Notes on Geometry

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Think Deeply of Simple Things
Motto of Ross Program at Ohio State University

Contents

1 Some Ways to Increase Understanding

When thinking about a problem or idea:

1. Can you build a physical model?
2. Can you build a virtual model?
3. What are some generalizations?
4. What are analogs in higher dimensions?
5. What are analogs on the sphere?
6. What are some physical ("real-life") applications?
7. What other areas of mathematics are needed to increase understanding?
8. What other areas of mathematics in turn benefit from this understanding?
9. What is its history?
10. How is it introduced and developed in the K–16 curriculum?
11. What questions can you ask a student to probe or enhance his/her understanding?
12. What are some good related outside readings?

2 Review

Exercise 2.0.1 Using some standard texts in geometry, re-familiarize yourself with some results in geometry.

Exercise 2.0.2 Illustrate some results with Geometer's Sketchpad or physical models. Some ideas:

1. Sum of the measures of the angles of a triangle.
2. Ratio of the circumference of a circle to its diameter. Ratio of the area of a circle to the square of its radius.
3. Measures of central and inscribed angles of circles.
4. Crossing secants of circles.
5. Pythagorean theorem.
6. Perpendicular bisectors, angle bisectors, medians and altitudes of triangles, and some associated circles.

Exercise 2.0.3 Explain the difference between an axiom, a postulate, a definition, a lemma, a theorem, and a corollary.

Exercise 2.0.4 According to the NCTM standards, identify where these results would be introduced and then further developed in the K–12 curriculum. What about the college curriculum?

3 Generating Questions to Think Deeply

3.1 Some Triangle Concurrencies

Exercise 3.1.1 Why are the perpendicular bisectors of a triangle concurrent?

Exercise 3.1.2 Why is a perpendicular bisector of a segment equal to the set of points equidistant from the endpoints of the segment?

Exercise 3.1.3 Why is the point of concurrency the center of the circumscribed circle?

Exercise 3.1.4 What is the analog in three-dimensions?

Exercise 3.1.5 Make some physical and virtual models of the two and three-dimensional versions.

Exercise 3.1.6 Why are the angle bisectors of a triangle concurrent?

Exercise 3.1.7 Why is an angle bisector of an angle equal to the set of points equidistant from the two rays of the angle?

Exercise 3.1.8 Why is the point of concurrency the center of the inscribed circle?

Exercise 3.1.9 What is the analog in three-dimensions?

Exercise 3.1.10 What is the analog to the angle bisector?

Exercise 3.1.11 Make some physical and virtual models of the two and three-dimensional versions.

Exercise 3.1.12 According to the NCTM standards, identify where these results would be introduced and then further developed in the K–12 curriculum. What about the college curriculum?

3.2 Triangle Congruence

Exercise 3.2.1 What are the standard triangle congruence theorems?

Exercise 3.2.2 In each case, why isn't the triangle determined with fewer pieces of information?

Exercise 3.2.3 Do other combinations work? If they don't work all of the time, when do they work?

Exercise 3.2.4 Why are the congruence theorems true?

Exercise 3.2.5 Given sufficient pieces of information to determine a triangle, how can the remaining pieces be determined? Via algebra? Via Sketchpad? Since congruent triangles have equal area, how can the area be determined from the given pieces of information?

Exercise 3.2.6 When are the given data consistent or inconsistent to determine a triangle?

Exercise 3.2.7 The coordinates of the three vertices determine the side lengths, and hence the triangle, so is there a formula for the area of the triangle in terms of the coordinates?

Exercise 3.2.8 What are some physical applications?

Exercise 3.2.9 Which results hold for triangles on a sphere? What are triangles on a sphere? How do we make sense of segments and segment congruence? How do we make sense of angles and angle congruence?

Exercise 3.2.10 What are analogous results in three dimensions? What is the analog of a triangle? What is an angle? What is angle measure? Are two angles congruent if and only if they have the same measure? Make models.

Exercise 3.2.11 What are analogous results for other two dimensional shapes? For quadrilaterals? Illustrate with Sketchpad.

Exercise 3.2.12 According to the NCTM standards, identify where these results would be introduced and then further developed in the K–12 curriculum. What about the college curriculum?

3.3 The Pythagorean Theorem

Exercise 3.3.1 Illustrate with Sketchpad. Illustrate (some special cases) with Polydron or other physical models.

Exercise 3.3.2 How is the theorem proved? What about the converse?

Exercise 3.3.3 What happens if instead of squares erected on the three sides of a right triangle, other shapes are used instead?

Exercise 3.3.4 What are some physical applications?

