Using Place Value Cards

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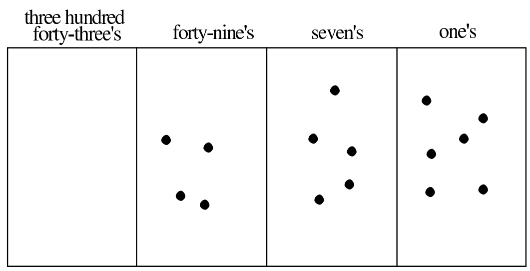
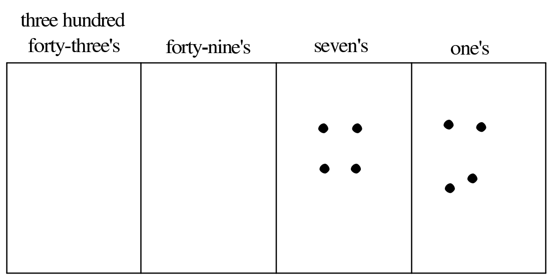
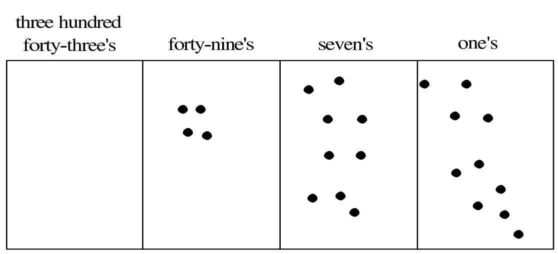
Problem: Use place value cards to evaluate $456_{seven} + 44_{seven}$.

Solution: We begin with two place value cards. The first card represents 456_{seven} and the second represents 44_{seven}.

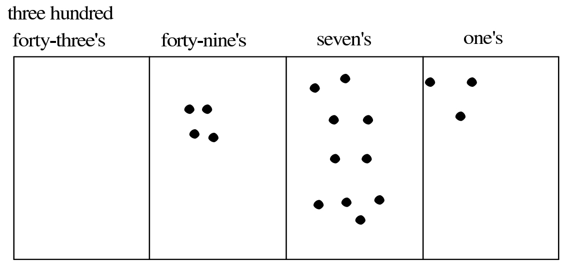




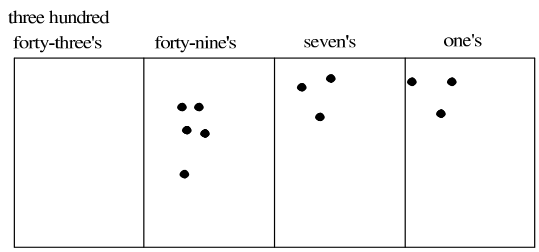
Since we are adding, we will need to combine the poker chips onto one place value card. This card is shown below.



Since we are working in base seven, no place can have more than six poker chips. Working from right to left, we see that there is a group of seven poker chips in the one's place. We exchange seven of these chips for one chip in the seven's place. The result is shown below.



Next we will need to exchange seven of the chips in the sevens place for one chip in the forty-nine's place. The result is shown below.



Since no place has more than six chips, we have arrived at the final answer: $456_{seven} + 44_{seven} = 533_{seven}$.

(*NOTE: The students should make exchanges from right to left so that they can be certain the algorithm will terminate after one pass. If students suggest that they make exchanges from left to right or in random order, ask them to add something similar to* $99+99+99+99+99+99+99+99+99+99+99$ *(base ten) by making exchanges from left to right. Then ask them to compare the different algorithms.*)

