3 Distance and Betweenness

3.1 The Metric Axioms

This is an outline some of the main results of Section 2.5 of the text.

Axiom D-1 — Existence: Each pair of points A, B is associated with a unique real number $AB \geq 0$, called the *distance* from A to B.

Axiom D-2 — Positive Definiteness: For all points A and B, $AB > 0$ unless $A = B$.

Axiom D-3 — Symmetry: For all points A and B, $AB = BA$.

Definition of Betweenness: For any three points A, B, and C in space, we say that B is *between* A and C, and we write A-B-C, if and only if A, B, and C are distinct, collinear points, and $AC = AB + BC$.

Definition: If A, B, C, and D are four distinct collinear points, let the betweenness relations A-B-C-D represent the composite of all four betweenness relations A-B-C, A-B-D, A-C-D, and B-C-D.

Theorem 2.5.1: If A-B-C, then C-B-A, and neither A-C-B nor B-A-C. (This is Theorem 1 of Kay, Section 2.5.)

Definition: Distance is said to satisfy the *Triangle Inequality* if $AB + BC \geq AC$ holds for all triples of points A, B, C. If Axioms D-1–D-3 and the Triangle Inequality hold, we have a *metric space*.

We won't require the Triangle Inequality to hold; it will be something we can prove later once we add some more axioms.

Theorem 2.5.2: If A-B-C, A-C-D, and the inequalities $AB + BD \geq AD$ and $BC + CD \geq BD$ hold, then A-B-C-D. (This is Theorem 2 of Kay, Section 2.5.)

Definition:

Segment AB**:** If A and B are distinct points, the *segment* AB is $\overline{AB} = \{A, B\} \cup \{C : A\text{-}C\text{-}B\}$. Points A and B are called the *endpoints* of the segment.

Ray AB**:** If A and B are distinct points, the *ray* AB is $\overrightarrow{AB} = \{A, B\} \cup \{C : A\text{-}C\text{-}B\} \cup \{D : A\text{-}B\text{-}D\}$. Point A is called the *endpoint* or *origin* of the ray.

Line AB **:** If A, B, and C are distinct points such that A-B-C, then $\overleftrightarrow{AB} = \overrightarrow{BA} \cup \overrightarrow{BC}$. Actually, this must be proven to be equivalent to the original definition of $\overleftrightarrow{AB}$ as the unique line containing both A and B—we will do this later.

Angle ABC**:** If A, B, and C are noncollinear points, the *angle* ABC is $\angle ABC = \overrightarrow{BA} \cup \overrightarrow{BC}$. Point B is called the *vertex* of the angle. Note that our definition of angle explicitly excludes the possibility that A, B, and C are collinear.

Theorem 2.5.3:

1. $\overline{AB} = \overline{BA}$.
2. $\overline{AB} \subseteq \overrightarrow{AB}$.
3. $\overrightarrow{AB} \subseteq \overleftrightarrow{AB}$.

(This is Theorem 3 of Kay, Section 2.5.)

Theorem 2.5.4: $\overrightarrow{AB} \cap \overrightarrow{BA} = \overline{AB}$. (This is Theorem 4 of Kay, Section 2.5.)

Model for Axioms I-1–I-5, D-1–D-3:

Points: Space consists of six points: $S = \{A, B, C, D, E, F\}$.

Lines: There are ten lines, each being a certain finite set of points:

$$\begin{gathered}\{A, B, C, D\}\\ \{A, E\}\\ \{B, E\}\\ \{C, E\}\\ \{D, E\}\\ \{A, F\}\\ \{B, F\}\\ \{C, F\}\\ \{D, F\}\\ \{E, F\}\end{gathered}$$

Planes: There are six planes, each being a certain finite set of points:

$$\{A, B, C, D, E\}$$
$$\{A, B, C, D, F\}$$
$$\{A, E, F\}$$
$$\{B, E, F\}$$
$$\{C, E, F\}$$
$$\{D, E, F\}$$

Distance: Distance between various pairs of points is defined by:

$$AE = BE = CE = DE = EF = 2$$
$$AF = BF = CF = DF = AC = 2$$
$$AB = BC = CD = AD = BD = 1$$
$$PQ = 0 \text{ if } P = Q$$
$$PQ = QP \text{ for all points } P \text{ and } Q$$

Observe that in this model, A-B-C holds but $\overrightarrow{BA} \cup \overrightarrow{BC} \neq \overleftrightarrow{AB}$. This shows that the following axiom cannot be proved from the preceding axioms, since it does not necessarily hold for a model even when the preceding axioms hold for that model. So this axiom is independent of the others.

Axiom D-4: Given any three points A, B, and C on line ℓ such that A-B-C, $\overrightarrow{BA} \cup \overrightarrow{BC} = \ell$.

The following Axiom is equivalent to D-4:

Axiom D-4': Given any four distinct collinear points A, B, C, and D such that A-B-C, then either D-A-B, A-D-B, B-D-C, or B-C-D. (You can prove this is equivalent to Axiom D-4.)

Exercise 3.1.1 Prove that Axiom D-4 implies Axiom D-4' and vice versa.

There is one more distance axiom, which we will encounter in Section 2.7 of Kay:

Axiom D-5 — Ruler Postulate: The points of each line ℓ may be assigned to real numbers x, $-\infty < x < \infty$, called *coordinates*, in such a manner that

1. Each point on ℓ is assigned to a unique coordinate.
2. Each coordinate is assigned to a unique point on ℓ.
3. Any two points on ℓ may be assigned to zero and a positive coordinate, respectively.
4. If points A and B on ℓ have coordinates a and b, respectively, then $AB = |a - b|$.