Exercise 3.3.5 What if the triangle is not right?

Exercise 3.3.6 Is there a three-dimensional analog? What is the analog of a right triangle? Test your conjectures with some specific examples.

Exercise 3.3.7 Is there a spherical analog? What is the analog of a right triangle? Test your conjectures with some specific examples.

Exercise 3.3.8 Is there an analog for quadrilaterals?

Exercise 3.3.9 What are some integer Pythagorean triples? How can they be found?

Exercise 3.3.10 If the two sides of the right triangle are parallel to the coordinate axes in a coordinate system, how can you get the length of the sides and the hypotenuse? How does this relate to the distance formula for the length of a segment? What if the two points are joined by a curve instead of a segment?

Exercise 3.3.11 According to the NCTM standards, identify where these results would be introduced and then further developed in the K–12 curriculum. What about the college curriculum?

3.4 Three-Dimensional Shapes

Exercise 3.4.1 What are three-dimensional analogs of circles, triangles, isosceles triangles, equilateral triangles, quadrilaterals, trapezoids, parallelograms, rectangles, rhombi, squares?

Exercise 3.4.2 What are three-dimensional analogs of regular polygons?

4 Axiomatic Systems

4.1 Features of Axiomatic Systems

One motivation for developing axiomatic systems is to determine precisely which properties of certain objects can be deduced from which other properties. The goal is to choose a certain fundamental set of properties (the *axioms*) from which the other properties of the objects can be deduced (e.g., as *theorems*). Apart from the properties given in the axioms, the objects are regarded as *undefined*.

As a powerful consequence, once you have shown that any particular collection of objects satisfies the axioms *however unintuitive or at variance with your preconceived notions these objects may be*, without any additional effort you may immediately conclude that all the theorems must also be true for these objects.

We want to choose our axioms wisely. We do not want them to lead to contradictions; i.e., we want the axioms to be *consistent*. We also strive for economy and want to avoid redundancy—not assuming any axiom that can be proved from the others; i.e., we want the axiomatic system to be *independent*. Finally, we may wish to insist that we be able to prove or disprove any statement about our objects from the axioms alone. If this is the case, we say that the axiomatic system is *complete*.

We can verify that an axiomatic system is consistent by finding a *model* for the axioms—a choice of objects that satisfy the axioms.

We can verify that a specified axiom is independent of the others by finding two models—one for which all of the axioms hold, and another for which the specified axiom is false but the other axioms are true.

We can verify that an axiomatic system is complete by showing that there is essentially only one model for it (all models are *isomorphic*); i.e., that the system is *categorical*.

For more details and examples, see Kay, Section 2.2.

4.2 Examples

Exercise 4.2.1 Let's look at three examples of axiomatic systems for a collection of committees selected from a set of people. (Of course, we may use the word "points" instead of "people" and "lines" instead of "committees," because other than the properties imposed by the axioms these terms are to be taken as undefined.) In each case, determine whether the axiomatic system is consistent or inconsistent. If it is consistent, determine whether the system is independent or redundant, complete or incomplete.

1. (a) There is a finite number of people.
   (b) Each committee consists of exactly two people.
   (c) Exactly one person is on an odd number of committees.
2. (a) There is a finite number of people.
   (b) Each committee consists of exactly two people.
   (c) No person serves on more than two committees.
   (d) The number of people who serve on exactly one committee is even.
3. (a) Each committee consists of exactly two people.
   (b) There are exactly six committees.
   (c) Each person serves on exactly three committees.

Exercise 4.2.2 Construct a polyhedron and then create list of consistent, independent, and complete properties to specify it, in terms of its faces, edges, vertices, angles, etc.

4.3 Another Example

Consider the following axiomatic system for people and committees.

1. Given any two distinct people, there is exactly one committee containing both of them.
2. Given any two distinct committees, they have exactly one member in common.
3. There exist four people, no three of which are contained in a common committee.
4. The total number of people is finite.

Exercise 4.3.1 Try to come up with some models for this axiomatic system. Which model could you find with the smallest number of people? Is this system independent?

Exercise 4.3.2 Prove that if there exists a committee with $q+1$ people, then every committee contains exactly $q+1$ members, every person is a member of exactly $q+1$ committees, there is a total of q^2+q+1 people, and there is a total of q^2+q+1 committees.

Exercise 4.3.3 Find a model containing exactly 13 people.

