers*

- In this section, Long and DeTemple define one-to-one correspondence, equivalent sets, the whole numbers and the ordering of the whole numbers.

- **Manipulatives:** On pages 87–89, Long and DeTemple discuss manipulatives that can be used to represent the whole numbers. We do have the cubes discussed at the top of page 88 and Cuisenaire rods which are similar to the number strips shown on page 88.

- **Pacing:** In retrospect, I do not think it was necessary to spend a whole lot of time on section 2. I also think you should skip the Hamming codes unless you really have a lot of extra time. You will probably need quite a bit of time on sections 3 and 4. The Hamming code discussion at the end of section 2 appears to be an attempt to show

the student that there are applications for set theory. I do believe that applications are important, but it is difficult to include them because of the pace of this course. I found that it was more important to devote extra time to the central ideas of the course which students will actually be teaching themselves.

Notes on Sections 2.3 and 2.4: *Addition and Subtraction of Whole Numbers* and *Multiplication and Division of Whole Numbers*

- In sections 3 and 4 Long and Temple define addition, subtraction, multiplication and division of whole numbers. It is important that students understand the different models of arithmetic discussed in these sections. Moreover, it is important that you define the arithmetic operations as they have done in the book because these definitions generalize easily.

- **Addition:**

  - I like to discuss the importance of the word "disjoint" in the definition for addition. This can be done by providing an example.

    An Activity Note: Sometimes it is useful to involve the class. Determine how many students have brown hair and how many students have green eyes. Ask them to determine how many students have brown hair or green eyes. If you are lucky, your sets will not be disjoint. Ask them how the problem would be different if we wanted to determine the number of students who have green eyes or blue eyes.

  - Students should use the set model when combining two groups and the number line model when looking at distances.

  - The properties on page 101 follow (easily) from the definition of whole number addition and the properties of sets given on page 78.

- **Subtraction:**

  - Students may think it is strange to define whole number subtraction by the missing addend model. I find that it is useful to remind them that they first learned division by the missing factor model.

  - If students want to define whole number subtraction by the take-away model, I ask them how they would define subtraction for the integers. It is difficult to think of taking away a negative number. We use the missing addend model to define subtraction because it generalizes easily to other number systems. We are looking ahead toward the big picture.

- **Multiplication:**

  - The easiest model for students appears to be the repeated addition model. That is familiar to them.

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- I think the multiplication tree model is difficult to understand because they do not put labels on the tree. When I labeled the nodes and the final outcome at each leaf of the tree, they seemed to understand it better. Multiplication trees will show up again in the second semester when they study probability.

- The Cartesian product model will show up again when students study probability in the second semester.

- The rectangular area model for multiplication is IMPORTANT. It is the only model given in this section for whole number multiplication which can be adjusted for rational number multiplication.

- I really like the way that the textbook uses the area model to explain the foil method to students.

- **Division:**

  - The partition model for division tends to be the easiest model for students to understand, though they will need to understand all three models given for division.

  - Many students are unaware that division is repeated subtraction. It is, therefore, not surprising that some of them will not remember how to do a long division problem without using a calculator. For them, the long division algorithm is simply an algorithm without meaning. The long division algorithm should flow naturally from this model.

  - The missing factor model is used to define division because, as with the missing addend model for subtraction, this definition will generalize easily to other number systems.

  - It is a personal pet peeve of mine that students do not know that you can divide zero by a nonzero number but you cannot divide any number by zero. I believe that this is because they do not fully understand division. Certainly, they can memorize this fact for the exam, but I think it is important that they be able to explain why this is the case using all three of the models for division. I have attached a sheet with my explanation for the three models. I do require that they provide these types of explanations on exams.