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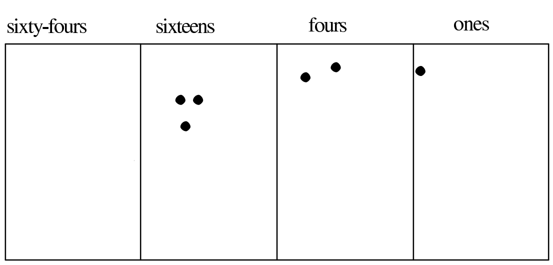
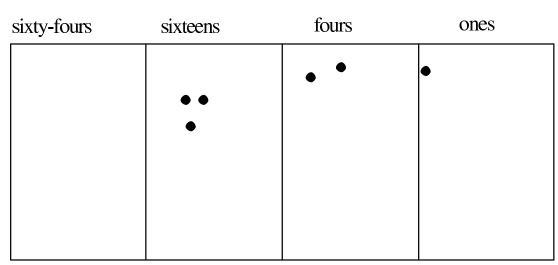
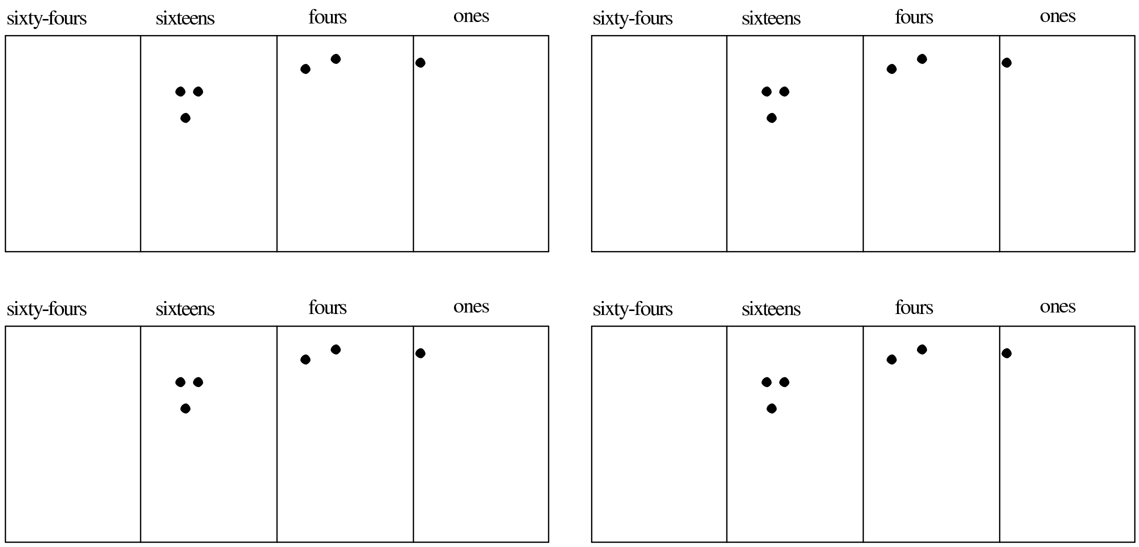
Problem: Use place value cards to evaluate $12_{four} \times 321_{four}$.

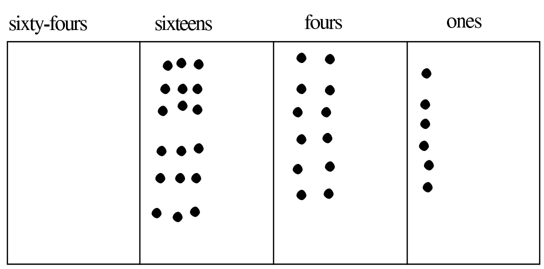
Solution: Here we will need to begin with six place value cards (because six$= 12_{four}$). Each of the six cards will need to represent 321_{four}.



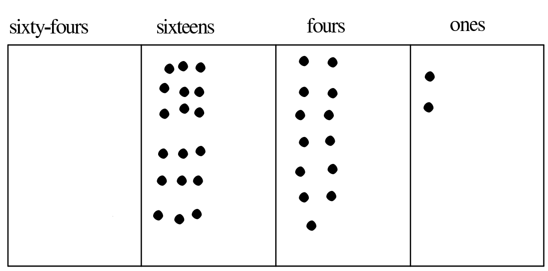




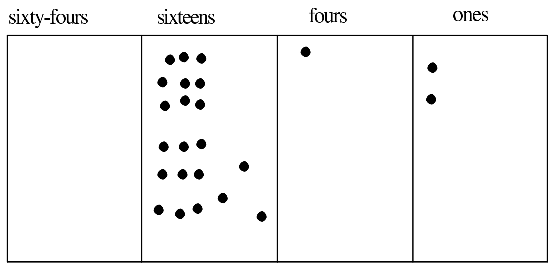
Since multiplication is repeated addition, and addition is the union of disjoint sets, we will need to combine all of the poker chips on one card.



