Class Notes

January 15

1. Worked on problem 4, Ducks and Cows. Made posters. Discussed the variety of solutions. Handout: Problems, page 1.
2. Went over the syllabus. Handout: Syllabus.
3. Discussed Polya's four phases in the context of the Ducks and Cows problem. Handout: Polya's Four Phases of Problem Solving.
4. Discussed Kentucky's Program of Studies. Handout: Kentucky's Program of Studies, plus Kentucky's Process Standards for Eighth Grade.
5. Discussed the NCTM Standards. Handout: NCTM National Mathematics Standards, plus summaries of the five NCTM Process Standards.
6. Began problem 6(a), Counterfeit Coins.

January 20

1. Discussed Counterfeit Coins problem (#6) and extensions to (a) to larger numbers of coins.
2. Introduced the National Library of Virtual Manipulatives website, nlvm.usu.edu/NLVM, including handout. Looked at the Coin problem (including the solution to the 9 coin problem and a demonstration of the 12 coin problem), Base Blocks Addition, Base Blocks Subtraction, Algebra Balance Scales, Algebra Balance Scales - Negatives, Bar Chart, Coin Tossing, and Spinners. Mentioned the difference between theoretical and experimental probability.
3. Brief discussion on Chapter 1.
4. Worked the Elevator Problem (#2), and discussed the variety of solutions. By the way, there is more on analyzing and converting units of measurement in Chapter 8.

January 22

1. Mentioned that Chapter 8 has a more extensive discussion of units of measurement and conversion.
2. Worked and discussed problems in Chapter 1: #7 (Race, p. 20) and #11 (Circular Table, p. 22). For the Circular Table problem, we mentioned some geometric theorems, including:
   (a) The Pythagorean Theorem: In a right triangle, the sum of the squares of the lengths of the two legs equals the square of the length of the hypotenuse.
   (b) If a tangent and a radius meet at a common point on the circumference of a circle, then they are perpendicular.
   (c) The sum of the measures of the interior angles of a quadrilateral equals 360 degrees.
   (d) Alternate interior angles formed by a transversal of two parallel lines are congruent and thus have equal measure.
   (e) The two diagonals of a square have equal length.
3. Worked and discussed Problem 1, page 7, Soccer Game.
   (a) Discussed the differences in solutions if one counts each high five made by a pair of people once, or if one counts all of the high fives made by each individual (and thus getting an answer that is twice as large).
   (b) Found two different approaches to counting the high fives among the 11 winning team members: Either $(11 \times 10)/2$ or $1 + 2 + 3 + \cdots + 8 + 9 + 10$.
   (c) Discussed "Gauss's method" for adding the integers from 1 to 100.

January 27 Class canceled due to the ice storm.

January 29 Class canceled due to the ice storm.

February 3

1. Worked on problem #6 on page 40. Discussed the nature of "systematic" lists. Gave an example of using a tree diagram to solve this problem. Showed how to organize the list in Excel, including checking the sums with color-coded cells.
2. Passed back Homework #1, together with a handout of a particular solution from one of the class members, and a handout on "Review of Problem Solutions." Everyone should add "Check your answer" to this handout!
3. Worked on problems in Homework #2. In particular, discussed the solution to problem #4. For a summary of trigonometry, see, for example, http://en.wikipedia.org/wiki/Trig.

