MA 327/ECO 327
Introduction to Game Theory
Fall 2018
Notes

Contents

1 Wednesday, August 22

1. Played the game of Fifteen and eventually figured out it was isomorphic to Tic-Tac-Toe—arrange the numbers 1 through 9 as in a magic square. Discussed some of the characteristics of this game.

2. Reviewed the syllabus and the scope of the course.

3. Played the game of 100 (start at 0 and alternately add whole numbers between 1 and 6 inclusive). We figured out that the winning strategy is always to move to a number that is 2 more than a multiple of 7.

4. For games with no ties, positions that a player attains are either winning or losing. A position is winning if it is a final winning position or if every move from it leads to a losing position. A position is losing if there exists a move from it that leads to a winning position.

5. Discussed the game of Pick-up-Bricks (Game 1.1 on page 1).

2 Friday, August 24

1. Played the game of Hex (also called Nash). Stated that this game cannot end in a tie, that it is known that the first player has a winning strategy, but that no one knows what this strategy is for large boards.

2. Section 1.1.

3. Examples of combinatorial games: Pick-Up-Bricks (start with 7), Chop, Chomp, Tic.

4. Definition of combinatorial game. 2-player, players L and R, a (finite) set of positions, a move rule indicating what positions either can move to from each position, a win rule specifying terminal positions and an outcome for each terminal position $+-$, $-+$, 00. To play, specify a starting position and a starting player and alternate moves.

5. Some games are normal play games: the player who cannot move loses. Many combinatorial games are up of this type. E.g., Pick-up-Bricks, Chop, Chomp.

6. No random elements. Full information—no hidden information.

7. Game of Tic.

8. Game tree. A representation of the game. Procedure: Build a Game Tree.

9. W-L-D game tree when outcomes of terminal positions are given, and each branch node labeled R or L—other game specifics are superfluous—could just play the game by moving directly on the W-L-D tree. Could have only one node—trivial tree.

10. Require that the game trees are finite.

11. Labeled the game tree for Tic. Each nodes is labeled with one of $+-$, $-+$, 00, and one move is specified at each node.

3 Monday, August 27

1. Played the game of Gale. We will see why this has a winning strategy for the first player.
2. Game trees may have only one node! (Such games are boring.)
3. Strategy: Specify a move choice at each node. The strategy might not be good, though. A winning strategy for R guarantees that R will win when following it. A drawing strategy for R guarantees that R will win or draw when following it (i.e., not lose). Note that if a strategy is followed, one may never get to certain nodes in the tree.
4. Procedure: Working Backwards. To label each node with $+-$, $-+$, or 00. Such a labeling determines who has a winning strategy or if both have a drawing strategy.
5. Comments on "labeling on the fly" while playing the game repeatedly, and on my program for tic-tac-toe—anyone want to code this?
6. Section 1.2.
7. Zermelo's Theorem. Every W-L-D game tree is one of three types.
   - $+-$. L has a winning strategy; i.e., by following this strategy, L will win no matter what R does.
   - $-+$. R has a winning strategy; i.e., by following this strategy, R will win no matter what L does.
   - 00. Both players have drawing strategies; i.e., by following this strategy, L will not lose no matter what R does, and R will not lose no matter what L does,
8. Proofs by induction. Induction:Proof::Recursion:Programming. You must prove the simplest case (base case), and you must show you can prove any complicated case from knowing the truth for less complicated cases (inductive step).
9. Proof of Zermelo's Theorem.

   We will label each node with a symbol and a choice of move, starting at the terminal nodes.

   Base case: One node.

   Inductive step. R is faced with possible moves. (Case for L is similar.)

- At least one move to a $-+$ subtree. Label is $-+$, and move is to this subtree.
- No moves are to $-+$, but there is at least one move to a 00 subtree. Label is 00, and move is to this subtree.
- All moves are to $+-$ subtrees. Label is $+-$, and move is to any subtree.