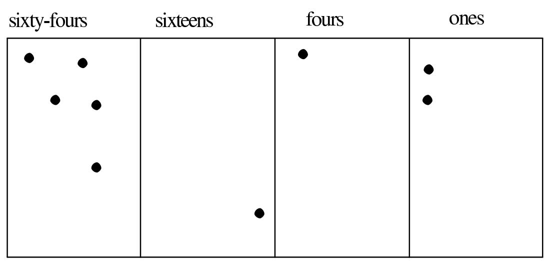
Because we are doing multiplication in base four, no place may have more than three poker chips. Working from right to left, we begin by exchanging four chips in the ones place for one chip in the fours place.



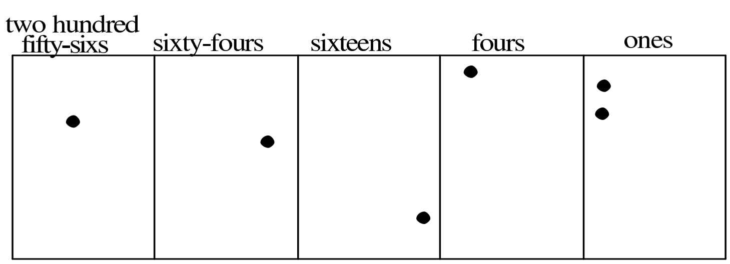
Next we exchange twelve chips in the fours place for three chips in the siexteens place.



Then we exchange twenty chips in the sixteens place for five chips in the sixty-fours place.



At this point, we see that we need another place since the sixty-fours place has more than three chips. So we add the the two hundred fifty-sixes place and exchange four chips in the sixty-fours place for one chip in the two-hundred fifty-sixes place.



We now see that we have made all possible exchanges. Therefore $21_{four} \times 321_{four} = 11112_{four}$.

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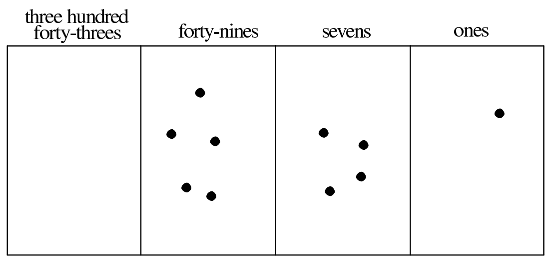
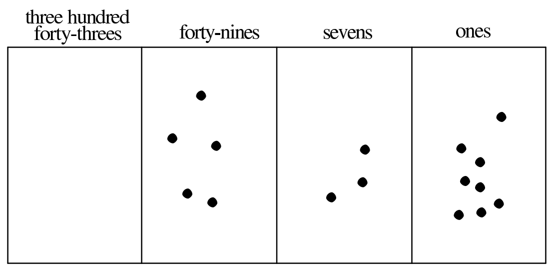
Problem: Use place value cards to evaluate $541_{seven} - 63_{seven}$.

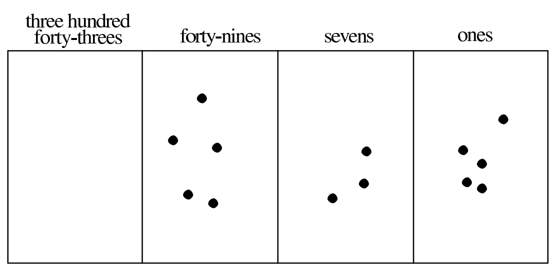
Solution: You only need one place value card for subtraction. Begin with the place value card that represents 541_{seven}



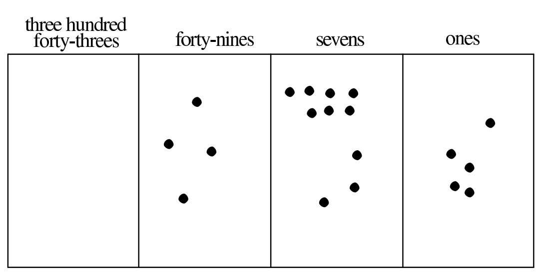
We need to remove three chips from the ones place. Since we do not have three chips in the ones place, we will need to exchange one chip in the sevens place for seven chips in the ones place.