February 5

1. Discussed Problem #1 on Homework #2. The important principle is that this problem can be solved with general H and L by exactly the same procedure as with specific values of H and L. Let D be the number of ducks and C be the number of cows. We get one equation based on heads, $D + C = H$, and one equation based on legs: $2D + 4C = L$. Subtract twice the first equation from the second: $2C = L - 2H$. Solve for C: $C = (L - 2H)/2 = L/2 - H$. Substitute into the first equation: $D + (L/2 - H) = H$. Solve for D: $D = 2H - L/2$.
2. Discussed Problem #5 on Homework #2. One solution: If n happens to be even and we compute the sum $1+2+3+\cdots+(n-2)+(n-1)+n$ by adding the first term to the last, the second to the next-to-last, etc., then we get $(n+1)+(n+1)+\cdots+(n+1)$ with $n/2$ terms. So the answer is $n(n+1)/2$. If n is odd then we have to be careful because there is an unmatched middle term—please think about the details of this. Another solution is to let $S = 1+2+3+\cdots+(n-2)+(n-1)+n$. Then S also equals $n+(n-1)+(n-2)+\cdots+3+2+1$. Adding these two expressions together yields $2S = (n+1)+\cdots+(n+1)$, with n terms. So $2S = n(n+1)$ and thus $S = n(n+1)/2$. Another solution is to remember that if $n+1$ people each give "high fives" to each other, then from discussion in class the number of high fives is $n+(n-1)+(n-2)+\cdots+3+2+1$. But the $n+1$ people each give high fives to n others. So the total number of high fives, counted from each person's point of view, is $n(n+1)$. But we have counted all high fives twice, so there are in fact only $n(n+1)/2$ high fives. Therefore, the number of high fives is both $1+2+3+\cdots+n$ and $n(n+1)/2$, so these must be equal to each other.
3. Discussed Problem #2.1, Chickens and Eggs. Handout: Problems 2.1–2.8. One solution: Let C be the number of chickens, E be the number of eggs, and D be the number of days. We are given that $(C, E, D) = (1.5, 1.5, 1.5)$ is a valid possibility. Using that C and E are proportional if D is held constant, we can multiply C and E each by $4/3$ to get the valid combination $(C, E, D) = (2, 2, 1.5)$. Now using that E and D are proportional if C is held constant, we can multiply E and D each by 6 to get the valid combination $(2, 12, 9)$. Another solution is to recognize that E is proportional to both C and D, so there is a constant k such that $E = kCD$. Substituting the given combination, we get $1.5 = k(1.5)(1.5)$. Solve for k: $k = 2/3$. Thus $E = (2/3)CD$. Now substitute $C = 2$ and $E = 12$: $12 = (2/3)(2)D$. Finish up by solving for D: $D = 12(3/2)(1/2) = 9$.
4. Collected Homework #2 and passed out Homework #3.
5. Began working on Problem #2.6, Bookshelf.

February 10

1. Announced that I am revising Homework #3 by removing the first two problems.
2. Briefly introduced Google Earth.
3. Discussed a family of problems that can be tackled with tree diagrams and systematic lists that illustrate the Multiplication Counting Principle. Discussed also how you can generate the systematic list from the tree. The problems below may may not be the exact ones we discussed.
   (a) If you have 3 math books, 2 science books, and 3 art books, how many ways can you choose one of each? From a tree diagram we can see the answer as $3 \cdot 2 \cdot 3$ because there are 3 choices for the math book, then 2 for the science book, then 3 for the art book.
   (b) If you have 4 books, how many ways can you order then on a shelf? From a tree diagram we can see the answer as $4 \cdot 3 \cdot 2 \cdot 1$ because there are 4 choices for the first book on the shelf, then 3 for the second book, then 2 for the third book, and only 1 for the fourth book.
   (c) If you have 7 books, how many ways can you select and order 3 on a shelf? From a tree diagrams we can see the answer as $7 \cdot 6 \cdot 5$ (three terms—one for each spot on the shelf) because there are 7 choices for the first book, then 6 choices for the second book, and finally 5 choices for the third book. This answer equals $\frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1}$ which is $\frac{7!}{4!}$.
   (d) If you have 7 books, how many ways can you select 3 to put into your backpack (order is not important). If we look at the tree diagram from the previous problem, we see that we are over counting combinations of books. For example, the set of three books $\{1, 2, 3\}$ comes in 6 orderings: 123, 132, 213, 231, 312, 321. From our earlier problem we know that this number 6 is 3!. So we have over-counted all of our sets 6 times. Therefore we can take our old answer and divide by $3 \cdot 2 \cdot 1$:

$$\frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1)(3 \cdot 2 \cdot 1)} = \frac{7!}{(4!)(3!)}.$$

