Example of Guessing a Formula

Suppose you want to guess the formula for

$$f(n) = 1^2 + 2^2 + 3^2 + \cdots + n^2, \ n \geq 0.$$

(We take $f(0) = 0$.) You can begin by making a table of values and looking at differences.

n	0		1		2		3		4		5
$f(n)$	0		1		5		14		30		55
		1		4		9		16		25	
			3		5		7		9		
				2		2		2			
					0		0				

The fact the the fourth row of differences appears to be zeros strongly suggests that the formula is given by a cubic polynomial. This means we will need to use four values to determine the function.

First Method. Design a polynomial that will match,say, the first four values. Start with

$$f(n) = c_0(n-1)(n-2)(n-3) + c_1 n(n-2)(n-3) + c_2 n(n-1)(n-3) + c_3 n(n-1)(n-2).$$

One by one, substitute values 0, 1, 2, and 3 for n and solve for c_0, c_1, c_2, c_3.

$$f(0) = 0 = c_0 \cdot (-6), \text{ so } c_0 = 0.$$

$$f(1) = 1 = c_1 \cdot 2, \text{ so } c_1 = 1/2.$$

$$f(2) = 5 = c_2 \cdot (-2), \text{ so } c_2 = -5/2.$$

$$f(3) = 14 = c_3 \cdot 6, \text{ so } c_3 = 14/6 = 7/3.$$

Using these values of the c_i we have our proposed formula

$$f(n) = 0(n-1)(n-2)(n-3) + \frac{1}{2}n(n-2)(n-3) - \frac{5}{2}n(n-1)(n-3) + \frac{7}{3}n(n-1)(n-2)$$

which we can simplify to

$$f(n) = \frac{1}{6}n(2n^2 + 3n + 1).$$

At this point it would be wise to check that this formula works for $n = 4$ and $n = 5$, and indeed it does. Now we may proceed to prove this formula by induction on n.

Second Method. Guessing that the formula is a cubic polynomial, we may write $f(n) = an^3 + bn^2 + cn + d$, substitute in four values for n, and solve a system of equations for a, b, c, d.

$$\begin{aligned} f(0):&\quad d = 0 \\ f(1):&\quad a + b + c + d = 1 \\ f(2):&\quad 8a + 4b + 2c + d = 5 \\ f(3):&\quad 27a + 9b + 3c + d = 14 \end{aligned}$$

I'll leave the details to you, but you should compute that $a = 1/3$, $b = 1/2$, $c = 1/6$, and $d = 0$. Again check that this formula works for $n = 4$ and $n = 5$, and then prove it by induction.