MA 341 — Log of Class Activities

Contents

1 Wednesday, August 27

1. Worked on problems 1–5 on the Lines handout.

   (a) What is a line?
   For the moment, let us regard a line as a set of points in the plane satisfying an equation of the form $ax + by = c$, where it is not the case that both a and b are zero.

   (b) Describe the line containing the points $(-2, 3)$ and $(5, -9)$.
   The slope will be $m = \frac{-9-3}{5-(-2)} = \frac{-12}{7}$*. Using the point-slope formula yields the equation* $y - 3 = \frac{-12}{7}(x + 2)$*, which can be expressed as* $12x + 7y = -3$.

   (c) Describe the line containing the points $(-2, 3)$ and $(2, 3)$.
   The equation for this line is $y = 3$.

   (d) Describe the line containing the points $(-2, 3)$ and $(-2, -9)$.
   The equation for this line is $x = -2$.

   (e) Describe the line containing the points (x_1, y_1) and (x_2, y_2).
   Assuming that $x_1 \neq x_2$ *we can use the point-slope equation to get*

   $$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1),$$

   which can be simplified:

   $$(y - y_1)(x_2 - x_1) = (y_2 - y_1)(x - x_1)$$
   $$(y_1 - y_2)x + (x_2 - x_1)y = x_2 y_1 - x_1 y_2.$$

   Now it turns out that this final equation also happens to yield the correct answer even if $x_1 = x_2$*, because in this case substitution yields*

   $$(y_1 - y_2)x = x_1 y_1 - x_1 y_2,$$

   and since we cannot have y_1 *also equal to* y_2*, we can divide both sides by* $y_1 - y_2$ *to get*

   $$x = x_1.$$

2. Went over the syllabus.

3. Passed out the SETI handout, asking for a decoding by Friday's class. You may enlist the help of others. Assume that this message came from within our solar system. The first line is simply a listing of the symbols to be used. I don't plan to collect this.

2 Friday, August 29

1. Showed the trailer for "Contact" and gave some hints for completing SETI. In particular, try to figure out exactly what physical thing the final part of the message is referring to.

2. Explained some of the instructions for Homework #1, due next Friday.

3. Demonstrated GeoGebra, including the file collineartest.ggb. I strongly recommend that you fetch and install GeoGebra on your computer.

4. Demonstrated SketchUp. I strongly recommend that you fetch and install SketchUp on your computer.

3 Wednesday, September 3

1. Finished up the SETI problem.

   From The Sixth Book of Mathematical Games from Scientific American by Martin Gardner. The final line gives the formula for the volume of a sphere. If we believe the message came from our solar system, then a natural unit of measure is the radius of the sun. One can then check that the sphere in question has the radius of the earth. Thus, if this final paragraph is to signify the origin of the message, it appears that it was sent by some other humans—perhaps a disappointment!

2. Worked on the Toves and Borogoves handout, problems 1 and 2.

   (a) Contact has been established with an alien race (the Carrollians) and they convey the following information to you about a mathematical structure of interest to them.

      A. There is a finite number of toves.
      B. There is a finite number of borogoves.
      C. Given any borogove there are exactly two different toves that gimble with it.

   Prove that the number of toves that gimble with an odd number of borogoves is even.

   Let m be the number of toves and n be the number of borogoves. For $i = 1, \ldots, m$ let a_i be the number of borogoves that tove i gimbles with. Then

   $$S = \sum_{i=1}^{m} a_i$$

   counts the total number of gimblings taking place. But since every borogove gimbles only twice, then we also have

   $$S = 2n.$$

   Thus S is an even number. This implies that the number of odd integers among $\{a_1, \ldots, a_m\}$ must be even.

   (b) What is the meaning of toves and borogoves? What does gimble mean? What representations, if any, did you create while working on this problem?