February 12

1. Exam #1 will be in class on Thursday, February 26.
2. Passed back Homework #2 and collected Homework #3.
3. Handout: Problems III.
4. Discussed Problem 3.1 (How Many Questions?). In particular, discussed a halving strategy that can determine a number among 2^n numbers using n questions. For example, a number from 0 to 127 can be determined in 7 questions. This is an example of using the strategy of "Eliminating Possibilities."
5. Discussed Problem 2.7 (Packing Books), part 4. In particular, discussed how the number of combinations is 2^{20} using a tree diagram and the Counting Principle.
6. Discussed Problem 2.8 (Counting Pairings) by modeling it as a sequence of decisions and using the Counting Principle. If men happen to be selecting women, then the first man has n choices, the second then has $n-1$ choices, the third man then has $n-2$ choices, and so on. So the total number of pairings is $n!$.
7. Discussed Problem 3.3 (Number Guessing Cards). Discovered that the guesser simply adds the first numbers on the selected cards. We did not determine why this works yet, though people began to see patterns on how which numbers went on which cards.
8. Discussed the Twenty-Seven Card trick. I will pass out a handout on this later; right now it is on the "Connections" part of the course website.

February 17

1. Mentioned that I expect you to know the statements of Polya's four main phases of problem solving for the Exam.
2. Mentioned that the problems from the book that I assigned for Homework #4 are all to be taken from Problem Set A, pp. 60–65.
3. Looked at the Sieve of Eratosthenes as an example of the application of the strategy of "eliminating possibilities"—see Problem 3.2 in my list of problems. The primes among the numbers from 2 to 100 can be found by eliminating all of the multiples of 2 except 2 itself, then all the multiples of 3 except 3 itself, then 5, and then 7. We can stop here because the next prime is 11, and any composite number that has no prime factors among 2, 3, 5, and 7 must have at least two factors 11 or higher, and hence must be at least 121. In general, and by the same reasoning, to find all the prime numbers from 2 to n we only need to eliminate multiples of primes from 2 up to $\sqrt{n}$. We illustrated this method using the National Library of Virtual Manipulatives. We also discussed why 1 is not regarded as a prime number—one motivation is being able to say that (up to ordering) every number has a unique factorization into primes, and being allowed to include 1 a multiple number of times would make this statement false. So perhaps a good definition to remember is that a positive integer is prime if it has exactly two positive factors.
4. Had some discussion about how one might organize an explanation of the Twenty-Seven Card Trick problem.
5. Worked on homework problems, with some class discussion of Problem 9 (To Tell the Truth).

February 19

1. Returned Homework #3 and collected Homework #4.
2. Handouts: Excel—Getting Started; Mathematical Magic—The Twenty-Seven Card Trick and the Eight Card Trick; Index of Learning Styles.
3. Discussed how to approach the solutions to Problems 2.6 and 2.7, including where the formulas ${}_mP_n = \frac{m!}{(m-n)!}$ and ${}_mC_n = \binom{m}{n} = \frac{m!}{n!(m-n)!}$ come from (how to derive them). Discussed that to solve part 4 of Problem 2.7, you could realize that for each of the 20 books you have two decisions (to take or not to take), so by the Multiplication Counting Principle there are $2 \cdot 2 \cdots 2$ choices (twenty terms), and hence 2^{20} outcomes. But you could also calculate the sum $\binom{20}{0} + \binom{20}{1} + \cdots + \binom{20}{20}$.
4. Discussed that a statement of the form "If A then B" does not imply "If B then A." However, it *does* imply "If B is not true, then A is not true."
5. Discussed how to begin solving Problem #11, page 63. If you have not yet read Chapter 3, you should do so, and particularly look at the discussion of "Down on the Farm" that begins on page 50.
6. Demonstrated how to use a systematic list of possibilities to solve Problem 3.7 (Watching TV). Using 0 to represent "not watching TV" and 1 to represent "watching TV", we can list all 32 possible combinations. They can actually be thought of as binary representations of numbers, so I am going to label them 0 to 31:

#	A	B	C	D	E
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	1	0
3	0	0	0	1	1
4	0	0	1	0	0
5	0	0	1	0	1
6	0	0	1	1	0
7	0	0	1	1	1
8	0	1	0	0	0
9	0	1	0	0	1
⋮	⋮	⋮	⋮	⋮	⋮

Then you can use the statements from the problem to cross out possibilities. For example, statement #2 eliminates rows 0, 4, 8, . . . ; statement #3 eliminates rows

0, 1, 2, 3, ...; statement #4 eliminates rows 2, 3, 4, 5, ... After all statements are processed, whatever remains are the only viable possibilities.

