MA310 EXAM #2 SOLUTIONS

1. Make the following conversions. Round your answers to two decimal places.

   (a) Convert 60 miles/hour to inches/second. Recall that 1 mile equals 5280 feet.

$$\frac{60\text{mi}}{1\text{hr}} \times \frac{5280\text{ft}}{1\text{mi}} \times \frac{12\text{in}}{1\text{ft}} \times \frac{1\text{hr}}{60\text{min}} \times \frac{1\text{min}}{60\text{sec}} = \frac{1056\text{in}}{\text{sec}}.$$

   (b) Convert 12 yard^3 to centimeter^3. Recall that one meter is approximately 39.37 inches.

$$12\text{yd}^3 \times \left(\frac{36\text{in}}{1\text{yd}}\right)^3 \times \left(\frac{1\text{m}}{39.37\text{in}}\right)^3 \times \left(\frac{100\text{cm}}{1\text{m}}\right)^3 \approx 9174713.34\text{cm}^3.$$

2. Find an equation for the given function:

x	y
0	3
1	-8
2	-9
3	0
4	19

Taking finite differences, we see that the second column of differences appears to be constant, so we seek a quadratic formula, $y = gx^2 + fx + e$.

$$\begin{array}{c|ccc}
x & y & & \\
\hline
0 & 3 & & \\
 & & -11 & \\
1 & -8 & & 10 \\
 & & -1 & \\
2 & -9 & & 10 \\
 & & 9 & \\
3 & 0 & & 10 \\
 & & 19 & \\
4 & 19 & &
\end{array}
\qquad
\begin{array}{c|lcc}
x & y & & \\
\hline
0 & e & & \\
 & & g+f & \\
1 & g+f+e & & 2g \\
 & & 3g+f & \\
2 & 4g+2f+e & & \\
 & & & \\
 & & & \\
 & & & \\
 & & &
\end{array}$$

Equating, we have

$$\begin{aligned}
&2g = 10, \text{ so } g = 5, \\
&g + f = -11, \text{ so } 5 + f = -11, \ f = -16, \\
&e = 3.
\end{aligned}$$

So the formula is $y = 5x^2 - 16x + 3$.

3. You are playing a two-person game that begins with a single pile of 15 pennies. A move consists of removing 1, 2, or 3 pennies. The person who takes the very last penny wins. What is the winning strategy? Why?

   If you can reach the position 4, then you can win, because then your opponent can only reach 3, 2, or 1, and you will win on the next move. So 4 is a goal position. If you can reach the position 8, then you can win, because then in the next move your opponent can only reach 7, 6, or 5, from which you can reach the goal position 4. So 8 is a goal position. If you can reach the position 12, then you can win, because then in the next move your opponent can only reach 11, 10, or 9, from which you can reach the goal position 8. So 12 is a goal position. Therefore, the first player can win by moving to position 12, and thereafter moving to positions 8, 4, and 0.

4. On a recent trip I drove from city A to city B. The trip took 3 hours and I averaged 50 miles/hour. I then drove on the same route from city B back to city A. My overall average for the entire round trip from A to B and back to A was 60 miles/hour. What was my average rate for the portion of the trip from B to A. Explain your answer. Warning: the answer is NOT 70 miles/hour.

   On the first leg, the average rate is $r_1 = 50$ mi/hr, the time is $t_1 = 3$ hr, and so the distance is $d_1 = r_1 \times t_1 = 150$ mi. On the round trip, the average rate is $r = 60$ mi/hr and the distance is $d = 2d_1 = 300$ mi. So the time is $t = 300/60 = 5$ hr. Therefore, on the second leg, the time is $t_2 = t - t_1 = 5 - 3 = 2$ hr, the distance is $d_2 = d_1 = 150$ mi, and so the average rate is $r_2 = 150/2 = 75$ mi/hr.

5. 120 small unpainted cubes are assembled into a large $4 \times 5 \times 6$ rectangular prism ("box"). All six faces of this large box are then painted.

   (a) How many of the small cubes remain unpainted? Why? These cubes consist of the inner core of $2 \times 3 \times 4 = 24$ cubes.

   (b) How many of the small cubes end up with paint on exactly one of their faces? Why? These cubes consist of the cubes on each of the six faces, not counting the cubes on the edges or corners. For each of the two 4×5 faces there are $2 \times 3 = 6$ cubes. For each of the two 4×6 faces there are $2 \times 4 = 8$ cubes. For each of the two 5×6 faces there are $3 \times 4 = 12$ cubes. The grand total is 52 cubes.

   (c) How many of the small cubes end up with paint on exactly two of their faces? Why? These cubes consist of the cubes on the 12 edges, not counting the 8 corner cubes. The four edges of length 4 each contribute 2 cubes. The four edges of length 5 each contribute 3 cubes. The four edges of length 6 each contribute 4 cubes. The grand total is 36 cubes.

   (d) How many of the small cubes end up with paint on exactly three of their faces? Why? These are the 8 corner cubes.

