Class Notes

Contents

1 Thursday, August 25

1. Used padlet to brainstorm on terms and concepts related to polygons.

2. Worked on learning names of class members.

3. Reviewed the syllabus.

4. Classified triangles and drew examples using a diagram (posted on website).

   A triangle is

   - *acute* if *all* angles have measure less than 90° (are acute).
   - *right* if *one* angle has measure 90° (is right).
   - *obtuse* if *one* angle has measure greater than 90° (is obtuse).
   - *scalene* if *no* two sides are congruent (have the same length).
   - *isosceles* if *at least two* sides are congruent.
   - *equilateral* if *all three* sides are congruent. Thus equilateral triangles are also isosceles.

5. Showed how to sketch polygons, including regular polygons, and measure lengths and angles with GeoGebra. Note: To turn on the grid, choose the Move tool, and then right-click on the background.

6. Classified triangles with symmetries.

   A triangle is

   - *scalene* if it has *no* line of reflectional symmetry.
   - *isosceles* if it has *at least one* line of reflectional symmetry.
   - *equilateral* if has *three* lines of reflectional symmetry. In this case it also has 120° and 240° rotational symmetry.

7. Classified quadrilaterals with a diagram (posted on website).

   A quadrilateral is

   - *convex* if for every pair of points on or in the interior of the quadrilateral the line segment joining these two points lies on or in the interior. Otherwise the quadrilateral is *nonconvex*.

3

- a *trapezoid* if it has *at least* one pair of opposite parallel sides (called the *bases*). Thus parallelograms are also trapezoids.

- an *isosceles trapezoid* if it is a trazoid in which the other two opposite sides are congruent *and* the two angles in each base pair are congruent. Thus not all parallelograms are isosceles trapezoids.

- a *parallelogram* if it has *two* pairs of opposite parallel sides.

- a *rectangle* if it has four right angles.

- a *square* if it has four right angles and all sides are congruent.

- a *kite* if the four sides can be partitioned into two pairs of congruent adjacent sides. Note: It is permissible that all four sides are congruent.

- a *rhombus* if all four sides are congruent.
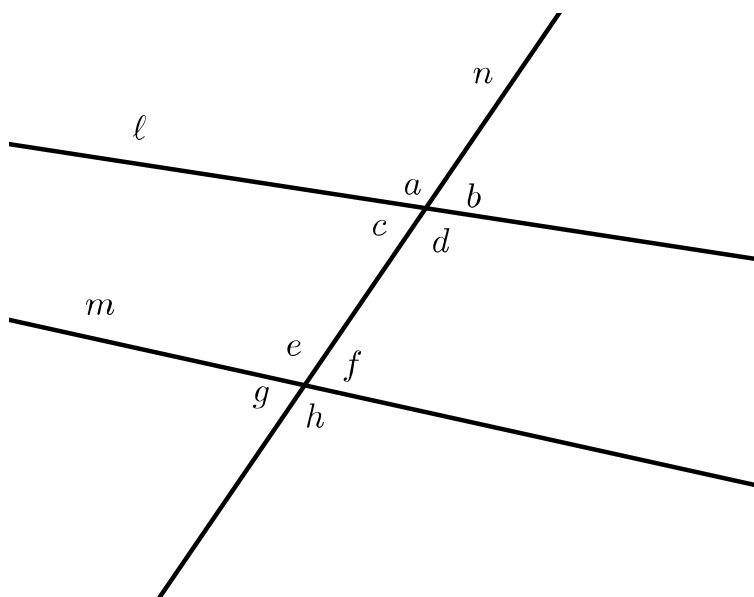
2 Tuesday, August 30

1. Answered some questions about the quadrilateral classofication chart. An arrow pointing from A to B means that every shape in class B is also in class A; that is to say, class B is a subset of class A. For example, the arrow from rectangles to squares means that every square is a rectangle.

2. Practiced measuring angles with protractors. Mentioned the website to create your own graph paper.

3. Mentioned the meaning of "1 radian": Wrap a curved arc with length equal to 1 radius along the circumference of a circle. The corresponding central angle has measure 1 radian, and we can now see why there are 2π radians in a circle, since the circumference equals $2\pi r$.

4. Tore off corners of triangles and quadrilaterals to motivate that the angle sum of triangles is $180°$ and the angle sum of quadrilaterals is $360°$.

5. Theorem: The sum of the measures of the interior angles in every triangle is $180°$. We did not prove this theorem yet.

