MA 310 — Homework #2

Solutions

Refer to "Problems" on the Course Website

1. Solve "Variations on Game of 100" part (1). In particular, identify all the key positions and justify that they are in fact key positions by proving that if you achieve a key position, then on the next move your opponent cannot achieve a key position, but in your move after that, you can achieve a key position.

   Solution. Let q and r be the quotient and remainder, respectively, when n is divided by $k+1$. We claim that the set of numbers of the form $r+(k+1)\ell$, $\ell = 0, 1, 2, \ldots, q$, are "good" or key positions (ones you wish to reach), while the other numbers from 0 to n are "bad" (ones you do not wish to reach), in the following sense: From a good position all single moves lead to bad positions, and from a bad position there exists a single move to a good position. (Note that n is a good position.)

   For suppose you achieve a position of the form $r+(k+1)j$. Then on his/her next move, your opponent can only achieve positions $(r+1)+(k+1)j$, $(r+2)+(k+1)j$, ..., or $(r+k)+(k+1)j$, and none of these are of the form $r+(k+1)\ell$. On the other hand, from each of these bad positions you can make a single move (adding $k, k-1, \ldots, 1$, respectively), and reach the position $r+(k+1)(\ell+1)$, another good position.

   Thus if $r \neq 0$ the first player can guarantee a win by moving to r and subsequently reaching good positions; but if $r = 0$ then the second player can guarantee a win by moving to r on the second move of the game and subsequently reaching good positions

2. Solve "Variations on Game of 100" part (2). In particular, carefully identify all the key positions and justify that they are in fact key positions by proving that if you achieve a key position, then on the next move your opponent cannot achieve a key position, but in your move after that, you can achieve a key position.

   Solution. Using the definition of key (or good) and non-key (or bad) positions, and working backwards carefully from 100, which is a good (or key) position, we obtain that both 99 and 98 are bad positions to reach, but 97 is good. Continuing, 96, 95, and 94 are all bad, but 93 is good. This pattern continues, yielding the key positions 100, 97, 93, 90, 86, 83, 79, 76, 72, 69, 65, 62, 58, 55, 51, 48, 44, 41,3 7, 34, 30, 27, 23, 20, 16, 13, 9,6 ,2. So the first player can win by moving to 2 and subsequently reaching key positions.

3. Solve "Penny Piles". In particular, identify all the key positions and justify that they are in fact key positions by proving that if you achieve a key position, then on the next

move your opponent cannot achieve a key position, but in your move after that, you can achieve a key position.

Solution. Using the definition of key (or good) and non-key (or bad) positions, and working up from the final position of $(0, 0, 0)$, it can be determined that the good positions are: (0,0,0), (0,1,1), (0,2,2), (0,3,3), (0,4,4), (0,5,5), (1,2,3), (1,4,5), (2,4,6), (2,5,7), (3,4,7), and (3,5,6), and their permutations. Thus the first player can win by always moving to one of these key positions.