MA 341 Homework #7

Due Wednesday, November 5 in Class

1. Handout on More Trigonometric Identities

   (a) Problem 1.

      i. Prove that area $(\triangle ABC) = \frac{1}{2}bc\sin A$.
      Solution. $\sin A = \frac{h}{b}$, so $h = b\sin A$. area $\triangle ABC = \frac{1}{2}hc = \frac{1}{2}bc\sin A$.

      ii. What is a formula for area $(\triangle ABC)$ using $\sin B$? Using $\sin C$? (Note: you will have to use the altitude from A or B).
      Solution.
      $$\text{area}\,(\triangle ABC) = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C$$

      iii. What is the relationship of these formulas to the SAS triangle congruence criterion?
      Solution. Since the triangle is determined up to congruence by SAS, then so is its area, and the formulas show exactly how—the area is half of the product of the two sides and the sine of the included angle.

   (b) Problem 2.

      i. We showed that the area of this triangle was given by three different formulas. What are they?
      Solution.
      $$\text{area}\,(\triangle ABC) = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C.$$

      ii. From these three formulas, prove the Law of Sines.
      Solution. Divide the above equations through by $\frac{1}{2}abc$.

   (c) Problem 3.

      i. Show that $c' = b\cos A$.
      Solution. Observe that $\cos A = c'/b$ from triangle $\triangle ACD$.

      ii. Observe the obvious fact that $c'' = c - c'$.
      Solution. This is clear from the diagram.

      iii. Verify that $h^2 = b^2 - (c')^2$.
      Solution. This follows from the Pythagorean Theorem applied to triangle $\triangle ACD$.

iv. Apply the Pythagorean Theorem to triangle $\triangle CDB$, then use the facts above to make the appropriate substitutions to prove the Law of Cosines.

Solution.

$$\begin{aligned} a^2 &= h^2 + (c'')^2 \\ &= b^2 - (c')^2 + (c - c')^2 \\ &= b^2 - (c')^2 + c^2 - 2cc' + (c')^2 \\ &= b^2 + c^2 - 2cc' \\ &= b^2 + c^2 - 2bc\cos A. \end{aligned}$$

v. What happens when you apply the Law of Cosines in the case that $\angle A$ is a right angle?

Solution. In this case $\cos A = 0$ and you get $a^2 = b^2 + c^2$, which is the Pythagorean Theorem.

(d) Problem 7.

Solution. Beginning with $a^2 = b^2 + c^2 - 2bc\cos A$, solve for $\cos A$.

$$\begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{(x_1^2 + y_1^2) + (x_2^2 + y_2^2) - [(x_2 - x_1)^2 + (y_2 - y_1)^2]}{2\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}} \\ &= \frac{x_1^2 + y_1^2 + x_2^2 + y_2^2 - (x_2^2 - 2x_1x_2 + x_1^2 + y_2^2 - 2y_1y_2 + y_1^2)}{2\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}} \\ &= \frac{x_1x_2 + y_1y_2}{\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}}. \end{aligned}$$

(e) Problem 8.

i. From Problem **??** we know that

$$\cos(\angle AOB) = \frac{x_1x_2 + y_1y_2}{\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}}$$

From this, prove that

$$\cos(\beta - \alpha) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Solution. Use the fact that $(x_1, y_1) = (\cos\alpha, \sin\alpha)$ and $x_1^2 + y_1^2 = 1$, and the analogous facts about β.

ii. Replace α with $-\alpha$ in the previous equation to prove

$$\cos(\beta + \alpha) = \cos\beta\cos\alpha - \sin\beta\sin\alpha.$$

Solution. $\cos(\beta + \alpha) = \cos(\beta - (-\alpha)) = \cos(-\alpha)\cos\beta + \sin(-\alpha)\sin\beta = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

iii. Replace β with $\pi/2 - \gamma$ and α with $-\delta$ in the previous equation to prove

$$\sin(\gamma + \delta) = \sin\gamma\cos\delta + \cos\gamma\sin\delta.$$

Solution.

$$\begin{aligned} \cos(\beta + \alpha) &= \cos\beta\cos\alpha - \sin\beta\sin\alpha \\ \cos(\pi/2 - \gamma - \delta) &= \cos(\pi/2 - \gamma)\cos(-\delta) - \sin(\pi/2 - \gamma)\sin(-\delta) \\ \sin(\gamma + \delta) &= \sin\gamma\cos\delta + \cos\gamma\sin\delta \end{aligned}$$

iv. Replace δ with $-\delta$ in the previous equation to prove

$$\sin(\gamma - \delta) = \sin\gamma\cos\delta - \cos\gamma\sin\delta.$$

Solution. $\sin(\gamma - \delta) = \sin(\gamma + (-\delta)) = \sin\gamma\cos(-\delta) + \cos\gamma\sin(-\delta) = \sin\gamma\cos\delta - \cos\gamma\sin\delta$.

(f) Problem 9.

Prove the *double angle* formulas:

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha.$$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha.$$

Solution. Set $\gamma = \alpha$ and $\delta = \alpha$ in the formula for $\sin(\gamma + \delta)$, and set $\beta = \alpha$ in the formula for $\cos(\alpha + \beta)$. Then use $\sin^2\alpha + \cos^2\alpha = 1$.

(g) Problem 10.

Prove the *half angle* formulas for angle $0 \leq \beta \leq \frac{\pi}{2}$.

$$\sin(\beta/2) = \sqrt{\frac{1 - \cos\beta}{2}}.$$

$$\cos(\beta/2) = \sqrt{\frac{1 + \cos\beta}{2}}.$$

Set $\alpha = \frac{\beta}{2}$ in the second two formulas for $\cos(2\alpha)$ and then solve for $\cos(\frac{\beta}{2})$ and $\sin(\frac{\beta}{2})$.

2. Given the line ℓ described by $ax + by + c = 0$, and the point $P(x_0, y_0)$, derive a formula, with justification, for the point Q, such that ℓ is the perpendicular bisector of the segment $\overline{PQ}$. Suggestion: Consider the line m through P that is perpendicular to ℓ, and express m in parametric form.

   Solution. Consider the parametric equation of a line m passing through P and in the direction (a, b), which is perpendicular to ℓ. The equation is

   $$(x_0, y_0) + t(a, b)$$

   or

   $$(x_0 + ta, y_0 + tb).$$

   To find M, the point of intersection of ℓ and m, substitute this expression into the equation of ℓ and solve for t.

$$a(x_0 + t^* a) + b(y_0 + t^* b) + c = 0$$
$$ax_0 + t^* a^2 + by_0 + t^* b^2 + c = 0$$

$$t^* = \frac{-ax_0 - by_0 - c}{a^2 + b^2}.$$

Note that $a^2 + b^2$ is not zero because in the equation of a line ℓ you cannot have both $a = b = 0$.

Since $t = t^*$ gives us the point M, we need to use $t = 2t^*$ to get the point Q on the other side of ℓ.

$$Q = (x_0 + 2t^* a, y_0 + 2t^* b) = \left(x_0 + \frac{-2a^2 x_0 - 2aby_0 - 2ac}{a^2 + b^2}, y_0 + \frac{-2abx_0 - 2b^2 y_0 - 2bc}{a^2 + b^2} \right).$$

Note: we can make this look a little nicer if we assume at the outset that we have divided the equation of the line through by $\sqrt{a^2 + b^2}$ so that we can assume $a^2 + b^2 = 1$. Then we have the formula

$$Q = ((1 - 2a^2)x_0 - 2aby_0 - 2ac, -2abx_0 + (1 - 2b^2)y_0 - 2bc).$$