Isometries

1 Identifying Isometries

1. Modeling isometries as dynamic maps.
2. GeoGebra files: isoguess1.ggb, isoguess2.ggb, isoguess3.ggb, isoguess4.ggb.
3. Guessing isometries.
4. What can you construct or trace to find the defining elements?

2 Three Reflection Theorem

1. A point is uniquely determined by its distances from three noncollinear points — model with GeoGebra.
2. Therefore an isometry is uniquely determined by its action on three noncollinear points.
3. Three Reflection Theorem: If T and T' are congruent triangles, then T can be mapped onto T' using at most three reflections. GeoGebra file: threereflections.ggb.
4. Therefore every isometry is the composition of zero, one, two, or three reflections.
5. Similarly, there is a Four Reflections Theorem for isometries in space.

3 Compositions of Isometries

1. Zero reflections is the identity isometry.

2. One reflection is a reflection.

3. The composition of two reflections in parallel lines is a translation perpendicular to the lines by a distance equal to twice the distance between the two lines. GeoGebra file: tworeflections.ggb.

4. The composition of two reflections in intersecting lines is a rotation around the point of intersection by an angle twice that of the angle of intersection. GeoGebra file: tworeflections.ggb.

5. The composition of three reflections is either a reflection or a glide reflection.

6. The composition of two rotations is either a rotation or a translation. GeoGebra file: tworotations. Express each rotation as a double reflection to determine the center and angle of rotation.

4 Kaleidoscopes

1. Perform repeated reflection, alternating between two given lines.
2. What angles of intersection between the two lines yield only finitely many images?
3. There are three-dimensional variants of this involving reflections in three intersecting planes.

5 Formulas for Isometries

1. Translation by the amount (p, q).
$$\begin{bmatrix} 1 & 0 & p \\ 0 & 1 & q \\ 0 & 0 & 1 \end{bmatrix}$$

2. Rotation by δ about the point (p, q).
$$\begin{bmatrix} c & -s & -pc + qs + p \\ s & c & -ps - qc + q \\ 0 & 0 & 1 \end{bmatrix}$$
where $c = \cos\delta$ and $s = \sin\delta$.

3. Both of these matrices are of the form
$$\begin{bmatrix} c & -s & u \\ s & c & v \\ 0 & 0 & 1 \end{bmatrix}$$
where $c^2 + s^2 = 1$. Further, given any matrix of the above form, one can solve for δ, p and q, so any such matrix is an isometry.

4. Both of these matrices have determinant equal to one. These are the *direct isometries.*

5. Reflection across the line with equation $px+qy+r=0$, assuming that $p^2+q^2=1$.

$$\begin{bmatrix} -p^2+q^2 & -2pq & -2pr \\ -2pq & p^2-q^2 & -2qr \\ 0 & 0 & 1 \end{bmatrix}$$

6. Glide reflection by reflecting across the line with equation $px+qy+r=0$ followed by translation by the amount $(tq,-tp)$, assuming that $p^2+q^2=1$.

$$\begin{bmatrix} -p^2+q^2 & -2pq & -2pr+tq \\ -2pq & p^2-q^2 & -2qr-tp \\ 0 & 0 & 1 \end{bmatrix}$$

7. Both of these matrices are of the form

$$\begin{bmatrix} c & s & u \\ s & -c & v \\ 0 & 0 & 1 \end{bmatrix}$$

   where $c^2+s^2=1$. Further, given any matrix of the above form, one can solve for t, p and q, so any such matrix is an isometry.

8. Both of these matrices have determinant equal to negative one. These are the *indirect isometries*.

9. Every isometry is in fact one of the above forms. See Exercise 8.12 in *Geometry for Middle School Teachers*.

6 Examples with Maxima

1. Maxima is a free computer algebra system.
2. Creating isometry matrices.
3. Inverses.
4. Compositions.
5. Solving for isometries.

7 Point-Line Incidence via Isometries

1. Let P be a point and R_P be the isometry that is a rotation about P by 180 degrees.

2. Let L be a line and R_L be the isometry that is a reflection in L.

3. Then P is incident to L if and only if

$$R_P \circ R_L = R_L \circ R_P,$$

   equivalently, if and only if

$$R_P R_L R_P R_L = I.$$

4. Verification geometrically.

5. Verification algebraically.

6. This idea can be expanded to give an axiomatic system for Euclidean geometry (or other geometries) in terms of groups.

8 Interlude—Mathematical and Physical Reflections

1. Why does the image in a mirror appear to be in the location defined by a mathematical reflection?
2. If the human eye detects a set of light rays that, when traced backward, appear to emanate from a common point, then the brain makes the interpretation that that common point is the origin of the light rays.
3. GeoGebra file: physicalreflection.ggb.
4. Because the atmosphere can bend the path of light rays, The same phenomenon accounts for the apparent presence of the sun just above the horizon after it has actually set below the horizon.

9 Applying Isometries to Figures Defined by Equations

1. We are accustomed to applying an isometry to a drawn figure in the plane, or to individual points via formulas, but what about to figures described by equations?

2. Example: Apply the translation by the amount (p, q) to the circle whose equation is $x^2 + y^2 = 100$. The isometry is given by $\overline{x} = x + p$, $\overline{y} = y + q$. Thus $x = \overline{x} - p$, $y = \overline{y} - q$. Substituting yields $(\overline{x} - p)^2 + (\overline{y} - q)^2 = 100$ for the equation of the translated circle.

3. In general, if we translate the graph of a function described by $y = f(x)$ by the amount (p, q) then we get the new graph described by $\overline{y} - q = f(\overline{x} - p)$. Thinking of things this way helps eliminate the necessity of memorizing rules for how to shift up or down or right or left.

