6 Triangles

6.1 Congruence

The following is a summary of Section 3.1 of Kay.

Definition: Given three noncollinear points A, B, C, $\triangle ABC$ is defined to be $\overline{AB} \cup \overline{BC} \cup \overline{AC}$. Refer to page 132 to confirm your understanding of the terms *angles of a triangle, sides, vertices, included angle between two sides, included side between two angles, and side opposite an angle.*

Definition: Given two triangles $\triangle ABC$ and $\triangle DEF$, we define a *correspondence* between the triangles, $\triangle ABC \leftrightarrow \triangle DEF$ to mean the particular pairing of the vertices

$$\begin{aligned} A &\leftrightarrow D \\ B &\leftrightarrow E \\ C &\leftrightarrow F \end{aligned}$$

We sometimes denote this correspondence by $ABC \leftrightarrow DEF$. Such a correspondence between the vertices induces a particular correspondence between the sides of the two triangles:

$$\begin{aligned} \overline{AB} &\leftrightarrow \overline{DE} \\ \overline{BC} &\leftrightarrow \overline{EF} \\ \overline{AC} &\leftrightarrow \overline{DF} \end{aligned}$$

and a correspondence between the angles of the two triangles:

$$\begin{aligned} \angle ABC &\leftrightarrow \angle DEF \\ \angle BCA &\leftrightarrow \angle EFD \\ \angle CAB &\leftrightarrow \angle FDE \end{aligned}$$

Definition:

1. Two segments $\overline{AB}$ and $\overline{CD}$ are said to be *congruent* if $AB = CD$. In this case we write $\overline{AB} \cong \overline{CD}$.

2. Two angles $\angle ABC$ and $\angle DEF$ are said to be *congruent* if $m\angle ABC = m\angle DEF$. In this case we write $\angle ABC \cong \angle DEF$.

3. Two triangles are congruent if there is some particular ordering of the vertices yielding a correspondence $\triangle ABC \leftrightarrow \triangle DEF$ such that each pair of corresponding sides is congruent and each pair of congruent angles are congruent. In this case we write $\triangle ABC \cong \triangle DEF$. So six congruences must hold:

$$\begin{gathered}\overline{AB} \cong \overline{DE}\\ \overline{BC} \cong \overline{EF}\\ \overline{AC} \cong \overline{DF}\\ \angle ABC \cong \angle DEF\\ \angle BCA \cong \angle EFD\\ \angle CAB \cong \angle FDE\end{gathered}$$

We say that *corresponding parts of congruent triangles are congruent,* sometimes abbreviated *CPCTC* (or *CPCFC* or *CPCF* when we consider congruence between more general figures).

Theorem: The congruence relation between triangles is

1. **Reflexive:** $\triangle ABC \cong \triangle ABC$.

2. **Symmetric:** If $\triangle ABC \cong \triangle DEF$, then $\triangle DEF \cong \triangle ABC$.

3. **Transitive:** If $\triangle ABC \cong \triangle DEF$ and $\triangle DEF \cong \triangle GHI$, then $\triangle ABC \cong \triangle GHI$.

(This theorem is on page 136 of Kay.)

Of course, these three properties hold for congruences among segments and among angles as well—a result we must prove and use when proving the above theorem.

6.2 SAS Congruence

The following is summarized from Sections 3.2–3.3 of Kay.

All of you probably remember some conditions that guarantee congruence between triangles. One of them is known as *SAS* ("side-angle-side"):

> **Axiom C-1:** Under the correspondence $ABC \leftrightarrow XYZ$, let two sides and the included angle of $\triangle ABC$ be congruent, respectively, to the corresponding two sides and the included angle of $\triangle XYZ$. (For example, $\overline{AB} \cong \overline{XY}$, $\overline{BC} \cong \overline{YZ}$, and $\angle ABC \cong \angle XYZ$.) Then $\triangle ABC \cong \triangle XYZ$ under that correspondence.

You might guess that since this is listed as an axiom, it cannot be proved from the other axioms that we have so far. We can confirm this by making a model that satisfies all the other axioms, but not Axiom C-1. This is called *taxicab geometry.*

1. Points are ordinary points (x, y, z) in three-dimensional Euclidean space $\mathbf{R}^3$.
2. Lines are ordinary lines in $\mathbf{R}^3$.
3. Planes are ordinary planes in $\mathbf{R}^3$.
4. Measures of angles are ordinary measures of angles in $\mathbf{R}^3$.
5. The distance between two points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ is defined to be

$$PQ = |x_2 - x_1| + |y_2 - y_1| + |z_2 - z_1|.$$

The book shows that for any three points, A-B-C in ordinary $\mathbf{R}^3$ if and only if A-B-C in taxicab geometry (this is Theorem 1 in Section 3.2).