10. Corollary. Working backwards produces the correct label on the root node.
11. Section 1.3.
12. Can often discuss strategies (choices of moves) without drawing entire game tree.
13. Symmetry. Example: Placing Pennies on a Round Table. Proposition 1.12. Consider an $m \times n$ position in Chop. If $n = m$, the second player has a winning strategy. If $n \neq m$, the first player has a winning strategy. The winning strategy is to always move to a square position.
14. Simultaneous chess. You can play two games of chess with two different people and win at least one of them, or else draw in both of them.

4 Wednesday, August 29

1. Demonstrated 4D-Tic-Tac-Toe. It turns out that the first player has a winning strategy for 3D $4 \times 4 \times 4$ Tic-Tac-Toe.

2. Pairing Strategies. Proposition 1.13. Consider a Pick-up-Bricks position of n bricks. If three divides n, the second player has a winning strategy. Otherwise the first player has a winning strategy. The winning strategy is to always moved to a position that is a multiple of three. This can be viewed as a pairing strategy. If the allowable move is to take one, two, or three bricks, then the winning strategy is to always move to position that is a multiple of four.

3. Strategy stealing. Proposition 1.15. For every rectangular position in Chomp except one by one, the first player has a winning strategy.

4. Proposition 1.16. The first player has a winning strategy in Hex. However, no one has a description of this strategy for all sizes of gameboards.

5. It turns out that you cannot tie in Gale, so a similar argument shows the first player has a winning strategy in Gale. An explicit pairing strategy exists.

6. Worked on Homework #1.

5 Friday, August 31

1. Section 2.1.

2. See handout LRNP.

3. Normal play combinatorial games. The winner is the last player to move. There are no ties. Pick-up-bricks, Chop, Chomp.

4. Cut-cake. Rectangular pieces of cake with horizontal and vertical lines marked. Louise can make vertical cuts. Richard can make horizontal cuts. This is an example of a partizan game. Games in which each player have the same options are called impartial games.

5. Domineering.

6. Representations of normal play games. For each position indicate which positions L can move to and which position R can move to. Each position can be equated to an ordered pair of sets of positions. Examples with Cut-Cake.

7. Types of positions. Depending upon who starts, either L or R will have a winning strategy by Zermelo's Theorem. This leads to the following classification.

   - Type L. Louise has a winning strategy whoever goes first.
   - Type R. Richard has a winning strategy whoever goes first.
   - Type N. The next player to move as a winning strategy.
   - Type P. The second or previous player has a winning strategy.

   Examples with Pick-Up-Bricks and Cut-Cake. We were unable to obtain any Cut-Cake positions of type N—are there any?

8. Determining type. Began discussing Proposition 2.3. If $\gamma = \{\alpha_1, \ldots, \alpha_m | \beta_1, \ldots, \beta_n\}$, the type of γ is given by:

	Some β_j is type R or P	All β_j are types L or N
Some α_i is type L or P	N	L
All α_i are types R or N	R	P

9. Thus can determine types of positions by working your way up from simpler positions.

6 Wednesday, September 5

1. Determining type. Discussed Proposition 2.3. If $\gamma = \{\alpha_1, \ldots, \alpha_m | \beta_1, \ldots, \beta_n\}$, the type of γ is given by:

	Some β_j is type R or P	All β_j are types L or N
Some α_i is type L or P	N	L
All α_i are types R or N	R	P

2. Thus can determine types of positions by working your way up from simpler positions. Example: 2×3 Cut-Cake.

3. Played "Two Pile Pick-up-Bricks" and determined the types of some positions.

4. Worked on types of some particular Cut-Cake positions, working up from simpler positions.

5. Section 2.2.

6. Sums of games. If α and β are positions in normal-play games, define $\alpha + \beta$ to be a new position consisting of the components α and β. To move in $\alpha + \beta$, a player chooses one of the components and makes a valid move in that component.

