Geometry Midterm

INSTRUCTIONS: This is a take-home exam. You may use my course notes, *Journey Through Genius*, and your course notes, but no other source of information or assistance, human or non-human, though you may certainly contact me if you have questions. You should strictly avoid communicating anything at all in any form to the other course members, not even comments like "I have been having trouble with the first problem but otherwise I am nearly finished." Your answers are due Sunday, October 21, at 11:00 pm, either uploaded to the Moodle site as a single file or emailed to lee@ms.uky.edu.

1. **Axiomatics.** Consider the following axiomatic system for committees of people.

   Axiom 1 Given any two people there is exactly one committee containing both of them (though the committee may contain other people as well).

   Axiom 2 Given any person A and any committee X not containing A, there is exactly one committee Y containing A but disjoint from X (not having any members in common with X).

   Axiom 3 There exist four people, no three of whom serve on the same committee.

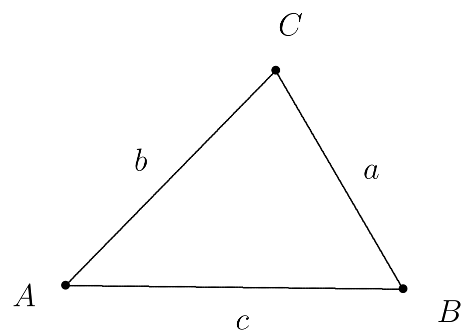
   Axiom 4 The total number of people is finite.

   (a) Prove that this system is consistent by finding a model.

   (b) Prove that each of the Axioms is independent of the others by finding appropriate models.

   (c) Prove that this system is not complete by finding another model that is not isomorphic to the one you found in (a).

2. **Angles, Trigonometry, and Area.** Consider triangle ΔABC:



You already found a formula for the area of a triangle in terms of the lengths of two sides and the measure of the included angle, which is therefore nicely associated with SAS congruence. Now prove the following formula for the area of a triangle in terms of the measures of two angles and the length of the included side:

$$\text{area}(\Delta ABC) = \frac{\frac{1}{2}c^2}{\cot A + \cot B}.$$

Suggestion: Begin with the "SAS area formula" and use the Law of Sines.

3. **Determinants and Area.** You already proved that the area of the triangle with vertices $(0,0)$, (x_1, y_1), and (x_2, y_2) is $\frac{1}{2}|x_1y_2 - x_2y_1|$. It can be proved that if these three points occur in counterclockwise order then $x_1y_2 - x_2y_1 > 0$, and if they occur in clockwise order then $x_1y_2 - x_2y_1 < 0$. You may use this fact in the following.

   (a) Prove that the area of the triangle with vertices $A = (x_1, y_1)$, $B = (x_2, y_2)$, and $C = (x_3, y_3)$ occurring in counterclockwise order is $\frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]$. Suggestion: Join each vertex to the origin (wherever it may be), and consider four cases:

      i. When the origin is inside the triangle.

      ii. When the origin is outside the triangle.

      iii. When the origin is one of the vertices of the triangle.

      iv. When the origin is in the interior of one of the edges of the triangle.

   (b) Conclude that the area of the triangle is:

$$\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}.$$

   (c) Armed with (a) prove that the area of a quadrilateral with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4) occurring in counterclockwise order is $\frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_1 - x_1y_4)]$. Suggestion: Divide the quadrilateral into two triangles.

4. **Inaccessible Distances.** Here is a simplified version of a very practical problem. After an earth tremor is detected at multiple sensors, determine the location and time of the event. For this problem we will assume that we are working in the plane, and that an event takes place at some location (x, y) at some time t. Assume that the wave from the event propagates at a rate of 1 unit of distance per minute. You have three detectors that detect the event:

   - Detector 1 at location $(0, 3)$ detects the event at 5 minutes past noon.
   - Detector 2 at location $(1, 5)$ detects the event at 6 minutes past noon.
   - Detector 3 at location $(5, 6)$ detects the event at 9 minutes past noon.

   From this information, determine the location and the time of the event. Suggestion: You can approach this problem algebraically as well as graphically with dynamic geometry software.