Continuing to Count — Binomial Coefficients

Notes:

- We will define $\binom{n}{k}$ to be 0 if it is not the case that $n \geq 0$ and $0 \leq k \leq n$.
- We define the falling factorial power $x^{\underline{k}}$ to be
$$x^{\underline{k}} = x(x-1)(x-2)\cdots(x-k+1) = \prod_{i=1}^{k-1}(x-i).$$
- We define the rising factorial power $x^{\overline{k}}$ to be
$$x^{\overline{k}} = x(x+1)(x+2)\cdots(x+k-1) = \prod_{i=1}^{k-1}(x+i).$$
- We take $n^{\underline{0}}$, $n^{\overline{0}}$, and 0! each to equal 1.

Questions:

1. How many subsets does an n element set have?
2. Without using formulas, explain why
$$\binom{n}{k} = \binom{n}{n-k}.$$
3. Without using formulas, explain why
$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$
Hint: Consider one particular element of an n element set, and count how subsets of size k will contain this element, and how many will not.
4. Now prove (3.) using the formula for $\binom{n}{k}$.
5. Returning to that grid problem from before, prove that the number of walks from $(0,0)$ to (m,n) taking only unit steps upward and to the right equals $\binom{m+n}{m}$. Can you find at least two different proofs?
6. Prove that if n is a nonnegative integer, then
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$
Approach #1: Think about $(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)$—how do you get terms like x^3y^5? Approach #2: Try a proof by induction.
7. What does $\sum_{k=0}^{n} \binom{n}{k}$ equal? Find two different approaches, one using (6.).
8. What does $\sum_{k=0}^{n} (-1)^k \binom{n}{k}$ equal? Suggestion: Use (6.).
9. Read "Summing on the Upper Index" and "Vandermonde's Convolution" on pages 141–2.