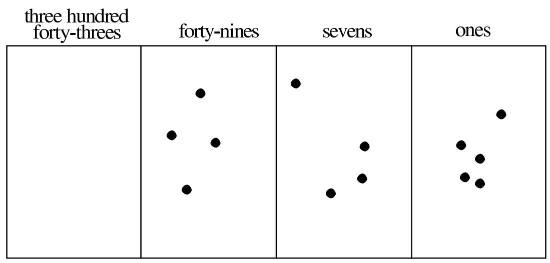
We can now remove three chips from the ones place.



Next we will need to remove six chips from the sevens place. Since there are not six chips in the sevens place, we will need to exchange one chip in the forty-nines place for seven chips in the sevens place.



We can now remove six chips from the sevens place.



Now that we have removed three chips from the ones place and six chips from the sevens place, we see theat $541_{seven} - 63_{seven} = 455_{seven}$.

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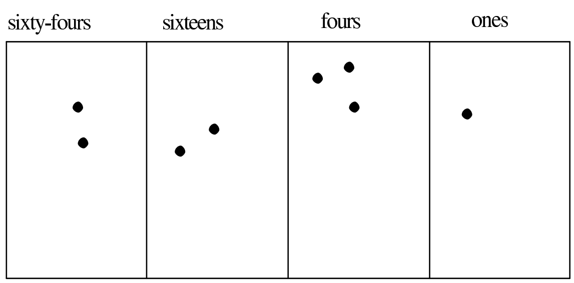
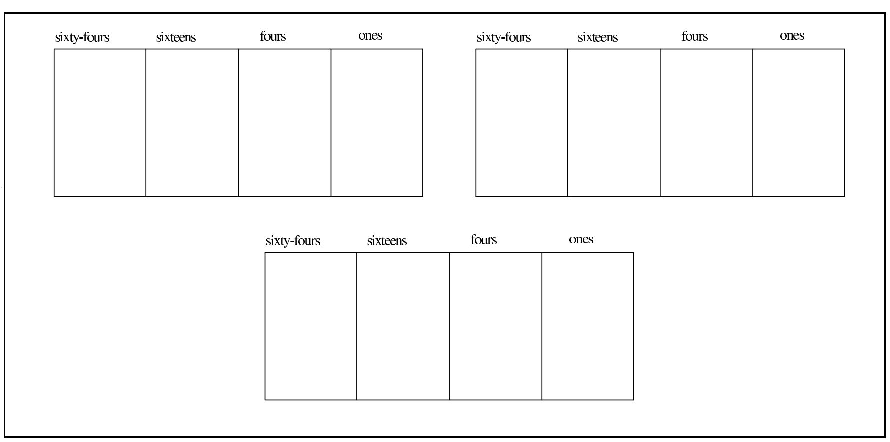
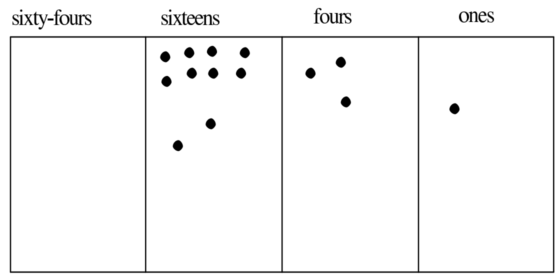
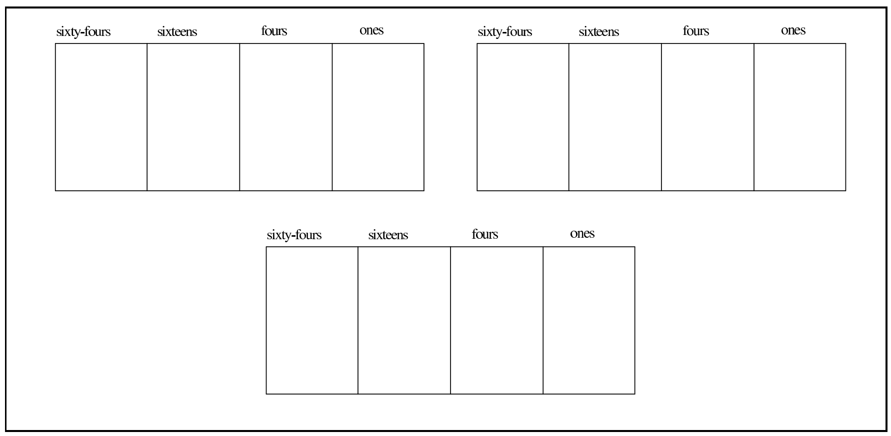
Problem: Use place value cards to evaluate $2231_{four} \div 3_{four}$.
Solution: We begin with a place value card that represents 2231_{four} and three blank cards since we are dividing by three.