6. Theorem: The sum of the measures of the interior angles in every quadrilateral is $360°$. We proved this by dissecting the quadrilateral into two triangles.

7. Theorem: Every polygon with n sides can be dissected using diagonals into $n - 2$ triangles. We did not prove this theorem for nonconvex polygons (though it is true), but for convex polygons we showed that you can pick a vertex and draw diagonals from that vertex to $n - 3$ other vertices, thus creating $n - 2$ triangles. We then used this theorem to prove the next.

8. Theorem: The sum of the measures of the interior angles in any polygon with n sides is $(n - 2)180°$. We had two proofs. For the first proof we dissected the polygon into $n - 2$ triangles. The sum of the interior angles of all of these triangles was seen to be the sum of the interior angles of the polygon. For the second proof (if the polygon is convex) we inserted a new point in the interior of the n-gon and joined this point to each of the n vertices, creating n triangles. We saw that the sum of the interior angles of all of these triangles was equal to $360°$ (around the inner point) plus the sum of the interior angles of the polygon. So this latter sum equals $n180° - 360° = (n - 2)180°$.

9. Theorem: The sum of the measures of the exterior angles in any polygon is $360°$. We argued this by "rotating and sliding" around the perimeter of the polygon, and then

argued it using the fact that at each vertex the exterior angle is supplementary to the interior angle, so we can use algebra to determine the exterior angle sum from the interior angle sum.

10. A *regular* convex polygon is one in which all sides are congruent and all interior angles are congruent. Notice that a regular n-sided polygon with have n lines of reflectional symmetry, and also rotational symmetry with angles that are multiples of $360°/n$. Theorem: The measure of each interior angle in a regular polygon with n sides is $\frac{n-2}{n}180°$.

11. Mentioned using regular polygons to create regular tessellations and semiregular tessellations.

12. Mentioned the structure and various components of the texts we are using.

3   Thursday, September 1

1. Recalled that when two lines cross, each pair of opposite angles forms a vertical pair of angles with equal measure, and each pair of adjacent angles is supplementary.

2. Consider the figure below.

Line n is a transversal to lines ℓ and m. Angles a and d form a pair of vertical angles, so $a = d$. Angles a and c form a pair of supplementary angles, so $a + c = 180°$.

Angles a and e form a pair of corresponding angles. Angles a and h form a pair of alternate exterior angles. Angles c and f form a pair of alternate interior angles. Angles c and e form a pair of consecutive interior angles.

3. We will assume the following theorems to be true:

Theorem 1. If x and y are a pair of corresponding angles in a transversal n of ℓ and m, and if ℓ and m are parallel, then $x = y$.
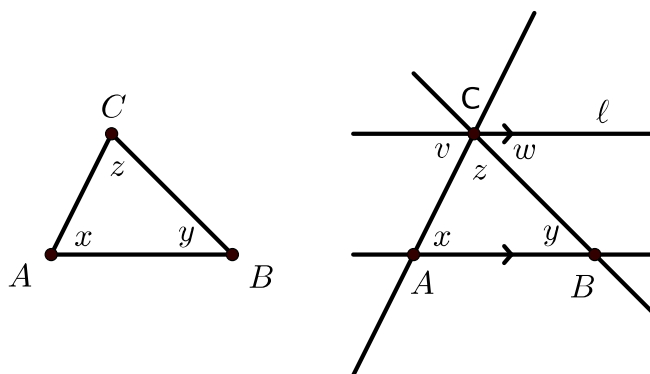
Theorem 2. If x and y are a pair of corresponding angles in a transversal n of ℓ and m, and if $x = y$, then ℓ and m are parallel.

We can use these two theorems, together with vertical angles and supplementary angles, to prove the following theorems.

- If x and y are a pair of alternate exterior angles in a transversal of parallel lines, then $x = y$.

- If x and y are a pair of alternate interior angles in a transversal of parallel lines, then $x = y$.

- If x and y are a pair of consecutive interior angles in a transversal of parallel lines, then x and y are supplementary.

- If x and y are a pair of alternate exterior angles in a transversal of lines ℓ and m, and $x = y$, then ℓ is parallel to m.

- If x and y are a pair of alternate interior angles in a transversal of lines ℓ and m, and $x = y$, then ℓ is parallel to m.

- If x and y are a pair of consecutive interior angles in a transversal of lines ℓ and m, and x and y are supplementary, then ℓ is parallel to m.

