Basic Facts on Equivalence

1. Two positions α and α' in normal-play games are equivalent if for every position β in any normal-play game, the two positions $\alpha + \beta$ and $\alpha' = \beta$ have the same type. Write $\alpha \equiv \alpha'$.

2. Not the same as isomorphic.

3. Proposition 2.10. If α, β, and γ are positions in normal-play games, then

   (a) $\alpha \equiv \alpha$ (Reflexive Property).

   (b) $\alpha \equiv \beta$ implies $\beta \equiv \alpha$ (Symmetric Property).

   (c) $\alpha \equiv \beta$ and $\beta \equiv \gamma$ implies $\alpha \equiv \gamma$ (Transitive Property).

   Thus we have an equivalence relation.

4. Proposition 2.11. If α is equivalent to α', then α and α' have the same type.

5. The converse statement is not a proposition. So equivalence is a finer distinction than type.

6. Proposition 2.12. If α, β, γ are positions in normal-play games, then

   (a) $\alpha + \beta \equiv \beta + \alpha$ (Commutative Property).

   (b) $(\alpha + \beta) + \gamma \equiv \alpha + (\beta + \gamma)$ (Associative Property).

7. Lemma 2.13. For positions of normal-play games,

   (a) If $\alpha \equiv \alpha'$, then $\alpha + \beta \equiv \alpha' + \beta$.

   (b) If $\alpha_i \equiv \alpha_i'$ for $1 \leq i \leq n$, then $\alpha_1 + \cdots + \alpha_n \equiv \alpha_1' + \cdots + \alpha_n'$.

   (c) If $\alpha_i \equiv \alpha_i'$ for $1 \leq i \leq m$ and $\beta_i \equiv \beta_i'$ for $1 \leq i \leq n$, then $\{\alpha_1, \ldots, \alpha_m | \beta_1, \ldots, \beta_n\} \equiv \{\alpha_1', \ldots, \alpha_m' | \beta_1', \ldots, \beta_n'\}$.

8. Lemma 2.14. If β is of type P, then $\alpha + \beta \equiv \alpha$. "Type P behaves like zero."

9. Proposition 2.15. If α and α' are type P, then $\alpha \equiv \alpha'$. "Uniqueness of zero."

10. Lemma 2.16. If $\alpha + \beta$ and $\alpha' + \beta$ are both type P then $\alpha \equiv \alpha'$. "Uniqueness of additive inverse."