7. Examples. Two Pile Pick-up-Bricks = Pick-up-Bricks + Pick-up-Bricks. Hex + Gale.

7 Friday, September 7

1. Played Sprouts. See `https://en.wikipedia.org/wiki/Sprouts_(game)`.

2. The type of a sum.

3. Proposition 2.6. If β is type P then α and $\alpha+\beta$ are the same type. This is an example of a determinate sum.

   Proof:

   Case 1: "L+P". L has the following winning strategy:

   If L goes first, make a winning strategy move in α. Thereafter, respond with the appropriate winning strategy move in whichever component R chooses to move in.

   If R goes first, L should then and thereafter respond with the appropriate winning strategy move in whichever component R chooses to move in.

   Case 2: "R+P". R has the following winning strategy. Essentially the same argument but with the rules of R and L reversed.

   Case 3: "N+P". The first player has the following winning strategy: Make a winning strategy move in α. Thereafter, respond with the appropriate winning strategy move in whichever component the opponent chooses to move in.

   Case 4: "P+P". The second player has the following winning strategy: After the first player moves, then and thereafter, respond with the appropriate winning strategy move in whichever component the opponent chooses to move in.

4. Proposition 2.7. If α and β are both type L, then $\alpha+\beta$ is type L. Similarly, if α and β are both type R, then $\alpha+\beta$ has type R. These are examples of determinate sums.

5. The types of other sums are ambiguous; i.e., indeterminate sums.

6. Domineering. $G = 2 \times 2$ has type N. $H_1 = 1 \times 2$ has type R. $H_2 = 1 \times 4$ has type R. $G + H_1$ has type N. $G + H_2$ has type R.

7. Equivalent games. Not the same as isomorphic—two games may be equivalent but not isomorphic.

8. Two positions α and α' in normal-play games are equivalent if for every position β in any normal-play game, the two positions $\alpha+\beta$ and $\alpha' = \beta$ have the same type. Write $\alpha \equiv \alpha'$.

9. All games of type P are equivalent.
10. Worked on homework.

8 Monday, September 10

1. Comments on Homework #1 solutions.
2. Section 2.3. See handout on Equivalence. This is an equivalence relation with certain algebraic properties.
3. If $\gamma = \{\alpha_1, \ldots, \alpha_m | \beta_1, \ldots, \beta_n\}$ then $-\gamma = \{-\beta_1, \ldots, -\beta_n | -\alpha_1, \ldots, -\alpha_m\}$.
4. Chapter 3. Restrict attention to impartial normal-play games. Pick-up-Bricks, Chop, Chomp, Sprouts, Game of 100, etc.
5. All positions are of type P or N, and the winning strategy is to always move to a P position.
6. Game of Nim, with one or more piles of stones. The game $*n$ is a single pile of n stones. Determined a few P positions: $*0$, $*m + *m$, $*1 + *2 + *3$.

9 Wednesday, September 12

1. Remember that normal play impartial games have only N and P positions. An N position is one for which there exists a move to a P position. A P position is one for which all moves lead to N positions. The player with a winning strategy is the one who can reach P positions at the end of their moves.

2. A position in multipile nim is a P position if and only if it is "super balanced" with respect to powers of two.

3. The nim sum $a \oplus b$ of two nonnegative integers a and b is determined by writing a and b in base 2 notation and then adding in base 2 with no carry; i.e., using "exclusive or." This can be extended to adding any number of negative integers.

4. The multipile nim position $*a_1 + \cdots + *a_n$ is a P position if and only if $a_1 \oplus \cdots \oplus a_n = 0$.

5. For an impartial game position α, for short we write $\alpha = \{\alpha_1, \ldots, \alpha_n\}$, where $\alpha_1, \ldots, \alpha_n$ are the positions either player can achieve in one move.

6. The MEX value, $MEX(\alpha)$ is then recursively defined to be the smallest nonnegative integer not in the set $\{MEX(\alpha_1), \ldots, MEX(\alpha_n)\}$. $MEX(\alpha)$ is also called the nim value or Sprague-Grundy value.