We worked on the first four in class; you should be able to prove the last two.

4. Now we can prove that the sum of the interior angles in every triangle is 180°. Let $\triangle ABC$ be a triangle. Choose, say, $\overline{AB}$ to be the base. Construct line ℓ through C parallel to the base $\overline{AB}$. Extend all three sides of the triangle to lines.

Since $\overleftrightarrow{AC}$ is a transversal to parallel lines ℓ and $\overleftrightarrow{AB}$, then alternate interior angles x and v are equal. Since $\overleftrightarrow{BC}$ is a transversal to parallel lines ℓ and $\overleftrightarrow{AB}$, then alternate interior angles y and w are equal. Since angles v, z, w together form a straight angle, then $v + w + z = 180°$. Therefore $x + y + z = 180°$
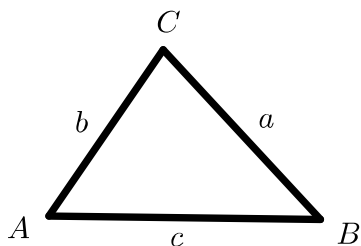
4 Tuesday, September 6

1. Discussed some homework problems.

   Triangle Inequality Theorem: If a, b, c are the side lengths of a triangle, then $a > b+c$, $b > a + c$, and $c > a + b$. Conversely, if a, b, c are positive numbers satisfying these three inequalities, then there is a triangle having these side lengths.

   Quadrilateral Inequality Theorem: If a, b, c, d are the side lengths of a triangle, then $a > b+c+d$, $b > a+c+d$, $c > a+b+d$, and $d > a+b+c$. Conversely, if a, b, c, d are positive numbers satisfying these four inequalities, then there is a quadrilateral having these side lengths.

   Problem #9a on page 77. We used GeoGebra to see that this triangle is impossible to construct. Then we used the Law of Sines to confirm the impossibility.

   Law of Sines: Consider the triangle given below.

   Then
   $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{sinC}{c}.$$

   If we substitute $c = 2.5$, $b = 1$, and $B = 40°$, then by the Law of Sines,
   $$\frac{\sin 40°}{1} = \frac{\sin C}{2.5}$$

   which implies $\sin C \approx 1.6$. But this is impossible since the sine of every angle lies between -1 and 1.

2. Played the Quadrilateral Game on pages 74–75 of Shapes and Designs using geoboards. Showed the virtual geoboard at `http://www.mathlearningcenter.org/web-apps/geoboard`.

3. Worked on Investigation 1 of Covering and Surrounding.

4. Examined the concepts of area and perimeter by constructing some polyominos.

5. Looked at the formulas for the area and perimeter of rectangles, and why these formulas make sense. If a rectangle has side lengths a and b, then its area is $A = ab$ and its perimeter is $P = 2a + 2b$.

6. Considered the problem of rectangles with minimum and maximum perimeter for a given area (Investigation 1.2). For example, consider the area $A = 24\,\text{m}^2$. With whole number sides, the rectangle with the minimum perimeter has dimensions $4\,\text{m} \times 6\,\text{m}$, and the rectangle with the maximum perimeter has dimensions $1\,\text{m} \times 24\,\text{m}$. If we allow noninteger side lengths, then we can use algebra to write the perimeter as a function of one side length x:

$$xy = 24,$$
$$y = 24/x,$$
$$P = 2x + 2y = 2x + 2(24/x) = 2x + 48/x.$$

We graphed this with Desmos, and saw that the minimum perimeter rectangle is a square with side length $\sqrt{24}\,\text{m}$, and that we can find rectangles with arbitrarily large perimeters.

5 Thursday, September 8

1. Mentioned website resources:

   (a) Describing and Defining Triangles — Mathematics Assessment Project.

   (b) Describing and Defining Quadrilaterals — Mathematics Assessment Project.

   (c) Applying Angle Theorems — Mathematics Assessment Project.

   (d) Pattern Shapes app.

   (e) iOrnament — iPad app for creating images with symmetries.

2. Discussed how to approach writing out solutions to problems. In particular, use full sentences and write in such a way that it is understandable to a friend or a student when read out loud. Include justifications for your statements, even if they are brief. Use drawing tools (like rulers or GeoGebra) for diagrams.

3. p.56 #16. Star.

4. p.57, #19–20. Tilings.

