MA 341 — Homework #3

Solutions

1. Find an example of two lines, ℓ_1 and ℓ_2, satisfying all of the following properties:

   (a) ℓ_1 is described by an equation of the form $a_1x + b_1y = c_1$ and ℓ_2 is described by an equation of the form $a_2x + b_2y = c_2$, where $a_1, a_2, b_1, b_2, c_1, c_2$ are all integers.

   (b) The lines ℓ_1 and ℓ_2 do not have the same slope.

   (c) The two lines intersect in a point $P(x, y)$ such that x and y are not both integers.

   Solution: One simple example is

   $$\ell_1 : x = 0$$

   $$\ell_2 : 2x = 1$$

   The latter line contains no integer points, so is the empty set and thus does not share any point with ℓ_1.

   If we wish to avoid lines that are empty of integer points, here is another example:

   $$\ell_1 : x = 1$$

   $$\ell_2 : 2x + 3y = 0$$

   Here the ordinary intersection point is $(1, -2/3)$, which does not have all coordinates being integer.

2. Refer to the handout on Geometrical Worlds. Use the solution to problem 1, above, of this homework, to explain why for the Integer Plane (2.2.3 on the handout) property 2 at the beginning of the handout is false.

   Solution: Consider the line ℓ with equation $x = 1$ and the point $P = (-3, 2)$. One line that contains P and does not intersect ℓ in an integer point has equation $x = -3$. But another, from the previous problem, is the line with equation $2x + 3y = 0$, since this second line also contains P and does not intersect ℓ in an integer point, and thus shares no POINT with ℓ.

   The remaining questions refer to the handout on Geometrical Worlds. In each case, determine whether properties 1 and 2 at the beginning of the handout are true or false, providing justification.

3. The Sphere, 2.2.4.

   Solution: Note that every LINE is the intersection of a plane through the origin O (the center of the sphere) with the sphere. If you choose any two different POINTS P, Q on the sphere, then there is always a plane through P, Q, O so there is always at least one LINE containing P, Q. If P, Q are not opposite each other on the sphere, then there is only one such plane, hence only one such LINE. But if P, Q are opposite each other, then P, Q, O all lie on a common line ℓ, and there are infinitely many planes containing ℓ, and hence infinitely many LINES containing P, Q. Thus the first property is false.

   Consider any two differnet LINES. Each is determined by a plane through O, and these two planes intersect in a line through O, which in turn intersects the sphere in a pair of POINTS on the sphere. So every pair of LINES intersects, which means the second property must be false.

4. The Inside-Out Plane 2.2.7. (GeoGebra is very helpful for explorations.)

   Solution: Choose any two different POINTS P, Q. First assume that neither one is the POINT Z. If P, Q, O are not collinear, then they determine a unique circle passing through them, which results in a unique LINE containing P, Q. If P, Q, O are collinear, then there is a unique line passing through them, which results in a unique LINE containing P, Q. Now assume that $Q = Z$. Then the unique line through P, O results in the unique LINE containing P, Z. So the first property is true.

   Let ℓ be a LINE and P be a POINT not on ℓ. Case 1: ℓ corresponds to a straight line through O and $P \neq Z$. Then there is a unique circle tangent to ℓ containing P, which corresponds to the desired LINE. Case 2: ℓ corresponds to a circle through O and $P \neq Z$. Then there is a unique circle through O tangent to the given circle containing P, which corresponds to the desired LINE. Case 3: ℓ corresponds to a straight line through O and $P \neq Z$. Then there is a unique line through O tangent to ℓ, which corresponds to the desired LINE. Case 4: ℓ corresponds to a straight line through O and $P = Z$. This case cannot occur since Z is then on ℓ. So the second property is true.

5. The Klein Disk, 2.2.9. (GeoGebra is very helpful for explorations.)

   Solution: Given any two different POINTS (points in the interior of the circle) there is always a unique chord of the circle (a LINE) containing both of them, so the first property is true.

   Given any LINE (chord) and POINT (point in the interior of the circle) not on that chord, then it is easy to see that there is always an infinite number of chords through the given point that do not intersect the given chord. So the second property is false.

6. The Projective Plane, 2.2.12.

   Solution: Consider any two different POINTS P, Q. These correspond to two ordinary different lines through O, which in turn are contained in a unique ordinary plane containing O. This plane is thus a unique LINE containing P, Q. So the first property is true.

   Note that any two different ordinary planes through O intersect in an ordinary line through O. So any two different LINES intersect in a common POINT, forcing the second property to be false.