7. A position is a P position if and only if it has MEX value 0.

8. We will prove that if α and β are two game positions with $MEX(\alpha) = a$ and $MEX(\beta) = b$, then $MEX(\alpha + \beta) = a \oplus b$.

9. There are many games for which the set of P positions or the MEX values are difficult to determine or unknown.

10 Friday, September 14

1. Nim sum properties: commutative, associative, identity, inverses, $a \oplus b = c$ implies $b = a \oplus c$.

2. If α and β are two game positions with $MEX(\alpha) = a$ and $MEX(\beta) = b$, then $MEX(\alpha + \beta) = a \oplus b$.

3. Two positions are equivalent if and only if they have the same MEX values.

4. $MEX(*n) = n$.

5. Every impartial game α is equivalent to some $*n$, where $n = MEX(\alpha)$.

11 Monday, September 17

1. Worked on homework problems.

12 Wednesday, September 19

1. Discussed an example of how to find winning move in a sum of impartial games using the MEX values.

2. Chapter 4. Introduced the game of Checker Stacks for which each position can be associated with a number or value. The negative of a position is obtained by interchanging the labels of all the checkers. The value of the sum of two games is the sum of their values. Based on these principles we determined the value of several positions.

13 Friday, September 21

1. Dyadic numbers.
2. The Simplicity Principle and how to use it to determine the number of a partizan game. Note that there are partizan games to which one cannot assign a number in this way.
3. Worked on some examples, including trying to find a pattern for a general pile of checker stacks.

14 Monday, September 24

1. See handout "Games That are Numbers."
2. Why addition works.
3. Worked on homework problems for Chapter 4.

15 Wednesday, September 26

1. Chapter 5. Introduced the notion of two-person zero-sum matrix games.

2. Pure strategies.

3. Saddlepoints. Not every game has a saddle point, but if it does, then there is an optimal pure strategy for both R and C.

4. Eliminating dominated rows and columns.

16 Friday, September 28

1. If you repeatedly eliminate dominated rows and columns, down to the point pf a 1×1 matrix, then that entry is a saddle point of the original problem. But not all saddle points can be obtained this way.

2. Discussed expected value.

3. Defined mixed strategy.

4. A particular mixed strategy might still be poor if your opponent determines what probabilities you are using for your choices.

17 Monday, October 1

1. Looked at mixed strategies for some particular games, like Morra.

2. R mixes her strategies by using probabilities p_i to take a weighted sum of the rows to get a row of expected values. So R's goal is to choose the probabilities so as to maximize the minimum entry in that row. Call this maximum value M^*.R's problem is to solve

   $$M^* = \max_p \min pA.$$

   C mixes his strategies by using probabilities q_j to take a weighted sum of the columns to get a column of expected values. C's goal is to choose the probabilities so as to minimize the maximum entry in that column. Call this minimum value m^*. So C's problem is to solve

   $$m^* = \min_q \max Aq.$$

   The optimal pair (p^*, q^*) constitue a von Neumann solution. The von Neumann Minimax Theorem asserts that $M^* = m^*$; i.e.,

   $$\max_p \min pA = \min_q \max Aq.$$

   This is called the value of the game.

3. There is a link to an online game solver on the course website.

4. Began discussing solving $2 \times n$ games using the graphical method.

18 Wednesday, October 3

1. Examined the graphical method for matrix games with two rows.

2. R's goal is to find a vertical line whose lowest intersection with the various line segments is as high as possible. Call this point $P^* = (p^*, v^*)$. R should use the mixed strategy $(p^*, 1 - p^*)$, and the value of the game is v^*.