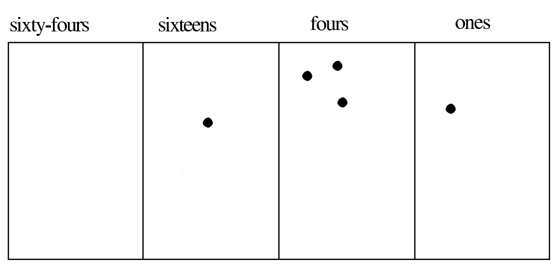


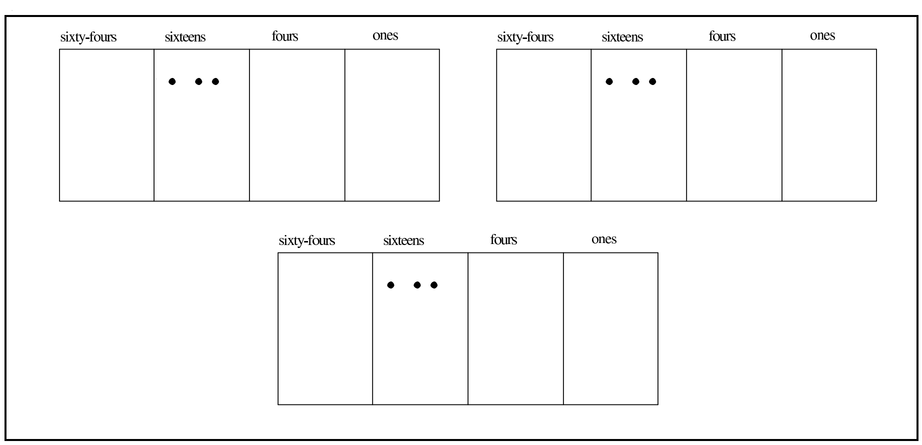
We begin in the sixty-fours column. We want to evenly distribute the chips in the sixty-fours column of the first place value card to the sixty-fours columns of the blank place value cards. We cannot do this because there are only two chips and we have three cards. Therefore we will need to exchange the two chips in the sixtyfours place for eight chips in the sixteens place.



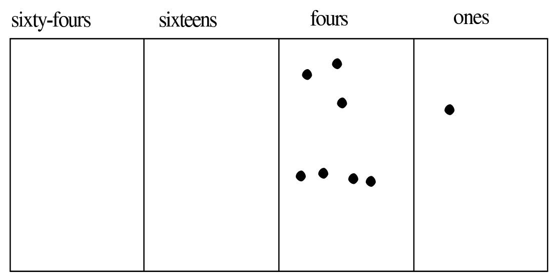


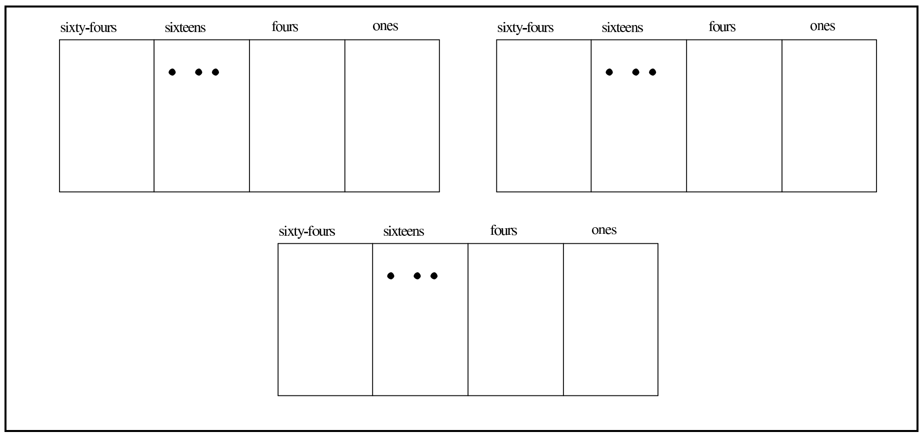
Now we need to evenly distribute the ten chips in the sixteens place to the sixteens place of the three blank cards. We can place three chips in sixteens place of each of the three cards. This will leave us with one chip in the sixteens place, three chips in the fours place, and one chip in the ones place of the original card.