February 24

1. Notes about the exam: Bring your student ID's. Part of the exam will offer choices of problems to select from. Be sure to know Polya's four stages of problem solving.
2. Professor Ben Braun will be guest instructor next Tuesday.
3. Passed back homework.
4. Answered questions.
5. Worked on Chapter 4, page 102, problem 1.
6. Worked on Chapter 4, page 96, problem 9. Mentioned that if you search the internet for "grid logic" you can find lots of examples of such logic puzzles.
7. Worked on Chapter 5, page 126, problems 1 and 2.
8. Worked on Chapter 5, page 127, problem 7.

February 26 Exam #1

March 3 Guest presenter: Professor Benjamin Braun

1. Discussed the Fibonacci sequence, both in recursive and closed form.
2. Worked on problems 10 and 11 on page 129.
3. Demonstrated with Maple how Pascal's Triangle arises from $(1+x)^n$, and looked at the triangle formed by the coefficients of $(1+2x)^n$.

March 5

1. Passed back exams and explained midterm grades.
2. A new homework assignment will be emailed out later today.
3. Worked on problems 3.1–3.3 in the handout on Quadratic Patterns of Change, noting connections with problems we have previously solved in this course. I will not be posting this handout on the website—if you did not get one yet, please see me.
4. Briefly demonstrated how to graph functions with the program GeoGebra, available at www.geogebra.org.
5. Worked on problem #51 on page 53.

March 10

1. Gave an example to show that you can choose any number a that you wish for the fourth term of the sequence $s_1 = 1$, $s_2 = 2$, $s_3 = 3$, $s_4 = a$, and come up with a formula for that sequence. For example, if you want the fourth term to be 12 (so the sequence is $1, 2, 3, 12, \ldots$), then the formula $s_n = \frac{4}{3}(n^3 - 6n^2 + 11n - 6) + n$ works! In general, for any choice of a you can use the formula $s_n = \frac{a-4}{6}(n^3 - 6n^2 + 11n - 6) + n$. In fact, for any finite number of terms you can create a formula that leads to a sequence starting with those terms. So when we are faced with a sequence and try to detect a pattern and come up with a formula that matches the part of the sequence we see so far, we must keep in mind that we are still making a good guess. We must use other means to justify why the formula is true for *all* values of n.

2. Demonstrated how to use finite differences to find a formula for a sequence, such as $a_n = 0, 1, 3, 6, 10, 15, \ldots$ for $n = 0, 1, 2, 3, 4, 5, \ldots$. Refer to Chapter 14 of the text. The first differences are $1, 2, 3, 4, 5, \ldots$ and the second differences are $1, 1, 1, 1, \ldots$. Because it appears that the second differences are all constant, we guess that our formula is given by a quadratic polynomial $s_n = gn^2 + fn + e$. (If the third differences were all constant, we would guess that we have a cubic polynomial, $s_n = hn^3 + gn^2 + fn + e$, and if the first differences were all constant, we would guess that we have a linear polynomial, $s_n = fn + e$.) We are trying to figure out the values of the three coefficients e, f, g.

   Method 1 for finding the coefficients: We are looking for three unknowns so we create three equations by substituting in three values of n. For example, we may use $n = 0, 1, 2$, to get:

   $$\begin{aligned} 0 &= 0g + 0f + e \\ 1 &= g + f + e \\ 3 &= 4g + 2f + e \end{aligned}$$

   The first equation gives $e = 0$. Substitute this into the second and third equations to get:

   $$\begin{aligned} 1 &= g + f \\ 3 &= 4g + 2f \end{aligned}$$

   Eliminate f by multiplying the first equation by -2 and adding the equations to get $1 = 2g$, so $g = \frac{1}{2}$. Substitute this value of g into $1 = g + f$ and solve to get $f = \frac{1}{2}$. Now we know e, f, g and we have our (by now familiar) formula $s_n = \frac{1}{2}n^2 + \frac{1}{2}n$.