6. Two waitresses, Robin and Jen, and their sons, Nicholas, Dustin, and Miles, just started collecting state quarters. To start their collection, they each acquired one quarter representing a different state: Rhode Island, Delaware, Massachusetts, New York, and Pennsylvania. Use the clues to match the names and the states. You do not

need to write out a full verbal explanation of your solution—just show your matrices (if you choose to use them).

(a) Nobody got a state quarter from a state whose name started with the same letter as his or her name.

(b) Robin and Jen work together. Their coworker Thelma gave one of them the Delaware state quarter.

(c) Robin told her son that the Pennsylvania quarter was worth \$2. Neither of them has it.

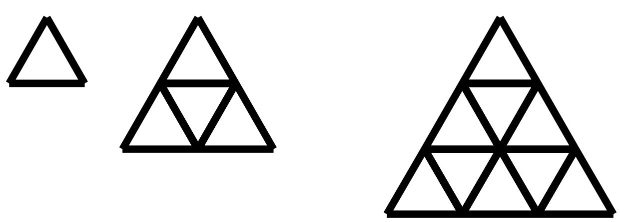
(d) One of Jen's two sons has the Massachusetts quarter.

(e) Miles got home from school and told his mom that he got the New York quarter. His brother Dustin wasn't home.

From statements (6c), (6d), and (6e) we see that Jen's sons are Miles and Dustin, and hence Robin's son is Nicholas. Once you realize this, you can fill in the chart using the various clues.

	RI	DE	MA	NY	PA
Robin	0	1	0	0	0
Jen	0	0	0	0	1
Nicholas	1	0	0	0	0
Dustin	0	0	1	0	0
Miles	0	0	0	1	0

7. One day I began making triangular patterns with toothpicks.



As you can see, the first pattern (which has a base of length one toothpick) uses a total of 3 toothpicks. The second pattern (which has a base of length two toothpicks) uses a total of 9 toothpicks. The third pattern (which has a base of length three toothpicks) uses a total of 18 toothpicks. How many toothpicks are required to make the hundredth pattern? Why?

One way to do this is to see from the diagrams that the numbers of toothpicks follow the pattern:

Pattern	Toothpicks
1	3
2	$3+6$
3	$3+6+9$
4	$3+6+9+12$

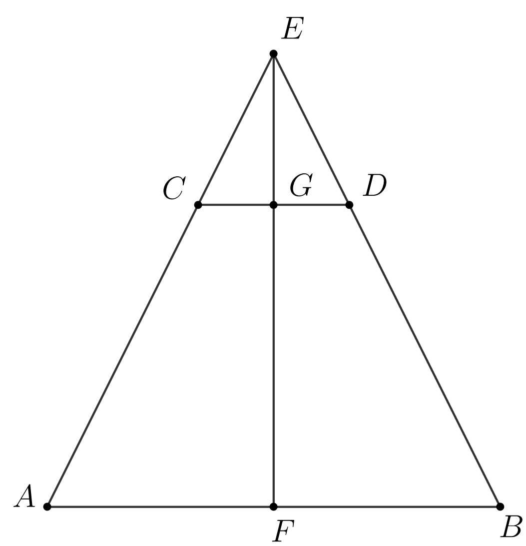
So Pattern #100 will have $3+6+9+12+\cdots+300$ toothpicks. This equals $3(1+2+3+\cdots+100) = 3\frac{100\times 101}{2} = 15150$.

8. Suppose $a_0, a_1, a_2, \ldots$ is a geometric sequence, $a_9 = 3a_7$, and $a_3 = 10$. Find the exact value of a_8 (do not express this as an approximation). Show your work.

   Because this is a geometric sequence we have $a_0 = a$, $a_1 = ar$, $a_2 = ar^2$, etc. In general $a_n = ar^n$. Now $a_9 = 3a_7$ so $ar^9 = 3ar^7$. Dividing by ar^7, $r^2 = 3$ so $r = 3^{1/2}$. Using $a_3 = 10$, we have $ar^3 = 10$, so $a \times 3^{3/2} = 10$, $a = 10 \times 3^{-3/2}$. Finally $a_8 = ar^8 = 10 \times 3^{-3/2} \times 3^{8/2} = 10 \times 3^{5/2} = 90 \times 3^{1/2}$.

9. You have a square-based pyramid. The base is a square of side length 15. From the top of this large pyramid you cut off a smaller square-based pyramid whose base is a square of side length 5. The height of the remaining solid is 12. What is the volume of the remaining solid? Show your work. Note: The volume of a pyramid is $\frac{1}{3}Bh$, where B is the area of the base and h is the height.

   Below is a diagram of the cross-section. $AB = 15$, $CD = 5$, and $FG = 12$.



   Let $x = EG$. Using similar triangles $\Delta CDE \sim \Delta ABE$ we have $\frac{x}{5} = \frac{x+12}{15}$. Solving, $15x = 5x + 60$ so $x = 6$. Now subtract the volume of the small pyramid from the volume of the large pyramid: $\frac{1}{3}15^2(18) - \frac{1}{3}5^2(6) = 1300$ cubic units.