   There is no particular meaning of these terms. The power of this is that no matter what meaning might be bestowed upon these terms, if the hypotheses are satisfied, then the conclusion is automatically true. Some examples generated from the class:

i. Borogoves: Prime numbers between, say, 1 and 10. Toves: Positive integers from 1 to 10. Gimble: A tove gimbles with a borogove if the tove is a divisor of the borogove. Conclusion: The number of positive integers dividing an odd number of the primes is even.

ii. Toves: Line segments. Borogoves: Angles formed by two of these line segments. Gimble: A line segment gimbles with an angle if that line segment is one of the two forming that angle. Conclusion: The number of line segments involved in an odd number of angles is even.

iii. Toves: Shoes. Borogoves: People in class. Gimble: A person gimbles with a shoe if that person is wearing that shoe. Conclusion: The number of shoes worn by an odd number of people is even; and hence, since every shoe is worn by only one person, the total number of shoes must be even.

One representation used in class was a diagram in which the toves were listed in a first column, the borogoves in a second column, and segments were drawn between gimbling toves and borogoves.

4 Friday, September 5

1. Collected homework.

2. Introduced the notions of *consistent* and *inconsistent*, *independent* and *dependent*, *complete* and *categorical* for axiomatic systems. Refer to the course notes, section 1.2.

3. Illustrated these notions with a axiomatic system for committees, in which

   A. There are exactly four people.

   B. There are exactly two committees.

   C. Each committee consists of exactly two people.

   D. No two committees have the same set of people as members.

   In this case the system is consistent, each axiom is independent of the others, but the system is not categorical.

5 Monday, September 8

1. Returned homework.
2. Worked on the handout on Committees.

6 Wednesday, September 10

1. Began working on the handout on Geometric Worlds, 2.2.5–2.2.10.

   For the sphere, 2.2.4, we observed that every great circle is determined by a plane that passes through the center O of the sphere. So if you choose two points A and B that are not opposite each other, then the points O, A, B determine a unique plane and hence a unique great circle. But if A and B are opposite each other (antipodal), like the north and south poles, then there are infinitely planes and hence infinitely many great circles containing both A and B. So Statement 1 is false for spheres. Also, since any two planes containing O will intersect in a line and hence create intersecting great circles, there exist no two great circles on the sphere that do not share points. So Statement 2 is also false.

   For the Poincaré disk, 2.2.10, we observed by experiment that Statement 1 was probably true. Regarding Statement 2, it appeared that given any line and any point not on that line, there would be infinitely many lines through the given point not meeting the given line.

2. In the process, demonstrated the spherical model and the Poincaré model using the PC software Wingeom.

7 Friday, September 12

1. Collected homework.

2. Discussed Section 1.8 of the course notes, including motivation for considering axiomatic systems.

3. Worked through why it makes sense that the derivative of the formula for the area of a circle is the formula for the perimeter, and similarly why the derivative of the formula for the volume of a sphere is the formula of the surface area. In the process we saw how Pascal's triangle comes into play when deriving the formula for the derivative of the formula for the volume of a sphere.

8 Monday, September 15

1. Worked on some of the homework problems, with some illustrations via GeoGebra.

2. Exam #1 will be on Wednesday, September 24.

9 Wednesday, September 17

1. Briefly discussed some points of the current homework.

2. I am working on preparing a review for Exam #1.

3. Worked on Sections 2.4–2.6 of the Course Notes. Sketched how one would approach Problem 2.4.8 in general, but we only worked through this for a particular pair of points, $(1, 2)$ and $(3, 5)$, showing that up to nonzero multiple there is only one line of the form $ax + by = c$ containing these two points. Mentioned that when you take the determinant of a 3×3 matrix, the determinant is nonzero if and only if the three column vectors are linearly independent, and further, the absolute value of the determinant is the volume of the parallelepiped generated by the three column vectors.

10 Friday, September 19

1. Returned and collected homework.

2. Discussed Section 2.7, Testing Collinearity. Elaborated on the matrix algebra interpretation of this, in terms of dependent and independent vectors.