5. p.58, #22. Regular polygons.

6. p.58, #24. Irregular polygons.

7. p.59, #25. Number of diagonals.

8. #11. Reversing sentences.

6   Tuesday, September 13

1. Announced Exam #1 will be on Thursday, September 29, on the material associated with the first two books.

2. Showed the video Optimal Potatoes, by Vi Hart.

3. Considered the problem estimating areas of arbitrary shapes by overlaying grids (#75 on p. 31).

4. Visually confirmed that a one inch square consists of 4 half inch squares and 16 quarter inch squares. We expect this, since

$$\frac{1}{2}\,\text{in} \times \frac{1}{2}\,\text{in} = \frac{1}{4}\,\text{in}^2,$$

and

$$\frac{1}{4}\,\text{in} \times \frac{1}{4}\,\text{in} = \frac{1}{16}\,\text{in}^2.$$

5. Estimated, then calculated, the number of square centimeters in one square inch.

6. Showed how to convert units by "multiplying by 1s". For example, to convert 50 miles per hour to feet per second,

$$\frac{50\,\text{mi}}{1\,\text{hr}} \times \frac{5280\,\text{ft}}{1\,\text{mi}} \times \frac{1\,\text{hr}}{60\,\text{min}} \times \frac{1\,\text{min}}{60\,\text{sec}} = \frac{50 \cdot 5280\,\text{ft}}{60 \cdot 60\,\text{sec}}.$$

As an example of converting square units, to convert 10 square yards to square feet,

$$10\,\text{yd}^2 \times \left(\frac{3\,\text{ft}}{1\,\text{yd}}\right)^2 = 90\,\text{ft}^2.$$

7. Considered the problem of rectangles with minimum and maximum area for a given perimeter (Investigation 1.3). Started working on this problem when the perimeter $P = 18\,\text{m}$.

7   Thursday, September 15

We worked through a sequence of area formulas. In each case we could use the formulas derived earlier to justify the ones derived later.

1. Took as a given that the area of a square of side length s equals x^2.

2. Reviewed why the area formula for rectangles makes sense if the side lengths are positive integers.

3. Derived the area formula for rectangles using a dissection "puzzle" — see the handout "Rectangle Area Problem."

4. Derived the area formula for triangles, considering three cases.

5. Derived the area formula for parallelograms.

6. Derived the area formula for trapezoids.

8 Tuesday, September 20

1. Exam #1 is on Thursday, September 29.

2. Estimated the area of a circle of radius r — see the handout "Circle Area Problem."

3. For a circle we motivated the formula $A = \frac{1}{2}Cr$ two different ways. Then we used $C = 2\pi r$ to get $A = \pi r^2$.

   (a) Cutting the circle into sectors (wedges) and rearrange to approximate a parallelogram. See `https://www.geogebra.org/m/fyqAUV22`.

   (b) Uncurling concentric circles to make a triangle. See `https://www.geogebra.org/m/Ga3e6tfV`.

4. Worked on homework problems.

5. Discussed why it makes sense that the volume of a prism is Bh—the area of the base multiplied by the height—by thinking of stacking up layers of cubes.

6. Reviewed some of the Kentucky Academic Standards for geometry to observe the relationship with what we are doing in this course.

7. Considered nets of some solids and worked on the problem of finding all nets of an open-topped cubical box.

9 Thursday, September 22

1. Made some comments on homework being returned.

2. Showed some additional resources on area, perimeter, and distance.

3. Demonstrated the program SketchUp, including examples from middle school students in Lexington.

4. Reviewed the eight Mathematical Practices from the Common Core State Standards for Math, and discussed examples of activities from our course that correlated to some of these practices.

10   Tuesday, September 27

1. Answered questions.

2. Used proportional reasoning to estimate the height of individuals from photos of them holding rulers. We noted that measurement inaccuracies lead to inaccurate answers.

3. Drew simple figures with at least one curved line on quarter-inch graph paper. We then made a similar figure with scale factor of 4 on one-inch graph paper, and another similar figure with scale factor 2 on half inch graph paper. Observed that if the scale factor from Figure A to Figure B is s, and the scale factor from Figure B to Figure C is t, then the scale factor from A to C is st.

11   Thursday, September 29

Exam #1.

12   Tuesday, October 4

1. Made a simple two-dimensional shape using Polydron, and then made a second one similar to the first with a scale factor of two, and then another with a scale factor of three. We saw that all lengths doubled, all areas quadrupled, but all angle measures remained the same. Used the reptile consisting of an "L" of three squares to make another with a scale factor of two and another with a scale factor of three.