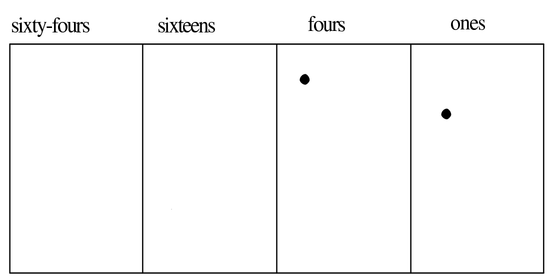


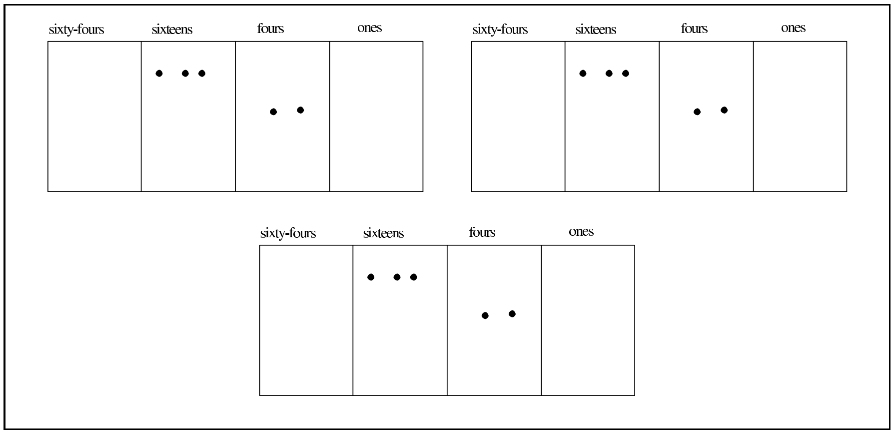
We cannot evenly distribute the one chip that remains in the sixteens place among the three cards, so we will exchange it for four chips in the fours place.



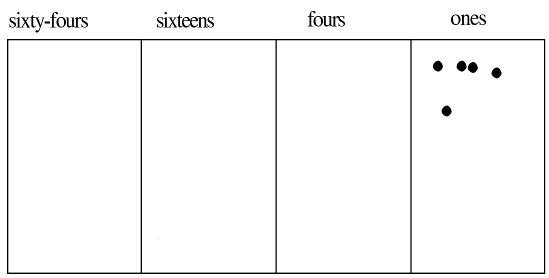


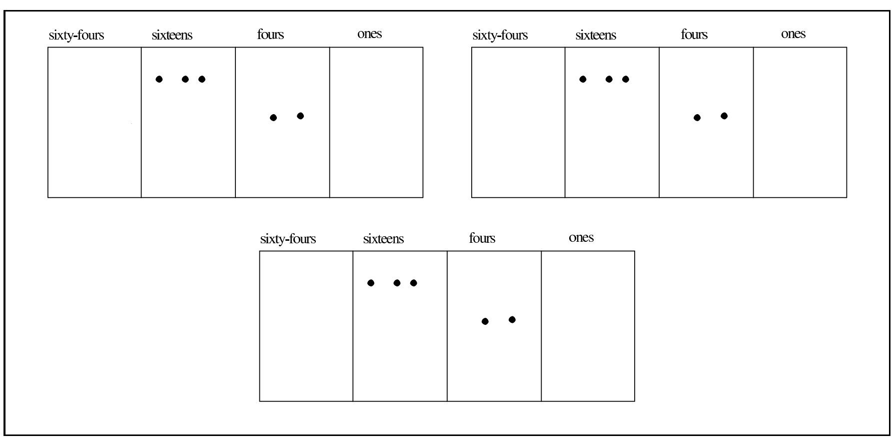
We now wish to evenly distribute the seven chips in the fours column into the fours columns of the three cards. We can place two chips in the fours column of each of the three cards. This will leave us with one chip in the fours column and one chip in the ones column.



