Central Kentucky High School Math Circle

4 March 2012
Partition Identities

Suppose that n is a positive integer. A *partition* of n is a way of writing n as a sum of positive integers, where we order the summands from least to greatest. For example, three partitions of 7 are

$$\begin{aligned} 7 &= 1+3+3 \\ 7 &= 1+2+2+2 \\ 7 &= 3+4\,. \end{aligned}$$

We do not consider sums like

$$7 = 2+1+2+2$$

because we want the terms in the sum to increase as we read them from left to right.

One thing mathematicians have been very interested in over time are *partition identities*, where a partition identity is a statement that the number of partitions of one kind are equal to the number of partitions of another kind. We will spend today investigating some lovely partition identities.

(1) The number of partitions of n into m parts equals the number of partitions of n whose greatest part is m.

(2) The number of partitions of n into at most m parts of size at most k equals the number of partitions of n into at most k parts of size at most m.

(3) The number of partitions of n that are self conjugate equals the number of partitions of n into distinct odd parts.

(4) (This one is due to Euler) The number of partitions of n into odd parts equals the number of partitions of n into distinct parts.

And here are two problems on *counting* partitions:

(8) How many partitions are there with at most m parts of size at most k? Use this setup to prove the recurrence for binomial coefficients ("Pascal's triangle").

(9) Let c_m denote the number of partitions whose kth part is at most $k-1$ (so their Ferrer's diagram fits into a "staircase"). Show that

$$c_{m+1} = c_0c_m + c_1c_{m-1} + \cdots + c_mc_0\,,$$

where we define $c_0 = 1$. The numbers c_m are called *Catalan numbers* and are given by the formula

$$c_m = \frac{1}{m+1}\binom{2m}{m}\,.$$