2. Worked on the problem of starting with a three-dimensional "building" made with Multilink cubes and creating a similar building with a scale factor of 2. We saw that the measures of angles did not change, the lengths were multiplied by the scale factor, the surface areas were multiplied by the square of the scale factor, and the volumes were multiplied by the cube of the scale factor. Because we can approximate irregular (but nonfractal) solids with cubes, the same scalings apply to them as well.

3. Mentioned an application of the above understandings to biology. Volume is associated with weight and heat production, area is associated with strength and heat escape, etc. See the essay "On Being the Right Size," posted to the course website.

4. Looked at the video Cosmic Voyage to visualize different scales of distance (link is on the course website).

13 Thursday, October 6

1. Demonstrated the Cosmic Eye app.

2. Discussed important concepts related to similar figures. Suppose figure A is similar to figure B with scale factor t from A to B. By definition, this means that for every distance or length a in A and the corresponding distance or length b in B, the ratio b/a always equals t. So if length p in A corresponds to length q in B, and length r in A corresponds to length s in B, we must have $b/a = q/p = s/r = t$. Notice that we can also determine for such similar figures that $bp = aq$ and so $a/p = b/q$. These ratios are ratios taken *within* each figure, and do not have to equal the scale factor. So, for example, the ratio of a door height to a door width might be 4/1 in both the model of a house and in the actual house, though the actual house may not be related to the model house by a scale factor of 4. Observed that corresponding angles in similar figures are congruent.

3. Note that the book describes how to make a similar figure with a scale factor of 2 using two rubber bands. This models a dilation. I modeled this with GeoGebra.

4. Worked on problems from the book, noting potential misconceptions, such as:

   (a) Confusing the mathematical meaning of "similar" to the everyday use of this word.

   (b) Incorrectly computing the scale factor based on scaling of areas instead of scaling of lengths.

   (c) Forgetting that distances between *all* pairs of corresponding points must be scaled by the same factor, not just line segments along the perimeter.

14   Tuesday, October 11

1. Discussed the dilation transformation. Each dilation f has a center of dilation, say, C, and a positive scale factor, say, s. Given any point A, the dilation maps A to the point B on the ray ray $\overrightarrow{CA}$ such that $CB = sCA$. If a dilation is applied to a figure, the result is a similar figure with scaling factor s. It is also possible to define dilations when $s < 0$; in this case the point B is placed on the ray $\overrightarrow{CA}$ such that $CB = |s|CA$.

2. Demonstrated dilations using GeoGebra.

3. If figure A can be made to coincide with figure B through a sequence of translations, rotations, reflections, and/or dilations, then A is similar to B.

4. For triangles $\triangle ABC$ and $\triangle DEF$, when we write $\triangle ABC \sim \triangle DEF$, we mean that under the specific correspondence of vertices $A \leftrightarrow D$, $B \leftrightarrow E$, and $C \leftrightarrow F$, we have that the two triangles are similar, and specifically

$$\frac{DE}{AB} = \frac{DF}{AC} = \frac{EF}{BC} = s,$$

where s is the scaling factor from $\triangle ABC$ to $\triangle DEF$. Note that if $\triangle ABC \sim \triangle DEF$, then $\mathrm{m}\angle A = \mathrm{m}\angle D$, $\mathrm{m}\angle B = \mathrm{m}\angle E$, and $\mathrm{m}\angle C = \mathrm{m}\angle F$.

5. The SSS Triangle Similarity Theorem: If triangles $\triangle ABC$ and $\triangle DEF$ are such that

$$\frac{DE}{AB} = \frac{DF}{AC} = \frac{EF}{BC} = s,$$

then $\triangle ABC \sim \triangle DEF$, and s is the scaling factor from $\triangle ABC$ to $\triangle DEF$.

6. The SAS Triangle Similarity Theorem. If triangles $\triangle ABC$ and $\triangle DEF$ are such that

$$\frac{DE}{AB} = \frac{DF}{AC} = s,$$

and $\mathrm{m}\angle A = \mathrm{m}\angle D$, then $\triangle ABC \sim \triangle DEF$, and s is the scaling factor from $\triangle ABC$ to $\triangle DEF$.

