Area and Volume Problems

1. Given the formula for the area of a square, derive the formula for the area of a rectangle of dimensions a and b.

2. Given the formula for the area of a rectangle, derive the formula for the area of a parallelogram of base b and height h.

3. Use calculus to derive the formula for the area of a parallelogram of base b and height h.

4. Derive the formula for the area of a triangle of base b and height h.

5. Derive the formula for the area of a trapezoid with bases b_1, b_2 and height h.

6. Derive the formula for the area of a regular polygon with perimeter P and apothem a.

7. Derive the formula for the area of a circle of radius r. (Via polygons, via "pie slices", and via "unrolling".)

8. Use calculus to derive the area of a circle of radius r.

9. Take the derivative of the formula for the area of a circle of radius r. What formula do you get? Why does this happen?

10. Describe a method using grids to approximate the areas of two-dimensional regions and volumes of three-dimensional regions. How can this lead to a definition of area and volume?

11. Use calculus to prove Cavalieri's principle: If two 2-dimensional regions have the property that all lines parallel to some fixed line that meet R_1 and R_2 do so in line segments having equal lengths, whose endpoints are the boundary points of R and R', then R and R' have the same area.

12. Two regions R_1, R_2 are *similar* if there is a one-to-one correspondence between the points of R_1 and the points of R_2, and a constant k, such that for every pair of points A, B in R_1 and corresponding pair of points A', B' in R_2, we have $AB = kA'B'$. If R_1 and R_2 are similar 2-dimensional regions, what is the relationship between the areas of R_1 and R_2? (If you are not certain, try this out with simple shapes first.)

13. What is the effect on area if a two-dimensional region is scaled by a factor of r parallel to the x-axis, and by a factor of s parallel to the y-axis?

14. What is the effect on volume if a three-dimensional region is scaled by a factor of r parallel to the x-axis, and by a factor of s parallel to the y-axis, and by a factor of t parallel to the z-axis?

15. Use calculus to prove Cavalieri's principle: If two 2-dimensional regions have the property that all planes parallel to some fixed plane that meet R_1 and R_2 do so in plane sections having equal areas, whose boundaries lie in the boundaries of R and R', then R and R' have the same volume.

16. Derive the formula for the volume of a prism whose base is a region of area B and whose height is h.

17. Use calculus to derive the formula for the volume of a pyramid whose base is a region of area B and whose height is h.

18. Use scaling arguments and Cavalieri's principle to derive the formula for the volume of a pyramid whose base is a region of area B and whose height is h, by relating the pyramid to a pyramid that is one-third of a cube.

19. Use calculus to derive the formula for the volume of a cylinder of radius r and height h.

20. Derive the formula of the surface area of a cylinder of radius r and height h.

21. Use calculus to derive the formula for the volume of a cone of radius r and height h.

22. Derive the formula for the surface area of a cone of radius r and height h.

23. Use calculus to derive the formula for the volume of a sphere of radius r.

24. Use calculus to derive the formula for the surface area of a sphere of radius r.

25. Take the derivative of the formula for the volume of a sphere of radius r. What formula do you get? Why does this happen?

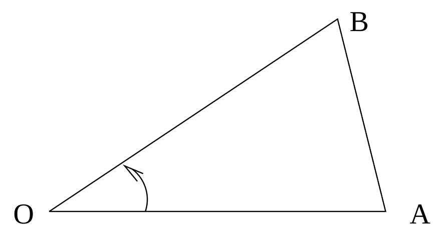
26. Use Cavalieri's principle to compare the volume of a hemisphere of radius r, with the volume of a cylinder of radius r and height r from which an inverted cone of radius r and height r has been removed.

27. If R_1 and R_2 are similar 3-dimensional regions, what is the relationship between the volumes of R_1 and R_2? Between the surface areas of R_1 and R_2?

28. Prove that if OAB is a triangle in $\mathbf{E}^2$ with coordinates $O = (0,0)$, $A = (x_1, y_1)$, $B = (x_2, y_2)$, then its area is

$$\frac{1}{2}\text{abs}\det\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$

Actually, it turns out that if the angle $\angle AOB$ is counterclockwise, measured from ray $\overrightarrow{OA}$ to $\overrightarrow{OB}$,



then we can drop the absolute value above, and the area is in fact just equal to the determinant

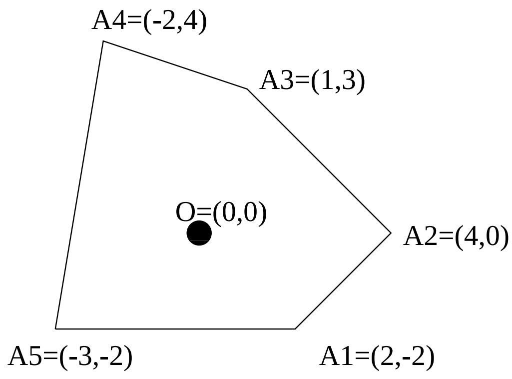
$$\frac{1}{2}\det\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$