3.2 Distance in $\mathbf{E}^2$

Definition: The distance AB between the points $A = (x_1, y_1)$ and (x_2, y_2) in $\mathbf{E}^2$ is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Exercise 3.2.1 Given the points A and B above, consider a third point $C = (x_2, y_1)$ and use triangle ABC to prove the distance formula from the Pythagorean theorem.

Exercise 3.2.2 Verify that Axioms D-1, D-2, and D-3 hold.

Exercise 3.2.3 Verify that Axiom D-4 (or Axiom D-4') holds.

Exercise 3.2.4 Verify that Axiom D-5 holds.

Exercise 3.2.5 Prove the Triangle Inequality holds for any three points A, B, C:

$$AC \leq AB + BC$$

3.3 Distance in $\mathbf{E}^3$

Definition: The distance AB between the points $A = (x_1, y_1, z_1)$ and (x_2, y_2, z_2) in $\mathbf{E}^3$ is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Exercise 3.3.1 Use the Pythagorean Theorem to prove this formula.

Exercise 3.3.2 Verify that Axioms D-1, D-2, and D-3 hold.

Exercise 3.3.3 Verify that Axiom D-4 (or Axiom D-4') holds.

Exercise 3.3.4 Verify that Axiom D-5 holds.

Exercise 3.3.5 Prove the Triangle Inequality holds.

3.4 Distance in $\mathbf{S}^2$

A good book about "living" in a spherical world is *Sphereland* by Burger.

Recall that $\mathbf{S}^2$ is a sphere of radius 1 centered at the origin. Let A and B be any two points on $\mathbf{S}^2$. Find a great circle containing both A and B (remember that great circles are "lines" in $\mathbf{S}^2$). This circle is divided into two arcs by the points A and B. Define the distance between A and B to be the length of the shorter of these arcs. Note that if A and B are not exactly opposite one another (antipodal), then there is a unique great circle containing both of them, so the distance between them is well-defined. If, on the other hand, A and B are antipodal, then there is an infinite number of great circles containing them, but the lengths of all the great-circular arcs joining A and B are the same. So even in this case the distance AB is well-defined.

Exercise 3.4.1 Which of the distance axioms D-1 – D-5 hold?

Exercise 3.4.2 Suppose $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ are two points on $\mathbf{S}^2$. Determine an explicit formula for the distance AB.

Exercise 3.4.3 Look up the latitude and longitude of Lexington, KY and Tokyo, Japan. The kilometer was first defined as $\frac{1}{10{,}000}$ of the distance from the North Pole to the equator of the Earth. Use this information to determine the distance between these two cities.

Exercise 3.4.4 Does the Triangle Inequality hold?

3.5 Distance in $\mathbf{U}^2$

Recall that POINTS in $\mathbf{U}^2$ are points on the unit sphere centered at the origin, except for the point $N = (0, 0, 1)$. For two points $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ in $\mathbf{U}^2$, define the distance AB to be

$$AB = \sqrt{(\frac{x_2}{1 - z_2} - \frac{x_1}{1 - z_1})^2 + (\frac{y_2}{1 - z_2} - \frac{y_1}{1 - z_1})^2}$$

Note that, peculiar that this definition may appear to be, it is well-defined because neither z_1 or z_2 equals 1.

Exercise 3.5.1 Verify that Axioms D-1 – D-3 hold.

Exercise 3.5.2 Prove that if A remains fixed and B moves on the LINE $\overleftrightarrow{AB}$ towards N, then AB tends to infinity.

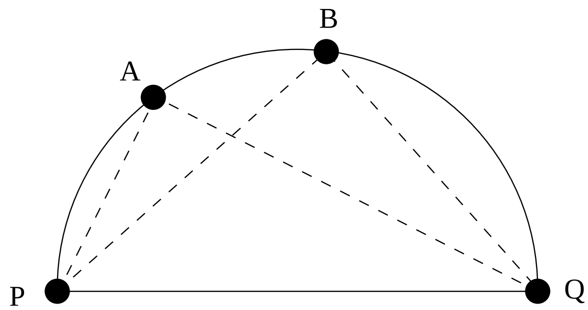
Exercise 3.5.3 Verify that Axiom D-4 (or Axiom D-4') holds.

Exercise 3.5.4 Verify that Axiom D-5 holds.

Exercise 3.5.5 Verify that the Triangle Inequality holds.

3.6 Distance in $\mathbf{H}^2$

Recall that POINTS in $\mathbf{H}^2$ are points (x, y, z) on the unit sphere centered at the origin such that $z > 1$; i.e, points in the upper hemisphere, excluding the equator. LINES in $\mathbf{H}^2$ are open half-circles perpendicular to the equator. For two points A, B, consider the unique semicircle that contains both of them, and let P and Q be the endpoints of the semicircle on the equator as shown below:



Define the distance AB to be

$$AB = \ln(\frac{AQ}{AP}\frac{BP}{BQ})$$

where AP, AQ, BP, and BQ are the ordinary lengths of line segments.

Exercise 3.6.1 Verify that Axioms D-1 – D-3 hold for this model.

Exercise 3.6.2 For two points A, C, consider the unique perpendicular semicircle that contains both of them, and let B be a point on the arc of the semicircle between A and C. Prove that A-B-C.

Exercise 3.6.3 Prove that if A remains fixed and B moves toward Q, then AB tends to infinity.

Exercise 3.6.4 Prove that Axioms D-4 and D-5 hold for this model.

Exercise 3.6.5 Does the Triangle Inequality hold for this model?

3.7 Distance in $\mathbf{P}^2$

Exercise 3.7.1 Motivated by your study of $\mathbf{S}^2$, give a reasonable formula for distance in $\mathbf{P}^2$ and discuss which of the Axioms D-1 – D-5 hold.