3. Discussed Section 2.1, including the proof of Theorem 2.1.2.

   Assume lines L and M intersect in more than one point. Suppose two different points A and B are contained in both lines. Then by the Axiom, since two points determine a unique line, lines L and M must be the same line, and hence not different lines.

4. Discussed the beginning of Section 2.8.1, Cramer's Rule.

   Multiply $a_1x+b_1y+c_1=0$ by b_2 and $a_2x+b_2y+c_2=0$ by b_1 and subtract. The result is $(a_1b_2-a_2b_1)x+b_2c_1-b_1c_2=0$ or

   $$(a_1b_2-a_2b_1)x=b_1c_2-b_2c_1=0. \qquad (*)$$

   Now solve for x, which you can do if $a_1b_2-a_2b_1\neq 0$:

   $$x=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}.$$

   These match the desired formula for x. The formula for y can be derived in a similar way.

   Note that if $a_1b_2-a_2b_1=0$, then the left-hand side of $(*)$ is zero, indicating that (a_2,b_2) is a multiple of (a_1,b_1). In this case, if the right-hand side of $(*)$ is also zero, then the two equations describe the same line and the lines are not different; if the right-hand side of $(*)$ is not zero, then the two equations have no solutions in common and the lines do not intersect.

11 Monday, September 22

1. Answered questions about the upcoming exam.

12 Wednesday, September 24

Exam #1.

13 Friday, September 26

1. Returned Exam #1.

2. Discussed how to use matrix multiplication and matrix inverses to express a system of two linear equations in two variables as a matrix equation and solve by multiplying by the inverse of the coefficient matrix. This provides another way of deriving Cramer's rule.

3. Discussed the parametric form of a line. Show how to convert a parametric form into a standard form by using a vector perpendicular to the direction vector of the parametric form.

4. Defined stereographic projection. Mentioned its use in the ancient navigation instrument called the astrolabe.

14 Monday, September 29

1. Discussed how to approach the first three homework problems. In the process, we derived the formula for (x, y, z) in terms of $(p, q, 0)$ in stereographic projection.

2. Began showing that the Euclidean plane, the punctured sphere, and the inside-out plane were isomorphic models.

15 Wednesday, October 1

Discussed the isomorphisms among the various Geometrical Worlds. See the handout "Relationships Among the Geometrical Worlds."

16 Friday, October 3

Worked on Problem 3.3.23 in the Course Notes. We ultimately solved this using calculus. We considered an arbitrary point $C = (x, 0)$ on the river bank and the function $f(x) = AC + BC$. We took the derivative and observed that it was always defined. We noted that $f'(x)$ was negative for very negative x and positive for very positive x. Since $f'(x)$ is continuous, by the Intermediate Value Theorem it is somewhere equal to 0. When we set the derivative to zero and considered the geometrical implications, we saw that the angle that $\overline{AC}$ makes with the river and the angle that $\overline{BC}$ makes with the river have the same cosines and are thus equal (since both will be less than 90 degrees when C lies on the stretch of the river between A and B). Thus at this point C "the angle of incidence equals the angle of reflection."

If we replace A with a light and B with an eye and the river bank with a mirror, knowing that light travels along a path of shortest time (and hence length when the speed of light is constant), we have thus derived a familiar law of physics.

I suggest that you go ahead and actually determine the coordinates of C, even though we did not do this in class.

17 Monday, October 6

Worked on the homework problems.

18 Wednesday, October 8

Continued working on the homework problems. In particular, I presented a detailed approach to writing up the solution to the first homework problem.

19 Friday, October 10

Worked on the handout on Polyhedra. In particular, we constructed most (all?) of the objects (see photo posted on course website) and saw how to get coordinates for a cube and a tetrahedron.