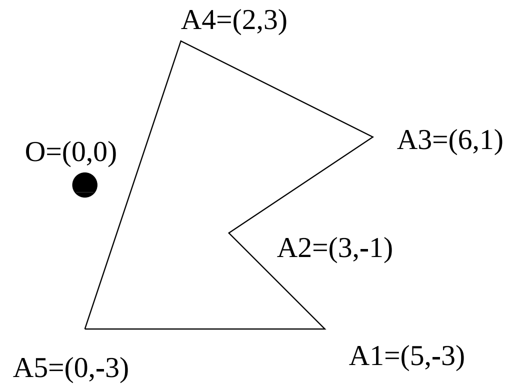
29. Now suppose that you have a convex polygon $A_1, A_2, \ldots, A_n$ in the plane $\mathbf{E}^2$ (listing its vertices in counterclockwise order) that contains the origin in its interior. Suppose that the vertex A_i has coordinates (x_i, y_i), $i = 1, \ldots, n$. Prove that the area of the polygon is

$$\frac{1}{2}\left(\det\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} + \det\begin{bmatrix} x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} + \cdots + \det\begin{bmatrix} x_{n-1} & y_{n-1} \\ x_n & y_n \end{bmatrix} + \det\begin{bmatrix} x_n & y_n \\ x_1 & y_1 \end{bmatrix}\right)$$

Actually, it turns out that this formula works even if the polygon is not convex, and even it does not contain the origin in its interior, as long as it does not cross itself.

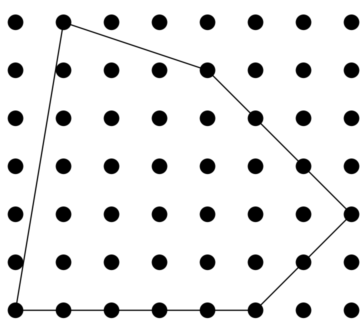
Use this formula to calculate the areas of the following polygons:

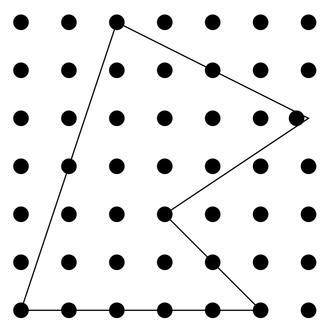




30. Suppose a polygon (not necessarily convex, but not self-crossing) has the property that each of its vertices has integer coordinates. Let B be the number of integer points on its boundary, and I be the number of integer points in its interior. Consider some examples and conjecture a formula for the area of the polygon in terms of B and I. This is called *Pick's Formula*.

    Use this formula to calculate the areas of the following polygons:

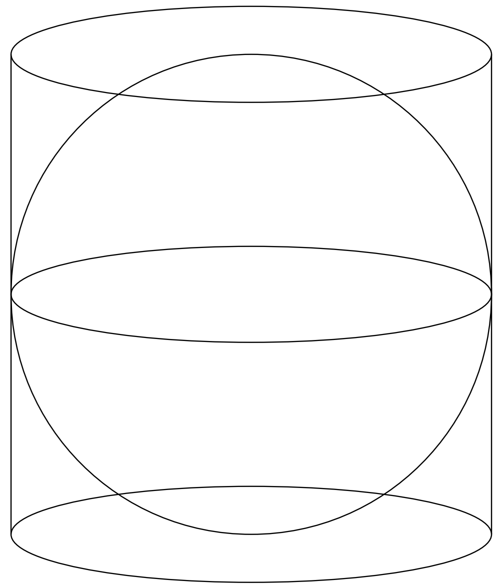




31. Recall that $\mathbf{S}^2$ is the sphere of radius 1 centered at the origin. Suppose that A and A' are any two points on $\mathbf{S}^2$ that are exactly opposite each other. Connect A and A' with two great half-circles that meet at A (and also at A') at an angle $\alpha < \pi$, where α is measured in radians. These two half-circles determine a region of the sphere called a *lune*. Explain why the area of this lune is 2α.

32. Use inclusion/exclusion to obtain the formula for the area of a spherical triangle with angles α, β, and γ (measured in radians) on a sphere of radius 1.

33. Suppose a sphere is inscribed in a cylinder whose radius is the radius of the sphere, and whose height is the diameter of the sphere.



Let R be any region on the sphere. Project this region out horizontally onto the cylinder to get the region R' on the cylinder. Use calculus to prove that R and R' have the same areas. This is one way to make maps of the earth that accurately represent areas of countries, although the shapes of the countries are distorted.

34. Assume that $OABC$ is a tetrahedron with coordinates $O = (0, 0, 0)$, $A = (x_1, y_1, z_1)$, $B = (x_2, y_2, z_2)$, $C = (x_3, y_3, z_3)$. Prove that its volume is

$$\frac{1}{6} \text{abs} \det \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$$

35. Try to find a formula analogous to that of problem (28) that applies to 3-dimensional polyhedra, all of whose faces are triangles. What could you do if not all faces are triangles?

36. Find a two-dimensional region with infinite perimeter and finite area.

37. Can you find a three-dimensional region with finite volume and infinite surface area?

38. Can you find a three-dimensional region with finite surface area and infinite volume?

39. Find a rectangle with perimeter 10 that has the greatest area.

40. Find a rectangle with area 10 that has the smallest perimeter.

41. Find a cylinder with volume 100 that has the smallest surface area.

42. Two cylinders of radius 1 intersect in a solid region. What is the volume of this region?

43. A bead is is created in a sphere by slicing off two opposite caps and drilling out a cylindrical hole through the center. The length of the cylinder is 10. What is the volume of the bead?

44. What is a planimeter and why does it work?