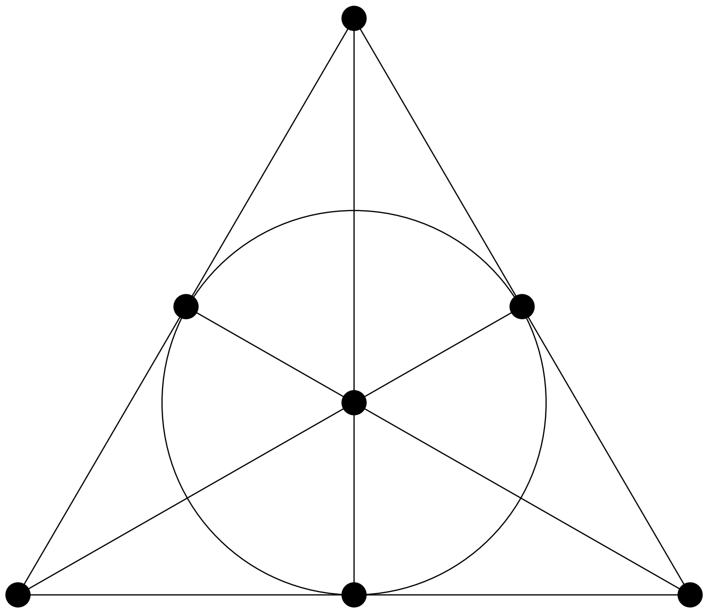
These axioms, replacing the words "people" with "points" and "committees" with "lines," define structures called *finite projective planes*. I have a game called *Configurations* that is designed to introduce the players to the existence, construction, and properties of finite projective planes. When I checked in January 2002 the game was available from WFF 'N PROOF Learning Games Associates, http://www.wff-n-proof.com, 402 E. Kirkwood, Fairfield, IA 52556, Phone (641) 472-0149, Fax (641) 472-0693, for a cost of $25.00.

Here are examples of some problems from this game:

Exercise 4.3.4 In each box below write a number from 1 to 7, subject to the two rules: (1) The three numbers in each column must be different; (2) the same pair of numbers must not occur in two different columns.

	Col 1	Col 2	Col 3	Col 4	Col 5	Col 6	Col 7
Row 1							
Row 2							
Row 3							

Exercise 4.3.5 Use the solution to the above problem to label the seven points of the following diagram with the numbers 1 through 7 so that the columns of the above problem correspond to the triples of points in the diagram below that lie on a common line or circle.



Exercise 4.3.6 Play the game of *SET*. (This game can be found, for example, on www.amazon.com under the name "SET Game" by SET Enterprises, Inc.) Find some ways to think of this game as a model for a set of axioms for points, lines, and planes.

4.4 Kirkman's Schoolgirl Problem

Exercise 4.4.1 Solve the following famous puzzle proposed by T. P. Kirkman in 1847:

> *A school-mistress is in the habit of taking her girls for a daily walk. The girls are fifteen in number, and are arranged in five rows of three each, so that each girl might have two companions. The problem is to dispose them so that for seven consecutive days no girl will walk with any of her school-fellows in any triplet more than once.* (Ball and Coxeter, *Mathematical Recreations and Essays,* University of Toronto Press, 1974, Chapter X.)

4.5 Euclidean Geometry

Exercise 4.5.1 Photocopy Kay, pp. A-56, A-57, A-62–A-66. Note that the Axioms fall into several groups: I, D, A, H, C, and P (Axiom P-1 is on page A-65). By cutting and pasting, group the theorems and corollaries (through the Two-Secant Theorem on page A-66) beneath the axioms upon which they depend.

Exercise 4.5.2 Study the proofs of the Exterior Angle Inequality and the Saccheri-Legendre Theorem on pages 156–160. Then study the proofs of the Euclidean Exterior Angle Theorem and the Corollary on pages 217–218. What is the significance of these results from the perspective of axiomatic systems?

Exercise 4.5.3 Which theorem provides the basis for the possibility of defining the six trigonometric functions for a right triangle?

Exercise 4.5.4 Which theorems provide the basis for the construction of Cartesian coordinate system?

Exercise 4.5.5 Select a theorem and make a directed graph showing which previous results, and ultimately which corollaries, it depends upon.

Exercise 4.5.6 How is the concept of an axiomatic system introduced and developed in the K–16 curriculum?

5 Points, Lines and Incidence

Exercise 5.0.7 How are the notions of points, lines, and incidence introduced and developed in the K–16 curriculum?

5.1 Incidence Axioms

Here are the Incidence Axioms, slightly reworded from Kay, Section 2.3. Lines and planes are certain subsets of points. We know nothing further about points, lines and planes beyond that which is specified in the axioms; i.e., they are the undefined terms.

Axiom I-1: Given two distinct points, there is exactly one line containing both of them.

