MA 310 — Homework #5

Solutions

Read Sections 2.4 of the text.

1. Determine a simple closed formula for the minimum number of steps it takes to solve the "Towers of Hanoi" puzzle with n disks (see Problems on the course website). Prove your answer. In particular, how does your proof show it cannot be solved in fewer steps?

   Solution. Let $f(n)$ be the minimum number of moves required to solve the puzzle with n disks, $n \geq 1$. Clearly $f(1) = 1$. Now consider the $k+1$ disk puzzle for some $k \geq 1$. In order to move the bottom disk, you must first move the upper k disks completely to some other peg. This is equivalent to solving the k disk puzzle and requires $f(k)$ moves. Then you move the bottom disk. Then you must bring the k disks back on top of the bottom disk by solving a k disk puzzle with another $f(k)$ moves. Altogether you have made $f(k)+1+f(k)$ moves to solve the $k+1$ disk puzzle. Thus we have proved the recursive formula

   $$f(k+1) = 2f(k)+1,\ k \geq 1,\ \text{and } f(1) = 1.$$

   We now prove by induction on $n \geq 1$ that $f(n) = 2^n - 1$. (This is the closed formula.)

   Base Case. If $n = 1$, then $2^n - 1$ equals 1, which is correct.

   Inductive Step. Assume $f(k) = 2^k - 1$ for some $k \geq 1$. Then

   $$\begin{aligned} f(k+1) &= 2f(k)+1 \text{ (using the recursive formula)} \\ &= 2(2^k - 1) + 1 \text{ (using the inductive hypothesis)} \\ &= 2^{k+1} - 2 + 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

   Thus the formula is also true for $n = k+1$.

   Therefore the formula is true for all $n \geq 1$ by induction.

2. Solve the problem "Thinking about Thinking—Hats" (see Problems on the course website).

   Solution. Call the three people A, B, C. Person A sees two other red hats. Person A can reason as follows. Suppose person A has a green hat. Then person B would see one red hat and one green hat, and also see that person C is raising his hand. B would then immediately conclude that B has a red hat. But since everybody is still standing around, A realizes that B cannot figure out his hat color. Therefore A must in fact have a red hat.

3. Solve problem 2.3.14 in the book.

   Solution. Assume to the contrary that $\log_{10} 2$ is rational. Then $\log_{10} 2 = \frac{a}{b}$ for some positive integers a and b (noting that $\log_{10} 2 \neq 0$). Thus $10^{a/b} = 2$, implying $10^a = 2^b$. But 10^a is divisible by 5, whereas 2^b is not. This contradiction shows that $\log_{10} 2$ must be irrational.

4. Solve problem 2.3.37(d) in the book.

   Solution. We are given that $f_0 = 0$, $f_1 = 1$, and thereafter $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. For convenience in notation, let

$$A = \frac{1+\sqrt{5}}{2} \text{ and } B = \frac{1-\sqrt{5}}{2}.$$

We will prove $f_n = \frac{1}{\sqrt{5}}(A^n - B_n)$ by induction on $n \geq 0$.

Base Case. Here we need to check two base cases, since each term depends on the two previous terms. For $n = 0$, $\frac{1}{\sqrt{5}}(A^0 - B^0) = 0$ as required. For $n = 1$,

$$\begin{aligned} \frac{1}{\sqrt{5}}(A^1 - B^1) &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) \\ &= \frac{1}{\sqrt{5}} \cdot \frac{2\sqrt{5}}{2} \\ &= 1, \text{ as required.} \end{aligned}$$

Inductive Step. Here we can use Strong Induction. Assume the formula is true for $n = 0, 1, 2, \ldots, k-1$ for some $k \geq 2$. Then

$$\begin{aligned} f_k &= f_{k-2} + f_{k-1} \\ &= \frac{1}{\sqrt{5}}\left(A^{k-2} - B^{k-2}\right) + \frac{1}{\sqrt{5}}\left(A^{k-1} - B^{k-1}\right) \\ &= \frac{1}{\sqrt{5}}\left(A^{k-2}(1+A) - B^{k-2}(1+B)\right) \end{aligned}$$

We will be done if it turns out that $1 + A = A^2$ and $1 + B = B^2$, so let's confirm these facts:

$$A^2 = \frac{1 + 2\sqrt{5} + 5}{4} = \frac{3+\sqrt{3}}{2} = 1 + A.$$

$$B^2 = \frac{1 - 2\sqrt{5} + 5}{4} = \frac{3 - \sqrt{3}}{2} = 1 + B.$$

This completes the inductive step.

Therefore the formula is true for all $n \geq 0$ by induction.

5. Solve problem 2.4.7 in the book. Suggestion: Draw graphs of distances from home as a function of time of day.

   Solution. The suggestion is not needed and might not be helpful. Think of this in terms of Sal. Sal's travel time for the round trip decreased by 10 minutes, so Sal's travel time for the first leg of the trip decreased by 5 minutes. Thus Sal meets Pat at 4:55 pm instead of 5:00 pm. This implies that Pat has been walking for 55 minutes.