8 Circles

8.1 Basic Results

This material is summarized from Section 3.8 of Kay.

Definition: A *circle* is the set of points in a plane which lie at a positive, fixed distance r from some fixed point O. The number r is called the *radius* (as well as any line segment joining point O to any point on the circle), and the fixed point O is called the *center* of the circle. A point P is said to be *interior* to the circle, or an *interior point*, whenever $OP < r$; if $OP > r$, then P is said to be an *exterior point.*

Look at the diagram in the book to clarify the definitions of the following terms: diameter, radius, chord, secant (line), tangent (line) and point of contact or tangency, central angle, inscribed angle, semicircle, angle inscribed in a semicircle, arc, subtended or intercepted arc or chord of an angle.

Lemma:

1. The center of a circle is the midpoint of any diameter.
2. The perpendicular bisector of any chord of a circle passes through the center.
3. A line passing through the center of a circle and perpendicular to a chord bisects the chord.
4. Two congruent central angles subtend congruent chords, and conversely.
5. Two chords equidistant from the center of a circle have equal lengths, and conversely.

Definition: A *minor arc* is the intersection of the circle with a central angle and its interior, a *semicircle* is the intersection of the circle with a closed half-plane whose edge passes through O, and a *major arc* of a circle is the intersection of the circle and a central angle and its exterior (that is, the complement of a minor arc, plus endpoints). If the endpoints of an arc are A and B, and C is any other point of the arc (which must be used in order to uniquely identify the arc), then we define the *measure* $m\ \overset{\frown}{ACB}$ of the arc as follows:

1. Minor arc: $m\ \overset{\frown}{ACB} = m\angle AOB$.
2. Semicircle: $m\ \overset{\frown}{ACB} = 180$.
3. Major arc: $m\ \overset{\frown}{ACB} = 360 - m\angle AOB$.

Given a circle with center O and ray $\overrightarrow{OP}$, let H_1 be one of the half-planes associated with $\overrightarrow{OP}$. Assign coordinates $0 \leq \theta < 180$ to $\overrightarrow{OP}$ and rays in this half-plane as before. Assign the coordinate 180 to the opposite ray of $\overrightarrow{OP}$. Assign coordinates $-180 < \theta < 0$ to the rays in the half-plane H_2 opposite to H_1, the negative of the coordinate that would have ordinarily been assigned with respect to H_2.

Lemma: For any arc $\overset{\frown}{ACB}$ on circle O, if P' lies in the complementary arc of $\overset{\frown}{ACB}$ and $a > b$ are the coordinates of rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$, respectively, relative to the half-planes determined by line $\overleftrightarrow{PP'}$, then $m\ \overset{\frown}{ACB} = a - b$.

Theorem 3.8.1: Suppose arcs $A_1 = \overset{\frown}{ADC}$ and $A_2 = \overset{\frown}{CEB}$ are any two arcs of circle O having just one point C in common, and such that their union, $A_1 \cup A_2 = \overset{\frown}{ACB}$, is also an arc. Then $m(A_1 \cup A_2) = mA_1 + mA_2$. (This is Theorem 1 of Section 3.8 of Kay.)

Theorem 3.8.2: A line is tangent to a circle iff it is perpendicular to the radius at the point of contact. (This is Theorem 2 of Section 3.8 of Kay.)

Theorem 3: If a line ℓ passes through an interior point A of a circle, it is a secant of the circle, intersecting the circle in precisely two points. (This is Theorem 3 of Section 3.8 of Kay.)

8.2 Circles on Spheres

Consider a circle of radius r (as measured along the surface of a sphere) on a sphere of radius 1. Determine a formula for the circumference and the spherical area of the circle.