Axiom I-2: Given three distinct noncollinear points (three points not contained in a common line), there is exactly one plane containing all three of them.

Axiom I-3: If two distinct points are contained in a plane, then any line containing both of these points is contained in that plane.

Axiom I-4: If two planes have a nonempty intersection, then their intersection is a line.

Axiom I-5: Space contains at least four noncoplanar points (four points not contained in a common plane) and contains at least three noncollinear points. Each plane contains at least three noncollinear points. Each line contains at least two distinct points.

5.2 Incidence Theorems

Notation: If A and B are two distinct points, then $\overleftrightarrow{AB}$ denotes the unique line containing both A and B.

Here are two theorems from Section 2.3, which can be proved directly from the Incidence Axioms.

Theorem 5.2.1 *If $C \in \overleftrightarrow{AB}$, $D \in \overleftrightarrow{AB}$ and $C \neq D$, then $\overleftrightarrow{CD} = \overleftrightarrow{AB}$.*

Theorem 5.2.2 *If two distinct lines ℓ and m meet (have nonempty intersection), then their intersection is a single point. If a line meets a plane and is not contained in that plane, their intersection is a single point.*

Exercise 5.2.1 Prove these Theorems.

Exercise 5.2.2 Do the Incidence Axioms imply that there must be an infinite number of points? If not, find two different models for the Incidence Axioms that each have a finite number of points. Construct physical or virtual models of them.

5.3 Geometrical Worlds

Exercise 5.3.1 Here are some geometrical "worlds." In each case we make certain choices on what we will call POINTS, LINES and PLANES. (I capitalize these words as a reminder these may not appear to be our "familiar" points, lines and planes.) In each case you should begin thinking about which of the incidence axioms hold for our choice of POINTS, LINES and PLANES. In particular, does Axiom I-1 hold? It would be helpful for experimentation to have some spherical surfaces to draw on, such as (very smooth) tennis balls, ping-pong balls, oranges or Lénárt spheres.

Also, consider the question of whether or not the following property holds:

For a given LINE and a given POINT not on that LINE, there is a unique LINE containing the given POINT that does not intersect the given LINE.

5.3.1 The Analytical Euclidean Plane: $\mathbf{E}^2$

POINTS: Ordered pairs (x, y) of real numbers; i.e., elements of $\mathbf{R}^2$.

LINES: Sets of points that satisfy an equation of the form $ax + by + c = 0$, where a, b and c are real numbers; and further a and b are not both zero.

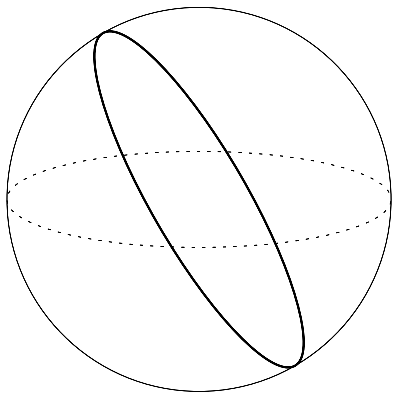
PLANES: There is only one PLANE; namely, the set of all of the POINTS.

5.3.2 The Sphere: $\mathbf{S}^2$

POINTS: All points in $\mathbf{R}^2$ that lie on a sphere of radius 1 centered at the origin.

LINES: Circles on the sphere that divide the sphere into two equal hemispheres. (Such circles are called *great circles.*)

PLANES: There is only one PLANE; namely, the set of all of the POINTS.

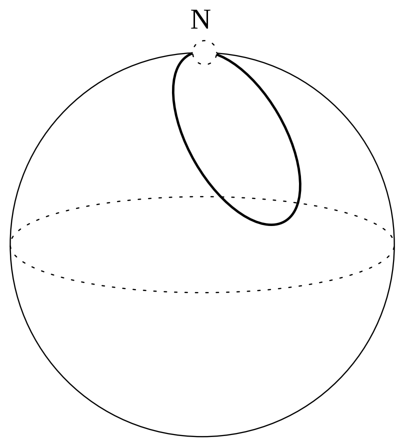


5.3.3 The Punctured Sphere: $\mathbf{U}^2$

POINTS: All points in $\mathbf{R}^2$ that lie on a sphere of radius 1 centered at the origin, with the exception of the point $N = (0, 0, 1)$ (the "North Pole"), which is excluded.

LINES: Circles on the sphere that pass through N, excluding the point N itself.

PLANES: There is only one PLANE; namely, the set of all of the POINTS.