3. C's goal is to find a weighted average of the various line segments to create a segment s^* such that its highest endpoint is as low as possible. Since all the line segments lie above P^*, then so must s^*. Thus L should not use any line segment (strategy) strictly above P^*, only using line segments (strategies) passing through P^*, with the goal of creating s^* with its highest and point (often both endpoints) at height v^*. The optimal weighted average corresponds to the optimal mixed strategy $q^* = (q_1^*, \ldots, q_n^*)$, and the value of the game is again seen to be v^*.

4. The above analysis can be extended into higher dimensions, and thus provides a sketch of the proof of the von Neumann Minimax Theorem.

5. All of the above can be adapted to games with two columns instead of two rows. High becomes low and low becomes high.

6. Some things to think about. How can the graphical method to help you identify a row that dominates another row? A column that dominates another column? A saddlepoint?

19 Friday, October 5

1. Exam review.

2. How to set up linear programs to solve zero-sum matrix games.

20 Monday, October 8

Exam #1 on Chapters 1–4.

21 Wednesday, October 10

1. Corollary. If for some p and q you have $\min pA = \max Aq$, then p is optimal for R and q is optimal for C. This equality is a "Certificate of Optimality."

2. Some methods of solving matrix games.

   (a) See if there is a saddle point.

   (b) Guess solutions p and q and confirm optimality using the above Certificate of Optimality.

   (c) Graphical method for $2 \times n$ or $m \times 2$.

   (d) If you know p is optimal for R with value $v = M^*$, compute pA. Then C can only call on (have nonzero q_j) for those j for which entry j of pA equals v. Then set up and solve equations for q. Similar statement given optimal q^*.

   (e) First sequentially eliminate dominated rows and columns, and then try one of the above methods.

   (f) Use a matrix game solver app — see the course website.

3. Matrix games can be solved by solving linear programa.

22 Friday, October 12

Worked on homework.

23 Monday, October 15

1. Non-simultaneous Morra. Game theory shows the value of knowledge.
2. Kuhn's Poker. Game theory confirms the value of bluffing.
3. See the file morrapoker.pdf in Canvas Files.
4. Introduction to general matrix games.
5. Utility and the von Neumann - Morgenstern lottery.
6. The Prisoner's Dilemma.
7. See the link to the Cooperation Game on the course website.
8. Pure strategies.
9. Eliminating rows and columns via domination.
10. Identifying a pure Nash equilibrium.
11. Movement diagrams.

24 Wednesday, October 17

1. Converting a game represented in tree form, with probabilistic elements and information sets, into a game represented in matrix form.

2. Converting the game represented in matrix form into a game represented in tree form.

25 Friday, October 19

1. Worked on homework.

2. Discussed the notion of a best response. Note that there may be more than one best response to p or to q.

3. A Nash equilibrium is a choice of p and q such that p is a best response to q and q is a best response to p. A game may have more than one Nash equilibrium.

4. Finding Nash equilibria for 2×2 matrix games. Use a motion diagram and/or elimination of dominated rows and columns to try to find a pure strategy Nash equilibrium. Otherwise, select p to equalize the expected values for C, and select q to equalize the expected values for R.

26 Monday, October 23

1. Application to evolutionary biology.

27 Wednesday, October 25 – Wednesday, November 7

1. Discussed evolutionary biology—section 8.2.
2. Discussed the Cournot duopoly—section 8.3.
3. Discussed Nash flow and the application of the Brouwer Fixed Point Theorem to confirm the existence of a Nash equilibrium—see section 9.4 and pages 328–330.
4. Nash arbitration. Cooperation enforced by an external agent. Think of prisoner's dilemma.
5. Security levels and the status quo point.
6. The payoff polygon. All the points achievable via cooperation.
7. The negotiation set.
8. Solution point selected by an arbitration scheme. Nash's axioms for Nash arbitration.
9. Proof that the Nash arbitration point is the unique point selected by Nash's axioms.
10. Application of Nash arbitration to the labor-management problem.

28 Friday, November 10

Exam #2 review.

29 Monday, November 12

Exam #2.

30 Wednesday, November 14

Worked on homework problems.