20 Monday, October 13

1. Exam #2 will be on Wednesday, October 22.

2. Demonstrated SketchUp. In particular, we constructed the cube and the tetrahedron based on what we did on Friday.

3. Worked on the Polyhedra handout, focusing on the 3333 polyhedron coordinates.

21 Wednesday, October 15

1. Demonstrated how to create guidelines to points with particular coordinates in SketchUp. Select thte Tape Measure tool, toggle "create guides" (see bottom of screen), click on the origin, move the cursor away from the origin but do not click, type ¡1,2,3¿ (for example), and press Enter. This creates a guideline from the origin to the point $(1, 2, 3)$.

2. Demonstrated how to subdivide segments. Select a segment, right-click the segment, select "Divide", type 5 (for example), and press Enter. This subdivides the segment into five pieces.

3. Continued working on coordinates, particular the problems in Homework #6. Note: Remember how to get the tetrahedron 333 from the cube 444. Think about creating 3333 by placing the vertices directly on the coordinate axes. Coordinates may be nicer when the object is centered at the origin. The last three polyhedra can be obtained from the first three by suitably cutting away material.

22 Friday, October 17

Finished up discussing how to get coordinates and sketches for Homework #6, which I will collect on Monday.

23 Monday, October 20

1. Collected homework.
2. Answered questions pertaining to Wednesday's exam.

24 Wednesday, October 22

Exam #2.

25 Friday, October 24

Discussed angles and trigonometric identities.

26 Monday, October 27

Continued with discussion of handout on the Unit Circle and Basic Trigonometric Identities. Passed out the handout on More Trigonometric Identities.

27 Wednesday, October 29

Worked on a motivation for the formula for the Taylor series; derived the Taylor series for e^x, $\sin x$, and $\cos x$, which converge for all real and complex values of x; observed (given the fact that for these series terms can be rearranged) that for real x, $e^{ix} = \cos x + i \sin x$; and in particular $e^{i\pi} + 1 = 0$.

28 Friday, October 31

Worked on homework. Noted in particular that the Law of Cosines generalizes the Pythagorean Theorem to triangles that are not necessarily right.

29 Monday, November 3

Developed the formula for rotating by an angle ϕ about the origin, and began creating some associated matrices.

30 Wednesday, November 5

1. Exam #3 is set for Monday, November 24.

2. Looked at two sets of eight matrices. The first set, let's call it G_1, consisted of all matrices for rotations about the origin by multiples of 45°. The second set, let's call it G_2, consisted of all matrices for rotations about the origin by multiples of 90° together with the matrices for reflections across the lines $y = 0$, $x = 0$, $y = x$, and $y = -x$. Both sets of matrices are closed under multiplication, have an identity under multiplication, are closed under multiplicative inverses, and satisfy the associative property, and so are groups. But they are not isomorphic, since different numbers of matrices are their own inverses in G_1 as opposed to G_2. We also practiced composing operations by multiplying matrices.

31 Friday, November 7

1. Showed parts of Cosmic Eye (an iPad app), Powers of Ten, and Cosmic Voyage. There are links on the course website.

2. Demonstrated the result of multiplying a 2×2 matrix by a vector, including the visualization of eigenvectors and eigenvalues. The GeoGebra file is on the course website.

3. Demonsrated the use of polar coordinates in modeling circular motion. There is a GeoGebra file on the course website. We experimented with various rates of rotation to see the difference between how many times a planet rotates around its axis in a year, and how many days an inhabitant of the planet will experience.

32 Monday, November 10

1. Introduced the notion of isometry.

2. Worked on ambigram puzzles. Motivation: regard the transparency as a representation of moving the entire plane back onto itself in a distance-preserving way.

3. Conducted demonstrations with members of the class, and then with GeoGebra, to consider the net effect of reflecting in a line, and then reflecting the result in a crossing line or a parallel line. In the case of crossing lines, it appeared that the net effect was a rotation about the point of intersection by an angle twice that of the angle formed by the lines. In the case of parallel lines, it appeared that the net effect was a translation perpendicular to the lines by a distance twice that of the distance between the lines.

4. President Capilouto offered some remarks to the class.

33 Wednesday, November 12

1. Discussed how to carry out addition and multiplication of complex numbers geometrically by regarding them as vectors in the plane.