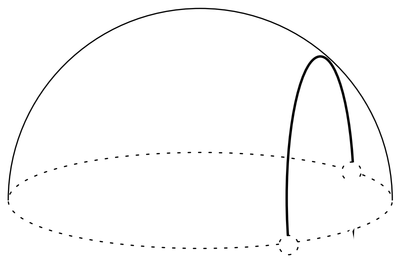


5.3.4 The Open Hemisphere: $\mathbf{H}^2$

POINTS: All points in $\mathbf{R}^2$ that lie on the upper hemisphere of radius 1 centered at the origin with strictly positive z-coordinate. (So the "equator" of points with z-coordinate equaling 0 is excluded.)

LINES: Semicircles (not including endpoints) on this hemisphere that are perpendicular to the "equator".

PLANES: There is only one PLANE; namely, the set of all of the POINTS.



5.3.5 The Projective Plane: $\mathbf{P}^2$

POINTS: All ordinary lines in $\mathbf{R}^3$ that pass through the origin.

LINES: All ordinary planes in $\mathbf{R}^3$ that pass through the origin.

PLANES: There is only one PLANE; namely, the set of all of the POINTS.

5.3.6 Analytical Euclidean Space: $\mathbf{E}^3$

POINTS: Ordered triples (x, y, z) of real numbers.

LINES: Sets of points of the form...

PLANES: Sets of points that satisfy an equation of the form $ax + by + cz + d = 0$, where a, b, c and d are real numbers; and further a, b and c are not all zero.

5.3.7 Analytical Euclidean 4-Space: $\mathbf{E}^4$

POINTS:

LINES:

PLANES:

5.3.8 Analytical Euclidean n-Space: $\mathbf{E}^n$

Here n is an integer greater than 3.

POINTS:

LINES:

PLANES:

6 Coordinates

Exercise 6.0.2 How is the notion of coordinates introduced and developed in the K–16 curriculum?

6.1 The Analytical Euclidean Plane $\mathbf{E}^2$

6.1.1 Equations of Lines

Suppose you have two distinct points (x_1, y_1) and (x_2, y_2) in $\mathbf{E}^2$. You probably already know several ways of getting an equation for the line containing them. For example, if $x_1 \neq x_2$, you can calculate the slope

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

and use the *point-slope* formula

$$y - y_1 = m(x - x_1).$$

Here is a variation on this formula that might be new to you, and works even if the line turns out to be vertical:

> An equation of the line containing (x_1, y_1) and (x_2, y_2) is
>
> $$(y_1 - y_2)x + (x_2 - x_1)y + (x_1 y_2 - x_2 y_1) = 0.$$

Exercise 6.1.1 Verify that this is the equation of a line. Where do you use the assumption that the two points are distinct?

Exercise 6.1.2 Verify that each of the two points satisfies the equation.

Exercise 6.1.3 Derive this formula by trying to solve the following two equations simultaneously for a, b and c, assuming that a and b are not both zero:

$$\begin{aligned} ax_1 + by_1 + c = 0 \\ ax_2 + by_2 + c = 0 \end{aligned}$$

Exercise 6.1.4 Explain how you can conclude from the previous problem that Axiom I-1 holds for $\mathbf{E}^2$.

Exercise 6.1.5 Suppose $A = (1, 2)$, $B = (1, 5)$, and $C = (2, -4)$. Use the formula to determine an equation for the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$.

6.1.2 Determinants

The following are formulas for *determinants* of arrays or *matrices* of numbers. We won't say more about determinants right now, but just learn the formulas:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (aei + bfg + cdh) - (afh + bdi + ceg).$$

Two other equivalent formulas for 3×3 matrices are:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - d\begin{vmatrix} b & c \\ h & i \end{vmatrix} + g\begin{vmatrix} b & c \\ e & f \end{vmatrix}.$$

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & \ell \\ m & n & o & p \end{vmatrix} = a \begin{vmatrix} f & g & h \\ j & k & \ell \\ n & o & p \end{vmatrix} - b \begin{vmatrix} e & g & h \\ i & j & \ell \\ m & o & p \end{vmatrix} + c \begin{vmatrix} e & f & h \\ i & j & \ell \\ m & n & p \end{vmatrix} - d \begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}.$$

Another equivalent formula is:

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & \ell \\ m & n & o & p \end{vmatrix} = a \begin{vmatrix} f & g & h \\ j & k & \ell \\ n & o & p \end{vmatrix} - e \begin{vmatrix} b & c & d \\ j & k & \ell \\ n & o & p \end{vmatrix} + i \begin{vmatrix} b & c & d \\ f & g & h \\ n & o & p \end{vmatrix} - m \begin{vmatrix} b & c & d \\ f & g & h \\ j & k & \ell \end{vmatrix}.$$

