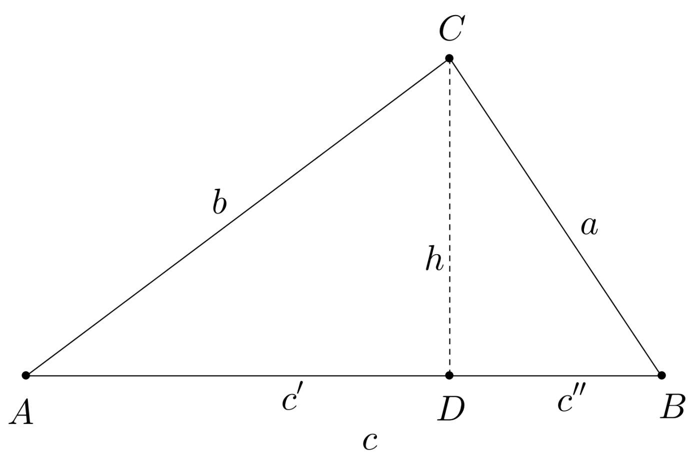
More Trigonometric Identities

1. In this problem we will use the triangle pictured below. In this triangle all angles have measure less than 90°; however, the results hold true for general triangles.

   The lengths of $\overline{BC}$, $\overline{AC}$ and $\overline{AB}$ are a, b and c, respectively. Segment $\overline{AD}$ has length c' and $\overline{DB}$ length c''. Segment $\overline{CD}$ is the altitude of the triangle from C, and has length h.



   The usual formula for the area of a triangle is $\frac{1}{2}$(base)(height), as you probably already know.

   (a) Prove that $\text{area}\,(\triangle ABC) = \frac{1}{2}bc\sin A$.

   (b) What is a formula for $\text{area}\,(\triangle ABC)$ using $\sin B$? Using $\sin C$? (Note: you will have to use the altitude from A or B).

   (c) What is the relationship of these formulas to the SAS triangle congruence criterion?

2. Using the same triangle, the *Law of Sines* is:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

   (a) We showed that the area of this triangle was given by three different formulas. What are they?

(b) From these three formulas, prove the Law of Sines.

3. The *Law of Cosines* is:

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

Using the above triangle:

(a) Show that $c' = b \cos A$.

(b) Observe the obvious fact that $c'' = c - c'$.

(c) Verify that $h^2 = b^2 - (c')^2$.

(d) Apply the Pythagorean Theorem to triangle $\triangle CDB$, then use the facts above to make the appropriate substitutions to prove the Law of Cosines.

(e) What happens when you apply the Law of Cosines in the case that $\angle A$ is a right angle?

4. Suppose for a triangle you are given the lengths of the three sides. How can you determine the measures of the three angles?

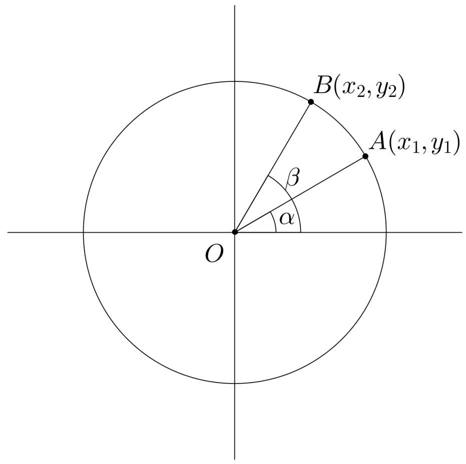
5. Suppose for a triangle you are given the lengths of two sides and the measure of the included angle. How can you determine the length of the other side, and the measures of the other two angles?

6. Suppose for a triangle you are given the measures of two angles and the length of the included side. How can you determine the measure of the other angle, and the lengths of the other two sides?

7. Assume that you have triangle $\triangle ABC$ such that the coordinates of the three (distinct) points A, B, and C are $(0, 0)$, (x_1, y_1), and (x_2, y_2), respectively. Use the Law of Cosines and the distance formula to prove that

$$\cos A = \frac{x_1 x_2 + y_1 y_2}{\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}}.$$

8. Assume that A and B are two points on the unit circle centered at the origin, with respective coordinates (x_1, y_1) and (x_2, y_2). Draw the line segments $\overline{OA}$ and $\overline{OB}$. Let α be the angle that $\overline{OA}$ makes with the positive x-axis, and β be the angle that $\overline{OB}$ makes with the positive x-axis.



(a) From Problem 7 we know that

$$\cos(\angle AOB) = \frac{x_1x_2 + y_1y_2}{\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}}$$

From this, prove that

$$\cos(\beta - \alpha) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

(b) Replace α with $-\alpha$ in the previous equation to prove

$$\cos(\beta + \alpha) = \cos\beta\cos\alpha - \sin\beta\sin\alpha.$$

(c) Replace β with $\pi/2 - \gamma$ and α with $-\delta$ in the previous equation to prove

$$\sin(\gamma + \delta) = \sin\gamma\cos\delta + \cos\gamma\sin\delta.$$

(d) Replace δ with $-\delta$ in the previous equation to prove

$$\sin(\gamma - \delta) = \sin\gamma\cos\delta - \cos\gamma\sin\delta.$$

The above four formulas are the trigonometric *angle sum* and *angle difference formulas.*

9. Prove the *double angle* formulas:

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha.$$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha.$$

10. Prove the *half angle* formulas for angle $0 \leq \beta \leq \frac{\pi}{2}$.

$$\sin(\beta/2) = \sqrt{\frac{1-\cos\beta}{2}}.$$

$$\cos(\beta/2) = \sqrt{\frac{1+\cos\beta}{2}}.$$