4. Completing the Square and the Quadratic Formula are merely using translations to simplify equations of parabolas!

5. Example: Consider the parabola described by

$$y = 2x^2 - 12x + 13.$$

Let's find a translation of the form $\overline{x} = x+p$ so that the equation of the translated parabola has no "x" term. Using $x = \overline{x} - p$ we get

$$y = 2(\overline{x} - p)^2 - 12(\overline{x} - p) + 13$$

which simplifies to

$$y = 2\overline{x}^2 + (-4p - 12)\overline{x} + (2p^2 + 12p + 13).$$

We want to choose p so that $-4p-12$ is zero, so choose $p = -3$. In other words, translating 3 units to the left results in a parabola centered on the y-axis. Then the equation further simplifies to

$$y = 2\overline{x}^2 - 5.$$

Finding the two roots (x-intercepts) is now easy:

$$\overline{x} = \pm\sqrt{\frac{5}{2}}.$$

Translating the parabola and its roots back to the original position gives

$$x = \overline{x} - p = 3 \pm \sqrt{\frac{5}{2}}.$$

6. Let's do this in general. Consider the parabola described by $y = ax^2+bx+c$. Using the translation $\overline{x} = x+p$ and substituting $x = \overline{x} - p$ this becomes

$$y = a(\overline{x} - p)^2 + b(\overline{x} - p) + c$$

which simplifies to

$$y = a\overline{x}^2 + (b - 2ap)\overline{x} + (ap^2 - bp + c).$$

We want $b - 2ap$ to be zero, so choosing $p = \frac{b}{2a}$ results in a parabola centered on the y-axis. With this choice of p the equation then simplifies to

$$y = a\overline{x}^2 + \frac{-b^2 + 4ac}{4a}.$$

The roots are simple to calculate:

$$\overline{x} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}.$$

Translating the parabola and its roots back to its original position gives the roots

$$x = \overline{x} - p = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a},$$

which is the quadratic formula.

7. Of course, we don't have to use only translations to simplify things. In general, if we have a figure described by an equation of the form $f(x, y) = 0$ we might consider applying an isometry with some rule mapping (x, y) to $(\overline{x}, \overline{y})$, and then substituting into the equation, choosing our isometry wisely so that afterwards things look simpler.

8. Hold on tight—I am now going to do an example of this that I was taught in my public high school (Baltimore County, Maryland) eleventh grade course in Trigonometry/Analytic Geometry by Mr. Laferty.

9. Problem: Identify the curve given by the equation

$$73x^2 + 52y^2 - 72xy - 290x + 280y + 325 = 0.$$

We will try a rotation about the origin so that after the rotation there will be no "xy" term. A rotation about the origin by an angle δ with $\sin\delta = s$ and $\cos\delta = c$ is given by:

$$\begin{aligned} \overline{x} &= cx - sy \\ \overline{y} &= sx + cy \end{aligned}$$

Solving for x and y is equivalent to rotating by $-\delta$, hence

$$\begin{aligned} x &= c\overline{x} + s\overline{y} \\ y &= -s\overline{x} + c\overline{y} \end{aligned}$$

If we substitute for x and y in the original equation, the coefficient of $\overline{xy}$ can be calculated to be

$$42cs - 72(c^2 - s^2).$$

We want this to equal zero, so we want

$$\frac{72}{42} = \frac{12}{7} = \frac{cs}{c^2 - s^2}.$$

But this latter expression, by the double angle formulas, equals

$$\frac{\frac{1}{2}\sin 2\delta}{\cos 2\delta} = \frac{1}{2}\tan 2\delta.$$

So $\tan 2\delta = 24/7$. Drawing a right triangle with legs 24 and 7, and using the Pythagorean Theorem to determine that the hypotenuse is 25, we calculate that $\cos 2\delta = 7/25$. Hence, using the half-angle formula,

$$s = \sin\delta = \sqrt{\frac{1-\cos 2\delta}{2}} = \frac{3}{5},$$

so $c = \cos\delta = \sqrt{1-\sin^2\delta} = 4/5$.

Using these values for c and s in the desired rotation, and substituting into the original equation, this simplifies to

$$100\overline{x}^2 - 400\overline{x} + 25\overline{y}^2 + 50\overline{y} + 325 = 0,$$

or

$$4\overline{x}^2 - 16\overline{x} + \overline{y}^2 + 2\overline{y} + 13 = 0.$$

Completing squares:

$$\begin{gathered} 4\overline{x}^2 - 16\overline{x} + 16 + \overline{y}^2 + 2\overline{y} + 1 + 13 = 16 + 1 \\ 4(\overline{x}-2)^2 + (\overline{y}+1)^2 = 4 \end{gathered}$$

So finally we have

$$\frac{(\overline{x}-2)^2}{1} + \frac{(\overline{y}+1)^2}{4} = 1,$$

which is the equation of an ellipse centered at the point $(2,-1)$. We could further translate it by the amount $(-2,1)$ to obtain the ellipse

$$\frac{\overline{x}^2}{1} + \frac{\overline{y}^2}{4} = 1$$

centered at the origin. Now it is possible to sketch the curve in its new position. By using the inverses of the translation and the rotation (in that order) we can sketch the curve in its original position.

10. One final comment—we can use the above technique to verify that certain figures have certain symmetries. For example, if we consider the figure described by the equation $xy = 100$ and apply the isometry $\overline{x} = -x, \overline{y} = -y$ we see that the equation is unchanged: $\overline{xy} = 100$. This this figure is symmetric under 180 degree rotation about the origin. Also, if we apply the isometry $\overline{x} = y, \overline{y} = x$ we again see that the equation is unchanged. Thus this figure is symmetric under the action of reflection across the line $y = x$.

11. Of course, all of these ideas can be extended into three and higher dimensions!