Exercise 6.1.6 Calculate the following determinants:

1.

$$\begin{vmatrix} -1 & 2 \\ 3 & -4 \end{vmatrix}$$

2.

$$\begin{vmatrix} 0 & 1 & 2 \\ -1 & 4 & 3 \\ -2 & 0 & 5 \end{vmatrix}$$

3.

$$\begin{vmatrix} -1 & 1 & 2 & -3 \\ 0 & -2 & 4 & 5 \\ 3 & 0 & 0 & -4 \\ 2 & 6 & 10 & -7 \end{vmatrix}$$

6.1.3 Equations of Lines via Determinants

Determinants can be used to express concisely the equation of a line determined by two points:

> An equation of the line containing the distinct points (x_1, y_1) and (x_2, y_2) is
>
> $$\begin{vmatrix} x & x_1 & x_2 \\ y & y_1 & y_2 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Exercise 6.1.7 Show that above statement is correct.

Exercise 6.1.8 Suppose $A = (1, 2)$, $B = (1, 5)$ and $C = (2, -4)$. Use this version of the formula to determine an equation for the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$.

6.1.4 Testing Collinearity

Three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear if and only if

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Exercise 6.1.9 Prove this statement. Suggestion: You might have to consider the special case that all three points are identical.

Exercise 6.1.10 Use this formula to show that the points $A = (1, 2)$, $B = (1, 5)$ and $C = (2, -4)$ are not collinear.

Exercise 6.1.11 Use this formula to show that the points $A = (1, 2)$, $B = (23/2, 5)$ and $C = (2, -4)$ are collinear.

6.1.5 Intersections of Lines

From a previous theorem we know that if two different lines intersect, then they intersect in exactly one point. Given two different lines in $\mathbf{E}^2$, how can we compute the coordinates of that point? One method is to use *Cramer's Rule*:

> If two different lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ intersect, then their point of intersection is given by:
>
> $$x = -\frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = -\frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

Exercise 6.1.12 Prove the above statement.

Exercise 6.1.13 What happens when you try to apply this formula to two lines that do not intersect, or to two equations describing the same line?

Exercise 6.1.14 Try to make sense of the following statement:

> If two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ intersect, then their point of intersection is given by:
>
> $$\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

6.1.6 Parametric Equations of Lines

Here is another useful description of a line determined by two points.

If (x_1, y_1) and (x_2, y_2) are two distinct points, then the line containing them is the set of points $\{(x_1, y_1) + t(u, v) : t \in \mathbf{R}\}$, where $u = x_2 - x_1$ and $v = y_2 - y_1$.

Exercise 6.1.15 Prove that this description is correct; i.e., prove that this set is exactly the same as the set of points on the line containing the original two points, as given by the earlier formula.

Exercise 6.1.16 What point on the line do you get when $t = 0$? When $t = 1$? When $t = 1/2$? When $t = 2$? When $t = -1$? Try plotting these points and explain their geometric relationship to the original two points.

Exercise 6.1.17 Suppose $A = (1, 2)$, $B = (1, 5)$ and $C = (2, -4)$. Use this description to obtain the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$.

6.2 Some Trigonometry

(This section was written with Sue Foege.)

6.2.1 Definitions of Sine and Cosine

The circle with center at the origin and radius of one unit is often called the *unit circle*. Recall that the equation of the unit circle is given by $x^2 + y^2 = 1$.

The trigonometric functions or ratios are often referred to as the *circular* functions. Let θ be an angle of rotation about the origin, measured from the positive x-axis, where a counterclockwise rotation produces a positive angle, as shown in Figure 1. The point $P(a, b)$ on the unit circle corresponds to θ.

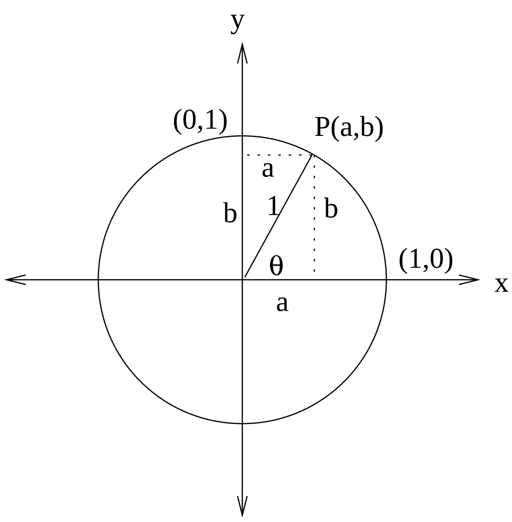


Figure 1: The Unit Circle — An Angle of Rotation

Definition: The *cosine* of θ, denoted $\cos\theta$, is the first, or x coordinate of the corresponding point P on the unit circle. In Figure 1, $a = \cos\theta$.