Justification. Translate, rotate, and/or reflect $\triangle ABC$ so that $\angle A$ coincides with $\angle D$, with point A now at point D, $\overrightarrow{AB} = \overrightarrow{DE}$, and $\overrightarrow{AC} = \overrightarrow{DF}$. Now dilate this triangle using center A and scaling factor s. As a result, B will move to E and C will move to F. Now the two triangles coincide. So the original triangle $\triangle ABC$ is similar to $\triangle DEF$.

7. Worked on homework problems.

15 Thursday, October 13

1. Be careful of units of distance and area when considering similar figures. For example, if a map scale is given as 1 inch equals 500 miles, the scale factor from the map to the real world is

$$\frac{500 \,\text{real mi}}{1 \,\text{map in}} \times \frac{5280 \,\text{real ft}}{1 \,\text{real mi}} \times \frac{12 \,\text{real in}}{1 \,\text{real ft}} = \frac{31,680,000 \,\text{real in}}{1 \,\text{map in}}.$$

If you are estimating area from the map, and you see that the area is 3 square inches on the map, then the actual area is

$$3 \,(\text{map in})^2 \times \left(\frac{500 \,\text{real mi}}{1 \,\text{map in}} \right)^2 = 750,000 \,(\text{real mi})^2.$$

2. Discussed the homework problem on the golden ratio. This ratio is related to the Fibonacci numbers and arises frequently in nature, as well as in art and architecture.

3. Analyzing transformation formulas. Apply each formula below to triangle $\triangle ABC$, where

$A = (0,0)$, $B = (3,0)$, and $C = (0,4)$. In each case answer these questions. Is the resulting triangle similar to the original one? If so, is the transformation a dilation? If it is a dilation, what is the center and the scaling factor? It is helpful to keep in mind that if a dilation is applied to a line segment, the resulting line segment will be parallel to the original one.

   (a) $(3x, 3y)$.
   (b) $(3x + 1, 3y - 1)$.
   (c) $(\frac{1}{2}x - 2, \frac{1}{2}y + 1)$.
   (d) $(x + 3, y + 4)$.
   (e) $(2x + 1, 3y - 2)$.
   (f) $(-x, -y)$.
   (g) $(-2x, 2y)$.
   (h) (y, x).
   (i) $(y, -x)$.
   (j) $(x + y, -x + y)$.

4. The AA Triangle Similarity Theorem. If triangles ΔABC and ΔDEF are such that $m\angle A = m\angle D$ and $m\angle B = m\angle E$, then $\Delta ABC \sim \Delta DEF$.

Justification. Translate, rotate, and/or reflect ΔABC so that $\angle A$ coincides with $\angle D$, with point A now at point D, $\overrightarrow{AB} = \overrightarrow{DE}$, and $\overrightarrow{AC} = \overrightarrow{DF}$. Now dilate this triangle using center A so that B will move to E. Because $m\angle B = m\angle E$, we have $\overrightarrow{BC} = \overrightarrow{EF}$. This implies that points C and F are equal. Now the two triangles coincide. So the original triangle ΔABC is similar to ΔDEF.

16   Tuesday, October 18

1. Answered some questions about the homework.

2. Mentioned the nice isometric drawing drawing tool: `https://illuminations.nctm.org/activity.aspx?id=4182`. Note that it can provide top, side, and front views.

3. Worked on volume and surface area formulas. Remember to be precise in your vocabulary. A nice way to draw some of these solids is with SketchUp.

4. General prism, not necessarily right, with irregular base.

$$V = Bh$$

where B is the area of the base and h is the height of the prism. This is motivated by approximating the prism with a stack of layers of cubes. We did not have a simple formula for the surface area.

5. General right prism with irregular base, which may or may not be a polygon.

$$V = Bh$$

$$S = 2B + Ph$$

where B is the area of the base, h is the height of the prism, and P is the perimeter of the base. The surface area formula comes from the two bases together with the rectangle resulting from "unwrapping" the lateral surface area.

6. Right prism with base being a regular polygon.

$$V = \frac{1}{2}Pah$$

$$S = Pa + Ph$$

where P is the perimeter of the base, a is the apothem of the base, and h is the height of the prism. The formulas come from the formulas in (5), remembering that the area of a regular polygon is $\frac{1}{2}Pa$.

7. Right rectangular prism.

$$V = xyz$$

$$S = 2(xy + xz + yz)$$

where the dimensions of this "box" are x, y, z.