Now consider the points

$$\begin{aligned} A &= (0,6,0) \\ B &= (0,0,0) \\ C &= (6,0,0) \\ X &= (-3,0,0) \\ Y &= (0,3,0) \\ Z &= (3,0,0) \end{aligned}$$

Check that $\triangle ABC$ is a right triangle with a right angle at B, and that $AB = BC = 6$. Check also that $\triangle XYZ$ is a right triangle with a right angle at Y, and that $XY = YZ = 6$. So two sides and an included angle of these two triangles are congruent, respectively, but the two triangles are not congruent because $AC = 12$ while $XZ = 6$.

6.3 ASA Congruence

The following is summarized from Section 3.3 of Kay.

Theorem 3.3.1 (ASA): If, under some correspondence, two angles and the included side of one triangle are congruent to the corresponding angles and included side of another, the triangles are congruent under that correspondence. (This is Theorem 1 of Section 3.3 of Kay.)

6.4 SSS Congruence

The following is summarized from Section 3.3 of Kay.

Lemma 3.3.A: Give a point A and a line ℓ, at most one line m perpendicular to ℓ contains A, and the point $B = \ell \cap m$, called the *foot* of the perpendicular m, is unique. (This is Lemma A of Section 3.3 of Kay.)

Corollary: A triangle can have at most one right angle.

Definition: A triangle $\triangle ABC$ is *isosceles* if at least two of its sides are congruent to each other. In this case, the two congruence sides are called the *legs* and the third side is called the *base*. The vertex opposite the base is sometimes called *the vertex* of the isosceles triangle.

Lemma 3.3.B: In $\triangle ABC$, if $\overline{AC} \cong \overline{BC}$, then $\angle A \cong \angle B$. (This is Lemma B of Section 3.3 of Kay.)

Lemma 3.3.C: In $\triangle ABC$, if $\angle A \cong \angle B$, then $\overline{AC} \cong \overline{BC}$. (This is Lemma C of Section 3.3 of Kay.)

From the above two lemmas, we know that a triangle is isosceles if and only if the base angles are congruent.

Definition: Suppose $\triangle ABC$ is an isosceles triangle and ℓ is a line. The following is a list of four possible *symmetry properties* of the line ℓ:

1. **PB:** ℓ is perpendicular to the base.
2. **BB:** ℓ bisects the base.
3. **PV:** ℓ passes through the vertex.
4. **BV:** ℓ bisects the vertex.

Theorem 3.3.2 (Isosceles Triangle Theorem:) In any isosceles triangle, if a line ℓ satisfies any two of the four symmetry properties, then it satisfies all four. In this case we call ℓ a *line of symmetry* for the triangle. (This is Theorem 2 of Section 3.3 of Kay.)

The proof has six parts:

1. Assume PV and BB. Prove PB and BV. (Done in the book.)
2. Assume PV and PB. Prove BB and BV. (Done in the book.)
3. Assume PV and BV. Prove PB and BB. (Homework problem in the book.)
4. Assume PB and BB. Prove PV and BV. (Homework problem in the book.)
5. Assume BV and BB. Prove PV and PB. (PV is immediate, so we are back in the first case.)
6. Assume BV and PB. Prove PV and BB. (PV is immediate, so we are back in the second case.

Theorem (Perpendicular Bisector Theorem): The set of all points equidistant from two fixed points A and B is the same as the set of all points on the perpendicular bisector of segment $\overline{AB}$. (This theorem is on page 155 of Kay.)

Theorem 3.3.3 (SSS Theorem): If, under some correspondence between their vertices, two triangles have the three sides of one congruent to the corresponding three sides of the other, then the triangles are congruent under that correspondence. (This is Theorem 3 of Section 3.3 of Kay.)

Theorem 3.3.4 (Existence of Perpendicular from an External Point): Let line ℓ and point A no on ℓ be given. Then there exists a unique line m perpendicular to ℓ and passing through A. (This is Theorem 4 of Section 3.3 of Kay.)