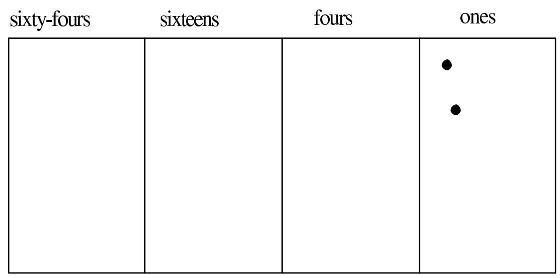


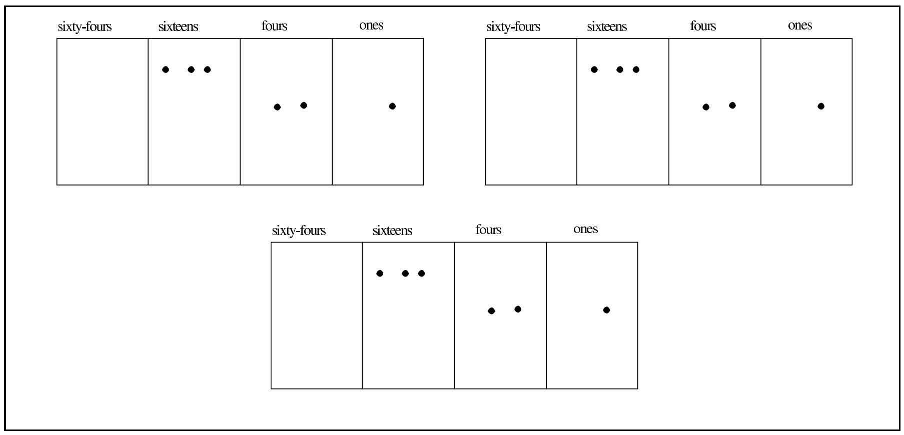
Once again, we cannot evenly distribute the one chip that remains in the fours column among the three cards. Therefore we will exchange it for four chips in the ones column.





Finally, we wish to evenly distribute the chips in the ones column into the ones columns of the three cards. We can place one chip in the ones column of each of the three cards. This will leave us with two chips in the ones column of our original card.





We cannot evenly distribute the two chips which remain in the ones column among the three cards. Moreover, we cannot make any more exchanges, so we have reached the end of the algorithm. Since each of the three cards represents 321_{four} and the original card represents 2_{four}, it follows that $2231_{four} \div 3_{four} = 321_{four}$ R 2_{four}.

Notes on Division: In my experience, division was the most difficult operation for the students to understand. There were two main difficulties they encountered. First, some of them wanted to represent the divisior on a place value card. That is, in their first attempt to solve the previous problem they had one card representing 2231_{four} and one card that representing 3_{four}. At this point they became stuck, for obvious reasons. The divisor needs to be represented with empty cards.

The second difficulty occured when the divisor has more than one digit. Even students who understood problems similar to $2231_{four} \div 3_{four}$ often became confused when trying to solve problems similar to $3131_{four} \div 12_{four}$. Some did not know where to begin, and others wanted to have twelve empty cards instead of six. They were still having difficulty differentiating between numbers and numerals. There were also some problems when the remainder was a multi-digit numeral. The remainder for $3131_{four} \div 12_{four}$ is five, which is represented 11_{four}. When they finally obtained the place value card that had five chips in the ones place, they saw that they could not evenly distribute the five chips among the six cards. Moreover, they could not make any more exchanges that would make it possible to distribute more chips. However, many of them forgot that 5 is not a numeral in base four. These students wrote $3131_{four} \div 12_{four} = 210_{four}$ R 5. When the algorithm was complete, they needed to make sure that all of the cards represented numerals in the appropriate base.

Since the base was four in this example, no card should have more than three chips in any place. The correct answer is $3131_{four} \div 12_{four} = 210_{four}$ R 11_{four}.