8. Right circular cylinder.

$$V = \pi r^2 h$$

$$S = 2\pi r^2 + 2\pi rh$$

where r is the radius of the base and h is the height of the cylinder. The formulas come from the formulas in (5), remembering that the area of a circle is πr^2 and the circumference ("perimeter") of a circle is $2\pi r$.

17   Thursday, October 20

1. Announced that Exam #2 will be on Thursday, November 3.

2. Discussed some homework.

3. Worked on Pyramid Puzzle I to see that for this particular pyramid, the volume is one third of a cube, and so in this case equals $\frac{1}{3}Bh$. It turns out that this formula works for general pyramids and cones. Used SketchUp to make a drawing.

4. Constructed pyramids using Polydron and derived the formula for the surface area when the base is a regular polygon and the apex is centered above the base:

$$S = \frac{1}{2}Pa + \frac{1}{2}P\sqrt{a^2 + h^2}.$$

18 Tuesday, October 25

1. Developed the formula for the surface area of a cone two different ways—one by approximation by a many-sided pyramid, and one by cutting the cone open into a shape that is the sector of a circle.

$$S = \pi r^2 + \pi r \sqrt{r^2 + h^2}$$

2. Similar Cross-Section Property. Similar cross-sections of cones and pyramids. Let F be a cone or pyramid of height h. Let P be a plane parallel to the base at a distance x from the apex. Then the cross-section $P \cap F$ is similar to the base, with a scaling factor of $s = x/h$. Thus, its area is $(x/h)^2 B$, where B is the area of the base. Used SketchUp to illustrate this.

19 Thursday, October 27

1. Moved date of Exam #2 to November 10.

2. Watched two videos of a middle school classroom in which the teacher has created an active learning environment. Mistakes are welcomed and are a source of learning, and students are engaged in developing understanding, solutions to problems, and many explanations.

20   Tuesday, November 1

1. One Direction Scaling Property. If you scale a solid by a factor of s in one direction only, then its volume is multiplied by s. (Think of stretching an approximating set of cubes.)

2. Cavalieri's Principle for volumes. From the Wikipedia article "Cavalieri's Principle":

   3-dimensional case: Suppose two regions in three-space (solids) are included between two parallel planes. If every plane parallel to these two planes intersects both regions in cross-sections of equal area, then the two regions have equal volumes. This is directly related to the calculus procedure of finding volumes by slicing.

   (There is also a 2-dimensional version: Suppose two regions in a plane are included between two parallel lines in that plane. If every line parallel to these two lines intersects both regions in line segments of equal length, then the two regions have equal areas.)

3. So if two pyramids or cones F and G have the same heights and have bases of the same areas, and P is as above, then the two cross-sections $P \cap F$ and $P \cap G$ have the same areas.

4. The special pyramid F^* in the pyramid puzzle has volume equal to $\frac{1}{3}B^*h^*$.

5. Let G be any pyramid or cone. Let B be its base area and h be its height. Then its volume is $\frac{1}{3}Bh$.

   Justification:

   Start with F^*. Its volume is $\frac{1}{3}B^*h^*$. Scale it by a factor s to get a pyramid F' with base area B' and height h', choosing s so that $B' = B$. Then the volume of F' is $s^3\frac{1}{3}B^*h^* = \frac{1}{3}s^2B^*sh^* = \frac{1}{3}B'h' = \frac{1}{3}Bh'$.

   Now scale F' in the direction of its height only by a factor of t to get a pyramid F'' with base area B and height h'', choosing t so that $h'' = h$. Then the volume of F'' is $t\frac{1}{3}Bh' = \frac{1}{3}Bth' = \frac{1}{3}Bh$.

   Now F'' and G have the same heights and have bases of the same areas. So by the Similar Cross-Section Property and Cavalieri's Principle, they have the same volumes.

6. Volume of sphere via Cavalieri
$$V = \frac{4}{3}\pi r^3.$$

   Justification: We used Cavalieri's Principle on two solids. One was an upper hemisphere of radius r. The other was a cylinder of radius r and height r from which

was removed an inverted cone of radius r and height r. We showed that at height x the cross-section of the hemisphere was a circle of radius $\sqrt{r^2 - x^2}$ and hence area $\pi(r^2 - x^2)$, and we showed that at height x the cross-section of the second solid was a "washer" consisting of an outer circle of radius r and an inner circle of radius x and hence an area of $\pi r^2 - \pi x^2$. Thus the volume of the hemisphere equals the volume of the cylinder minus the volume of the cone:

$$\pi r^2 r - \frac{1}{3}\pi r^2 r = \frac{2}{3}\pi r^3.$$

21 Thursday, November 3

1. Motivated the formula $V = \frac{1}{3}Sr$ for a sphere by dissecting it into skinny pyramids. Then used this to prove $S = 4\pi r^2$.