Method 2 for finding the coefficients is described in great detail in Chapter 14 of the text, and usually is much faster than Method 1, so you should all read and understand this well.

3. Worked on Homework #5.

March 12

1. Various students presented their solutions to problems in Homework #5, and we saw some alternate solutions to some of these problems.
2. Discussed the meaning of statements like: "Show that for any positive integer n, the sum of the positive integers from 1 to n equals $\frac{n(n+1)}{2}$." It is *not* enough to select one value of n, say, 5, and show that $1 + 2 + 3 + 4 + 5 = \frac{5(6)}{2}$. Rather, one must show that the formula is going to hold true for *all* values of n. A reasonable way to avoid this pitfall in most cases is to replace the word "any" with "all".
3. Changed the due date for Homework #5 to Tuesday, March 24.

March 24

1. Announced the date for the second exam: April 16, in class.
2. Discussed making sense of division—12 divided by 4, 12 divided by 1/2, 1 divided by 2/3. Then watched a video from *Connecting Mathematical Ideas* by Boaler and Humphreys in which the above problem is discussed by a seventh grade math class.
3. Worked on determining the volume and the surface area of a paper cup that is not a cylinder.
4. Collected Homework #5. The next homework assignment will be sent out by email.

March 26

1. Passed out the assignment for Homework #6, but changed the due date to Thursday, April 2.
2. Passed out an announcement for the origami/art/mathematics exhibit coming up at UK at the end of the month. See www.uky.edu/Libraries/Origami.
3. Discussed the problem of finding the volume of a cup. One proposal (a nice idea which we will see later does not give the correct answer) is to multiply the height of the cup by the average of the areas of the two bases (top and bottom). We began thinking of the cup as a cone with a smaller cone cut off. By looking at a cross-section, we determined some similar right triangles and set up a process to find the height of the cone that is cut off, and hence the height of the full cone. Once we have these measurements, we can subtract volumes to get the volume of the cup. We recalled that the volume of a cone of height h and radius r is $\frac{1}{3}\pi r^2 h$. (We also recalled that the volume of a right rectangular prism, a "brick", is length$\times$width$\times$height, and the volume of a cylinder of height h and radius r is $\pi r^2 h$.) During the course of this discussion we also commented on how to convert units of length, area, and volume; e.g., inches to centimeters, square inches to square centimeters, and cubic inches to cubic centimeters, and the like.
4. Watched some clips from the video *The Hobart Shakespeareans*. There is an associated book called *Teach Like Your Hair's on Fire*. You can find further information about both on Amazon.com.

March 31

1. Went over the solution to the problem of determining the volume and the surface area of a cup.
2. Worked on homework problems.
3. Reminder: Exam #2 will be on Thursday, April 16.

April 2

1. Reminder that Exam #2 will be on Thursday, April 16.
2. Collected Homework #6.
3. Passed out new installment of problems—Problems IV.
4. Passed out announcement of the Mathematics of Juggling event, Wednesday, April 8, 4 pm, CB 118, with pizza and a drawing for an iPod.
5. Discussed the piano problem from Homework #6, including a demo of the website with the virtual keyboard. Some advantages of this problem: using mathematics to model and solve an unfamiliar problem, making connections to another area of study (music), reviewing properties of geometric sequences (a staple of middle and high school), and practicing with quantities like $2^{1/12}$ that are not as commonly used in lessons as integers, simple fractions, and square roots of whole numbers, but which are heavily used in college math, science, engineering, and technology.
6. Discussed why the cup problem (page 180, #13) exemplifies the problem solving strategy of "Identifying Subproblems"; in this case, similar triangles, solving equations, using the formula for the volume of a cone, and casting the problem in terms of subtracting a small cone from a large cone.
7. Discussed the rate problem (page 156, #11). Some students solved this by the "Guess and Check" problem solving strategy, and some by algebra. One advantage of Guess and Check is that the mathematics used may turn out to be simpler. One disadvantage to Guess and Check is that the problem might have more than one answer, and you might only find one of them. Also, if the problem has no answer, then you may spend a lot of time without realizing and understanding this. One way to solve this problem by algebra: let a be the unknown bike rate and b be the unknown car rate. Then you can set up two equations—one for the total time, and the other for the total distance—and solve them simultaneously.
8. Discussed the unit conversion problem (page 212, #4) from the chapter on the problem solving strategy "Analyze the Units". In particular we noted that "passenger-miles" is the product of the number of passengers with the number of miles that they traveled as a group, and $\$1.59\frac{9}{10} = 1.599$ dollars.
9. Demonstrated the game "100" (problem 4.1 of my handout), realizing that the first player to get to 93 can win, and thus the first player getting to 86 can win.
10. Demonstrated the game "Fifteen" (problem 4.4 of my handout), revealing that it is the same as playing tic-tac-toe on a board labeled with numbers in a magic