6.5 Triangle Calculations in $\mathbf{R}^2$

6.5.1 The Area of a Triangle Using Sines

It's time to put some algebra and trigonometry to use. In this problem we will use the triangle in Figure 1. In this triangle all angles have measure less than 90°, however, the results can be proven to be true for general triangles.

The lengths of $\overline{BC}$, $\overline{AC}$ and $\overline{AB}$ are a, b and c, respectively. Segment $\overline{AD}$ has length c' and $\overline{DB}$ length c''. Segment $\overline{CD}$ is the altitude of the triangle from C, and has length h.

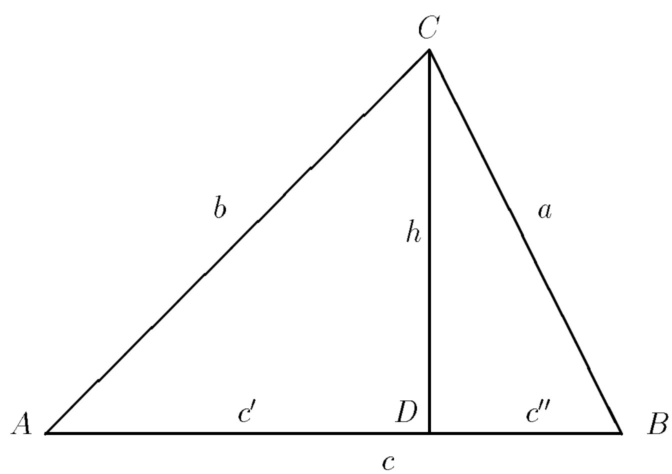


Figure 1: Triangle ABC

The usual formula for the area of a triangle is $\frac{1}{2}$(base)(height).

1. Using the given labeling, Area(ABC) =________________________________
2. Since triangle ADC is a right triangle, $\sin A =$____________ so $h =$____________
3. Thus, Area(ABC) $= \frac{1}{2}ch =$ ________________________________

4. What is a formula for Area(ABC) using $\sin B$? Using $\sin C$?? (Note: you will have to use the altitude from A or B).

The conclusion we have made is that the area of a triangle is one-half the product of the lengths of any two sides and the sine of the included angle.

6.5.2 The Law of Sines

Using the triangle in Section 6.5.1, the *Law of Sines* is:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The result holds for arbitrary triangles, but we shall prove it for the triangle ABC in Section 6.5.1.

1. We showed that the area of this triangle was given by three different formulas. What are they?
2. From these three formulas, prove the Law of Sines.

6.5.3 The Law of Cosines

Using the triangle in Section 6.5.1, the *Law of Cosines* is:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

1. Show that $c' = b \cos A$.
2. Verify that $c'' = c - c'$.
3. Verify that $h^2 = b^2 - (c')^2$.
4. Apply the Pythagorean Theorem to triangle CDB, then use the facts above to make the appropriate substitutions to prove the Law of Cosines.
5. One last question: What happens when you Apply the Law of Cosines in the case that $\angle A$ is a right angle?

6.5.4 Heron's Formula

In this section, we will put some of our results together to prove yet another formula for the area of a triangle.

For notation, refer to Figure 1 in Section 6.5.1.

Let $s = \frac{1}{2}(a + b + c)$ (s is half the perimeter). Heron's Formula for the area of triangle ABC is:

$$\text{area}(ABC) = \sqrt{s(s-a)(s-b)(s-c)}$$

First we gather some facts:

1. $\text{area}(ABC) = \frac{1}{2}bc \sin A$ (See Section 6.5.1).
2. $\sin^2 A + \cos^2 A = 1$, thus $\sin A = \sqrt{1 - \cos^2 A}$.
3. $a^2 = b^2 + c^2 - 2bc \cos A$, thus
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$
(This is the Law of Cosines, proven in Section 6.5.3.)
4. $s - a = \frac{1}{2}(-a + b + c)$, $s - b = \frac{1}{2}(a - b + c)$, and $s - c = \frac{1}{2}(a + b - c)$.