2. Proved the Pythagorean Theorem three different ways.

   (a) By drawing the altitude from the right angle and using similar triangles.

   (b) By drawing four copies of the right triangle in a square frame and using algebra (thanks to Rebecca).

   (c) By placing four copies of the right triangle in a square frame two different ways.

22 Tuesday, November 8

Election Day — no class.

23 Thursday, November 10

Exam #2.

24   Tuesday, November 15

1. Discussed how to use similarity of right triangles with a common acute angle to motivate and define the basic trig ratios, and also justify $\sin^2 x + \cos^2 x = 1$ (using the Pythagorean Theorem).

2. Worked on Looking for Squares (2.1 of *Looking for Pythagoras*). This led to seeing how to construct some numbers like $\sqrt{2}$ and $\sqrt{5}$ geometrically, as well as to yet another proof of the Pythagorean Theorem ("folding in" right triangles into a square of side length c).

3. Reviewed the statement of the Pythagorean Theorem and its converse (3.1–3.2, 3.4).

   Pythagorean Theorem: IF T is a right triangle with side lengths a and b and hypotenuse c, THEN $a^2 + b^2 = c^2$.

   Converse: IF T is a triangle with side lengths a, b, c with $a^2 + b^2 = c^2$, THEN T is a right triangle and the hypotenuse has length c.

   It is not always the case that the converse of a theorem is true. For example,

   TRUE: IF n is a prime larger than 2, THEN n is odd.

   FALSE: If n is an odd integer greater than 2, then n is a prime.

4. Saw how the Pythagorean Theorem leads to the distance formula (3.3).

25   Thursday, November 17

1. Worked on homework.

2. Saw how the Pythagorean Theorem leads to equations of circles (5.3).

3. Representing fractions as decimals, and understanding why they repeat (4.2).

4. Representing decimals as fractions (4.3).

26   Tuesday, November 22

Watch the following videos:

1. Snowflakes, Starflakes, and Swirlflakes, `https://www.youtube.com/watch?v=8EmhGOQ-DNQ`.

2. Sphereflakes, `https://www.youtube.com/watch?v=toKu2-qzJeM`.

3. Cookie Shapes, `https://www.youtube.com/watch?v=_n1126GoxbU`.

4. Origami Proof of the Pythagorean Theorem, `https://www.youtube.com/watch?v=z6lL83wl31E`.

5. Doodling in Math: Spirals, Fibonacci, and Being a Plant, `https://www.youtube.com/watch?v=ahXIMUkSXX0`, `https://www.youtube.com/watch?v=lOIP_Z_-OHs`, and `https://www.youtube.com/watch?v=14-NdQwKz9w`

27 Tuesday, November 29

1. Discussed symmetries of finite figures and of strip patterns, using the slides `http://www.ms.uky.edu/~lee/ma111fa11/slides11good.pdf`, posted on the course website.

2. Answered questions about the final project.

3. Tried to solve Scott Kim's ambigram problems, `http://www.scottkim.com.previewc40.carrierzone.com/thinkinggames/nctm2011handout.pdf`, posted on the course website.

28 Thursday, December 1

1. Reflections. Definition. Applying to figures on paper. Constructing using Geoboards. Constructing using Miras. Modeling with students using meter stick mirrors. Constructing using GeoGebra.

2. Rotations. Definition. Applying to figures on paper. Constructing using Geoboards. Constructing using GeoGebra.

3. Translations. Definition. Applying to figures on paper. Constructing using GeoGebra.

4. Demonstrated that the result of successive reflections in two parallel lines is a translation in a direction perpendicular to the lines by an amount equal to twice the distance between the lines.

29 Tuesday, December 6

Worked on "Picturing Functions".

30 Thursday, December 8

1. Defined congruence in terms of translations, rotations, and reflections.

2. Worked on the "Isometry Identification Problems."

3. Examined some of the Kentucky Mathematics Deconstructed Standards o see how they related to some of the material and goals of the course.