square configuration:

8	1	6
3	5	7
4	9	2

.

11. Began working on "School Lockers" (problem 4.5 of my handout).

April 7

1. Worked on homework problems. In particular, when consider the problems involving games, mentioned the notion of "goal position". A goal position is a position that you want to achieve at the end of your move. So, for example, for the game of "100", the number 100 is a goal position (obviously). But 93 is also a goal position, because if you can achieve 93 at the end of your move, your opponent will not be able to achieve 100 in his/her next move, but no matter what move he/she makes, you will then be able to respond in such a way to achieve 100. Similarly, 86 is a goal position, because if you can achieve 86 at the end of your move, your opponent will not be able to achieve 93 in his/her next move, but no matter what move he/she makes, you will then be able to respond in such a way to achieve the goal position 93.
2. Some hints about the other homework problems: (1) Penny Piles: Find all the goal positions. Play it many times with friends. (2) School Lockers: Think about particular locker numbers and the resulting status of the locker. How can you easily tell the final status of locker 72? Of locker 36? (3) Problem in the Text: You don't really want to add up all these numbers by hand. Think of ways of organizing the problem to cut your work way down.

April 9

1. Went over the homework problems. In particular: The goal positions for the "100" problem are 100,93,86,79,...,9,2. The goal positions for the "Penny Piles" problem are 000,011,022,033,044,055,123,145,246,356,347,257. The solution to the "School Locker" has to do with the fact that factors of a number come in pairs—if the number is not a square, the each pair has two different numbers, but if the number is a square, there is exactly one pair with a repeated factor. The difference between the sum of the first 500 even numbers and the sum of the first 500 odd numbers can be calculated in several ways: (a) Add everything up by hand and subtract (not to be recommended); use a Gauss-like technique to find these two sums, and then subtract; or the quickest way is to realize that you can calculate $(1-0)+(3-2)+(5-4)+\cdots+(999-998)$, which is $1+1+1+\cdots+1=500$.
2. Worked on Problems V, #5.1(1–2), 5.2, and 5.3. Handout: Problems V. We used a "Polydron" set to do this.
3. Collected Homework #7.
4. Sent out by email: Exam #2 Preparation handout

April 14

1. Worked on practice problems—see handout—and answered questions in review for the exam on Thursday.

April 16 Exam #2

April 21

1. Passed back Exam #2 and a handout of solutions.
2. Demonstrated reflections, rotations, and translations using the National Library of Virtual Manipulatives, nlvm.usu.edu/NLVM.
3. Worked on handout "Investigation 2: Symmetry Transformations". Note that there are accompanying "Labsheets" for some of the figures. The goal is to write short, precise answers.
    (a) Worked on all the problems on pages 27–35.
    (b) For pages 36–46 worked on problems #10–15, 17, 19–23, 27, 31, 33.
    (c) For page 47 worked on problems #1–4.
4. Passed out a handout of various patterns that will be used on Thursday.

April 23

1. Discussed glide reflections.
2. Analyzed the symmetries of various wallpaper patterns.

April 28

1. Discussed Homework #8 and collected it.
2. Demonstrated some simple uses of the program GeoGebra, available for free from www.geogebra.org.
3. Demonstrated combining reflections with meter sticks as mirrors and members of the class as objects and reflections.
4. Discussed some pattern symmetries.

April 20

1. Demonstrated analysis of wallpaper patterns using transparencies.
2. Handout on the program GeoGebra.
3. Answered lingering questions.