Thus, area$(ABC) = \frac{1}{2}bc\sin A$ (fact 1)

$$= \tfrac{1}{2}\sqrt{1-\cos^2 A} \text{ (fact 2)}$$

$$= \tfrac{1}{2}bc\sqrt{1-\left(\tfrac{b^2+c^2-a^2}{2bc}\right)^2} \text{ (fact 3)}$$

$$= \tfrac{1}{4}\sqrt{4b^2c^2-(b^2+c^2-a^2)^2}$$

$$= \tfrac{1}{4}\sqrt{4b^2c^2-(b^4+c^4+a^4+2b^2c^2-2a^2b^2-2a^2c^2)}$$

$$= \tfrac{1}{4}\sqrt{2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4} \qquad (*)$$

Now we work on the right hand side of Heron's Formula:

$$\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{\tfrac{1}{2}(a+b+c)\tfrac{1}{2}(-a+b+c)\tfrac{1}{2}(a-b+c)\tfrac{1}{2}(a+b-c)}$$

$$= \tfrac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)} \qquad (**)$$

To finish the proof, we must show that $(*) = (**)$.

1. Show that $(a+b+c)(-a+b+c)(a-b+c)(a+b-c) = 2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4$.

6.5.5 A Cosine Formula

In this section we remind you of a nice formula to get the cosine of the angle using coordinates of points.

Assume that you have triangle ABC such that the coordinates of the three (distinct) points A, B, and C are $(0,0)$, (x_1, y_1), and (x_2, y_2), respectively. The Law of Cosines can be used to prove that

$$\boxed{\cos A = \frac{x_1 x_2 + y_1 y_2}{\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}}.}$$

1. Use the Law of Cosines to prove this formula. Recall that the length of a line segment joining points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

6.5.6 Determinantal Area Formula

Referring to the triangle ABC of the previous section, we can prove another area formula.

$$\text{area}(ABC) = \frac{1}{2}|x_1y_2 - x_2y_1| = \frac{1}{2}\left\| \begin{matrix} x_1 & x_2 \\ y_1 & y_2 \end{matrix} \right\|$$

(I am using the notation $\| \cdot \|$ to denote the absolute value of the determinant.)

1. Use $\text{area}(ABC) = \frac{1}{2}bc\sin A$, the cosine formula from the previous section, and $\sin^2 A + \cos^2 A = 1$ to prove this formula.

6.6 Inequality Theorems

This material is summarized from Sections 3.4 and 3.5 of the Kay.

Theorem 3.4.1 (Exterior Angle Inequality): An exterior angle of a triangle has angle measure greater than that of either opposite interior angle. (This is Theorem 1 of Section 3.4 of Kay.)

Corollary: The sum of the measures of any two angles of a triangle is less than 180.

Corollary: A triangle can have at most one right or obtuse angle.

Corollary: The base angles of an isosceles triangle are acute.

Theorem 3.4.2 (Saccheri-Legendre): The sum of the measures of the three angles of any triangle is less than or equal to 180. (This is Theorem 2 of Section 4.2 of Kay.)

Corollary (Scalene Inequality): If one side of a triangle has greater length than another side, then the angle opposite the longer side has the great angle measure, and conversely.

Corollary: If a triangle has an obtuse or a right angle, then the opposite side is the longest side of the triangle.

Corollary: The hypotenuse of a right triangle is greater in length that either leg.

Theorem 3.5.1 (Triangle Inequality): In any triangle, the sum of the lengths of any two sides exceeds the length of the third side. For distinct points A, B, C, $AB + BC \geq AC$, with equality if and only if A-B-C. (This is Theorem 1 of Section 3.5 of Kay.)

Corollary (Median Inequality): If M is the midpoint of side $\overline{BC}$ in $\triangle ABC$, then $AM < \frac{1}{2}(AB + AC)$.

Theorem 3.5.2 (SAS Inequality): For triangles $\triangle ABC$ and $\triangle XYZ$, if $AB = XY$, $AC = XZ$, then $m\angle A > m\angle X$ if and only if $BC > YZ$. (This is Theorem 2 of Section 3.5 of Kay.)