2. As a consequence, solved problems like finding two square roots of i.

34 Friday, November 14

1. Exam #3 will be a take-home set of problems, to be distributed Monday, November 24.

2. Justified the geometric description of addition and multiplication of complex numbers.

35 Monday, November 17

1. Discussed how to use functions of complex variables for translation: Fix $w \in \mathbf{C}$ and define $f(z) = z + w$.

2. Discussed how to use functions of complex variables for rotation by an angle δ about the origin: Fix $w = \text{c}is\delta$ and define $f(z) = wz$. Further, if $w = rcis\delta$, then $f(z) = wz$ performs a rotation by δ about the origin together with a scaling by r.

3. Saw how we can set up a correspondence between complex numbers $x + iy = rcis\delta$ and matrices of the form
$$\begin{bmatrix} x & -y \\ y & x \end{bmatrix} = \begin{bmatrix} r\cos\delta & -r\sin\delta \\ r\sin\delta & r\cos\delta \end{bmatrix}.$$
This correspondence is an isomorphism that correctly represents complex number addition and multiplication.

4. Introduced the four types of isometries.

5. Worked on problem 2 of Homework #8.

36 Wednesday, November 19

1. Analyzed the eight GeoGebra Isometry Identification Problems on the course website. We did this by drawing some additional elements and using the trace option strategically.

2. Stated without proof (yet) that all isometries are one of these four types.

3. Developed a 3×3 matrix for translation by (p, q).

4. Developed a 3×3 matrix for rotation by angle δ about the origin.

5. Proposed a plan for developing 3×3 matrix for rotation about an arbitrary point.

37 Friday, November 21

1. Derived the general rotation matrix two different ways—via matrices and via complex numbers. Almost finished deriving the general reflection matrix, but I made an error near the end which I need to correct.

38 Monday, November 24

1. Finished the derivation of the reflection formula and matrix.

2. Derived the glide reflection matrix.

3. Showed that if you are given a matrix of the form

$$\begin{bmatrix} c & s & a \\ s & -c & b \\ 0 & 0 & 1 \end{bmatrix}$$

   where $s^2 + c^2 = 1$, then it is a reflection or glide reflection matrix, and you can recover the line of reflection and the amount of translation. Let's refer to such matrices as being of Type II.

   Reminder: every matrix of the form

$$\begin{bmatrix} c & -s & a \\ s & c & b \\ 0 & 0 & 1 \end{bmatrix}$$

   where $s^2+c^2 = 1$ is either a translation matrix or a rotation matrix, and we can recover the translation vector, or the center and angle of rotation, from the matrix. Let's refer to such matrices as being of Type I.

39 Monday, December 1

1. Used the distance formula to prove that translations and rotations are in fact isometries. The proof for reflections and glide reflections is almost the same.

2. Proved that if you know how an isometry acts on three particular noncollinear points A, B, C, then you can determine how that isometry acts on any other point P. Thus an isometry is completely specified by its action on three particular noncollinear points.

40 Wednesday, December 3

1. Used a sequence of problems to highlight some concepts useful for Exam #3.

2. Used the Law of Cosines to prove that isometries (which preserve distances) also preserve angle measures.

41 Friday, December 5

1. Proved the Three Reflections Theorem. Illustrated with GeoGebra. Use perpendicular bisector, then angle bisector, then reflect across a segment knowing the two circle intersection theorem.

2. Proved that composing two Type II matrices results in a Type I; hence the composition of two reflections is either a translation or a rotation.

42 Monday, December 8

1. Proved that every isometry is one of the four we have considered.
2. Defined congruence of figures via isometries.
3. Proved that two line segments are congruent if and only if they have the same lengths.
4. Proved that two angles are congruent if and only if they have the same measures.
5. Sketched the proof that two triangles are congruent if and only if there is a correspondence between the sets of vertices such that all corresponding sides are congruent and all corresponding angles are congruent.