Definition: The *sine* of θ, denoted $\sin\theta$, is the second, or y coordinate of the corresponding point P on the unit circle. In Figure 1, $b = \sin\theta$.

Exercise 6.2.1 Since $(\cos\theta, \sin\theta)$ is a point on the unit circle, what equation do these coordinates satisfy? (Note: this is a trig identity that you are familiar with!)

Exercise 6.2.2 What is the largest value of $\sin\theta$? Give three values of θ where this maximum value is attained.

Exercise 6.2.3 What is the smallest value of $\sin\theta$? Give three values of θ where this minimum is attained.

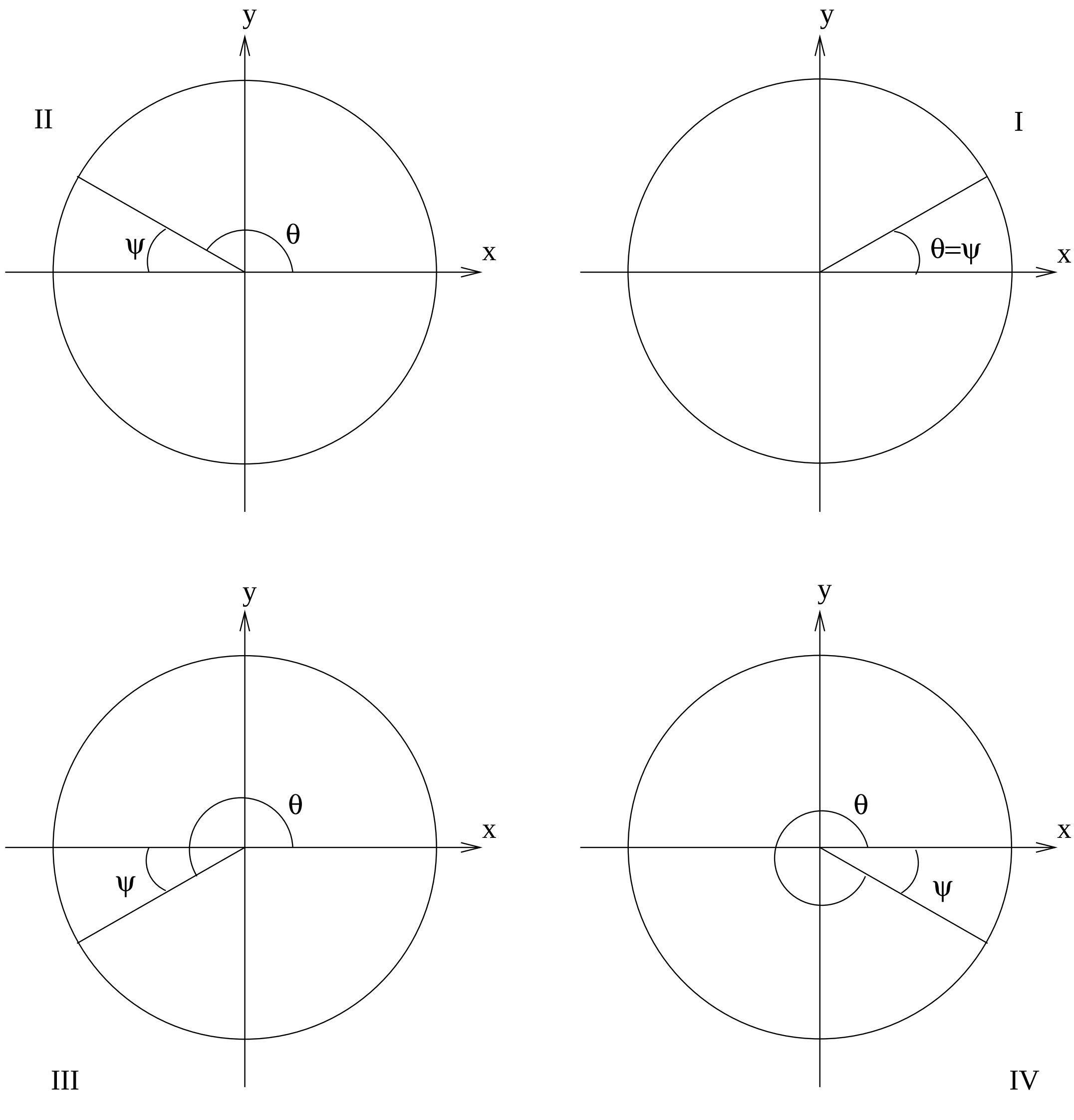
Exercise 6.2.4 Place your finger on the point $(1, 0)$, and trace the unit circle, counterclockwise, stopping at the point $(-1, 0)$. While doing this, how does the second coordinate of the point on the unit circle change? With the same question in mind, continue tracing around the circle.