6.7 Spherical Triangles

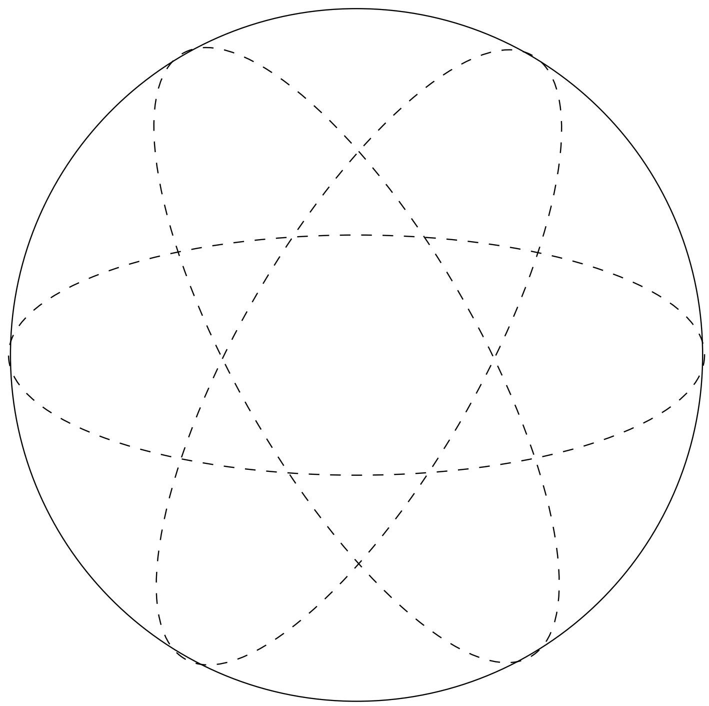
1. Suppose H_a, H_b, and H_c are three regions in the plane. Use Venn diagrams to show the following equality.

$$(*) \quad \begin{aligned} &\text{area}\,(H_a \cup H_b \cup H_c) \\ &= \text{area}\,(H_a) + \text{area}\,(H_b) + \text{area}\,(H_c) \\ &-\text{area}\,(H_a \cap H_b) - \text{area}\,(H_a \cap H_c) - \text{area}\,(H_b \cap H_c) \\ &+\text{area}\,(H_a \cap H_b \cap H_c) \end{aligned}$$

2. What is the area of a sphere of radius 1?

3. A *spherical sector* is a region on a sphere determined by the intersection of two hemispheres, bounded by two great circles Q and Q'. Assume that the sphere has radius 1. Express the area of the sector as a function of the measure α of the angle between Q and Q'. (Assume that the angle is expressed in radians.)

4. Now consider a spherical triangle $\triangle ABC$ on a sphere of radius 1. (See the following diagram.) Let the measures of the angles A, B, and C be α, β, and γ, respectively. Let the sides opposite vertices A, B, and C be labeled a, b, and c, respectively. Each side determines a great circle Q_a, Q_b, and Q_c, respectively. These great circles each determine a hemisphere H_a, H_b, and H_c, respectively, that contains the triangle $\triangle ABC$. The three great circles intersect in pairs on the opposite side of the sphere to determine three points A', B', and C' exactly opposite the points A, B, and C, respectively.



Use the formula $(*)$ (which also works for regions on the sphere) to find a formula for the area of spherical triangle $\triangle ABC$ in terms of α, β, and γ.

6.8 More on Angle Sums of Triangles

Recall that we know that the angle sum of any triangle is less than or equal to 180.

Suppose $\triangle ABC$ is a triangle. Let us say that this is a triangle *with defect* if its angle sum is strictly less than 180, and is a triangle *without defect* if its angle sum is equal to 180.

Define a *rectangle* to be a convex quadrilateral with four right angles.

1. Prove that if a triangle without defect exists, then at least one right triangle without defect exists. Suggestion: Divide the given triangle into two right triangles by constructing the altitude from the largest angle.
2. Prove that if a triangle with defect exists, then at least one right triangle with defect exists.
3. Prove that if a rectangle exists, then at least one right triangle without defect exists.
4. Prove that if a right triangle without defect exists, then at least one rectangle exists.
5. Prove that if a rectangle exists with side lengths a and b, and k is any positive integer, then a rectangle exists with side lengths ka and kb. (You can use a more informal argument for this.)
6. Prove that if there exists a right triangle with defect, then no rectangle exists. Suggestion: Let $\triangle ABC$ be a right triangle with defect, having right angle at A. Assume a rectangle exists. Then a rectangle $ADEF$ can be constructed that is large enough so that the two legs of the triangle are contained in two of the sides, $\overline{AD}$ and $\overline{AF}$, of the rectangle. Divide the rectangle into triangles by constructing segments from B and C to the corner E of the rectangle opposite A.
7. Put the above results together to prove:
   (a) If a triangle without defect exists, then every triangle is without defect, and rectangles exist.
   (b) If a triangle with defect exists, then every triangle is with defect, and no rectangles exist.