MA 310 — Homework #8

Solutions

1. Solve the problem "The Average Traveler" part 1.

   Solution. Let d be the distance between the two cities. Make a table of rate, time, and distance.

	From A to B	From B to A	Round Trip
Distance	d	d	$2d$
Time	t_1	t_2	$t_1 + t_2$
Rate	30	r_2	60

   From this we see that
   $$\frac{d}{t_1} = 30 \text{ so } t_1 = \frac{d}{30}.$$
   $$60 = \frac{2d}{t_1 + t_2} = \frac{2d}{\frac{d}{30} + t_2} = \frac{60d}{d + 30t_2}$$
   So
   $$1 = \frac{d}{d + 30t_2} \text{ and so } d + 30t_2 = d.$$
   This implies that $t_2 = 0$, so unless the traveler can drive from B to A in no time at all, it is impossible to average 60 miles per hour for the round trip.

2. Solve the problem "8 Card Trick", explaining exactly what the magician does, and also why it works.

   Solution: When the eight cards are in a stack, face down, number the positions of the cards 0 to 7 from top to bottom. Now we must see where the chosen card can be after dealing and then gathering the two piles. If the chosen pile is placed face down on the non-chosen pile, call this 0. If the chosen pile is placed face down beneath the non-chosen pile, call this 1.

   (a) First gathering is of type 0.
   Chosen card is among positions $0, 1, 2, 3$.

      i. Second gathering is of type 0.
      Chosen card is among positions $0, 1$.

         A. Third gathering is of type 0.
         Chosen card is at position 0.

B. Third gathering is of type 1.
Chosen card is at position 4.

ii. Second gathering is of type 1.
Chosen card is among positions $4, 5$.

A. Third gathering is of type 0.
Chosen card is at position 2.

B. Third gathering is of type 1.
Chosen card is at position 6.

(b) First gathering is of type 1.
Chosen card is among positions $4, 5, 6, 7$.

i. Second gathering is of type 0.
Chosen card is among positions $2, 3$.

A. Third gathering is of type 0.
Chosen card is at position 1.

B. Third gathering is of type 1.
Chosen card is at position 5.

ii. Second gathering is of type 1.
Chosen card is among positions $6, 7$.

A. Third gathering is of type 0.
Chosen card is at position 3.

B. Third gathering is of type 1.
Chosen card is at position 7.

In all cases, if the sequence of the gathering types is abc, then the card is at position cba, regarded as a number in binary.

3. Extra Credit: Solve the "27 Card Trick" with the added complication that you ask for a integer from 0 to 26 in advance, and force the selected card at the end of the trick to be the next one after this number of cards. Explain what the magician does and why it works.

4. Solve the problem "Counterfeit Coins" part 1.

Solution. Partition the coins into three sets of three. Call these sets A, B, and C. Weight A against B. If either set is heavier than the other, then the counterfeit coin is in that heavier set of three. If they balance, then the counterfeit coin is in set C. Now, take the set of three coins containing the counterfeit coin and weigh two coins against each other. If either is heavier, then that heavier coin is the counterfeit coin. If they balance, then the third coin is counterfeit.

5. Solve the problem "Counterfeit Coins" part 2 using a proof by induction.

   Solution. Proof by induction that if you have 3^n coins, where $n \geq 1$, then the counterfeit coin can be found in n weighings.

   Base case: $n = 1$. In this case you have 3 coins, and the counterfeit coin can be found in one weighing by weighing any two coins against each other.

   Inductive step: Assume that $k \geq 1$ and that the counterfeit coin can be found among 3^k coins in k weighings. Now assume you have 3^{k+1} coins. Partition the coins into three sets of 3^k coins each. Call these sets A, B, and C. Weight A against B. If either set is heavier than the other, then the counterfeit coin is in that heavier set of three. If they balance, then the counterfeit coin is in set C. So with one weighing you have reduced the problem to that of finding the counterfeit coin among 3^k coins, which can be done in k more weighings by the inductive hypothesis. So the counterfeit coin can be found using $k + 1$ weighings total.

   Therefore the statement is proved for all $n \geq 1$ by induction.

6. Solve the problem "Weird Multiplication I".

   Solution: As you carry out the divisions, make note of the remainders next to each number in the left-hand column, and also rewrite the numbers in the right-hand column in terms of the original number multiplied by a power of 2.

$$
\begin{array}{cccc}
0 & \cancel{18} & \cancel{14} & \cancel{2^0 \cdot 14} \\
1 & 9 & 28 & 2^1 \cdot 14 \\
0 & \cancel{4} & \cancel{56} & \cancel{2^2 \cdot 14} \\
0 & \cancel{2} & \cancel{112} & \cancel{2^3 \cdot 14} \\
1 & 1 & 224 & 2^4 \cdot 14 \\
\hline
 & & 252 &
\end{array}
\qquad
\begin{array}{cccc}
1 & 13 & 25 & 2^0 \cdot 25 \\
0 & \cancel{6} & \cancel{50} & \cancel{2^1 \cdot 25} \\
1 & 3 & 100 & 2^2 \cdot 25 \\
1 & 1 & 200 & 2^3 \cdot 25 \\
\hline
 & & 325 &
\end{array}
$$

   Then it is evident that the second number is being multiplied by precisely the powers of two used to express the first number in binary. For example, 18 in binary is 10010, meaning that $18 = 2^4 + 2^1$. Then $18 \cdot 14 = (2^4 + 2^1) \cdot 14 = 2^4 \cdot 14 + 2^1 \cdot 14$, which is precisely what the process of Weird Multiplication is doing.