Exercise 6.2.5 How does the first coordinate of the point change on this journey?

Exercise 6.2.6 Since the second coordinate of the point on the unit circle is the sine of the corresponding angle of rotation, and the first coordinate is the cosine of this angle, we can use the previous two problems to sketch the graphs of the sine and the cosine as a function of the angle. Sketch both graphs.

Exercise 6.2.7 Is circular a good description of the sine and cosine functions?

6.2.2 Angles in Different Quadrants



The above figure shows four possible positions of the angle θ—one in each of the four quadrants. Another angle, Ψ, is also marked. This is sometimes called the *reference angle* for θ. In each case, $\sin\theta = \pm\sin\Psi$ and $\cos\theta = \pm\cos\Psi$. Also, $\tan\theta = \pm\tan\Psi$, recalling that $\tan\theta = \sin\theta/\cos\theta$. The choice of $\pm$ depends upon the quadrant.

Exercise 6.2.8 Fill in the chart:

1. Quadrant I
   (a) $\sin\theta = \sin\Psi$.
   (b) $\cos\theta = \cos\Psi$.
   (c) $\tan\theta = \tan\Psi$.
2. Quadrant II
   (a) $\sin\theta =$
   (b) $\cos\theta =$
   (c) $\tan\theta =$
3. Quadrant III
   (a) $\sin\theta =$
   (b) $\cos\theta =$
   (c) $\tan\theta =$
4. Quadrant IV
   (a) $\sin\theta =$
   (b) $\cos\theta =$
   (c) $\tan\theta =$

One mnemonic device to remember the signs of the trig functions sine, cosine and tangent (sine/cosine) in the four quadrants is **A S**mart **T**rig **C**lass:

A **All** three trig functions are positive in Quadrant I.

S Only **Sine** is positive in Quadrant II; the others are negative.

T Only **Tangent** is positive in Quadrant III; the others are negative.

C Only **Cosine** is positive in Quadrant IV; the others are negative.

6.2.3 Some Familiar Angles

Here is a mnemonic to remember the sine and cosine of some common angles:

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	$\sqrt{0}/2$	$\sqrt{1}/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	$\sqrt{4}/2$
$\cos\theta$	$\sqrt{4}/2$	$\sqrt{3}/2$	$\sqrt{2}/2$	$\sqrt{1}/2$	$\sqrt{0}/2$

Exercise 6.2.9 Convince yourself that these values are correct by considering $45° - 45° - 90°$ triangles and $30° - 60° - 90°$ triangles.

6.2.4 Some Basic Identities

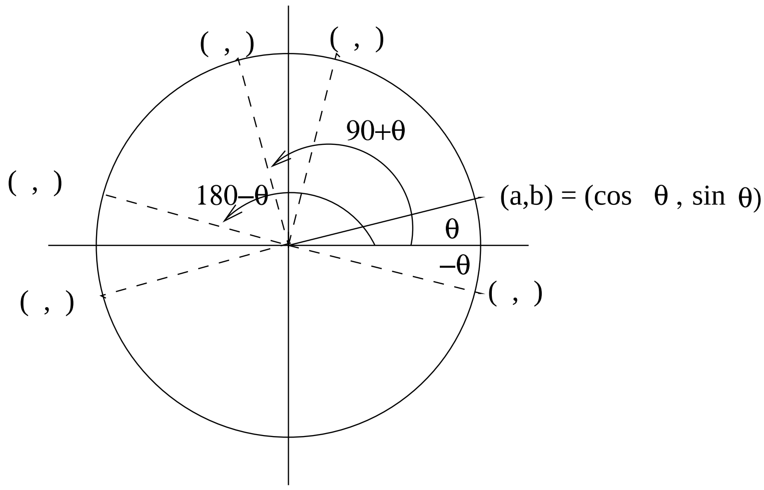


Figure 2: Unit Circle — $a = \cos\theta$, $b = \sin\theta$

Exercise 6.2.10 Using the coordinates a and b, fill in the missing coordinates in Figure 2. Since $a = \cos\theta$ and $b = \sin\theta$, we easily get the sine and cosine of other angles, such as $-\theta$, $90° + \theta$, $180° + \theta$, $90° - \theta$, and $180° - \theta$. For example, $\sin(-\theta) = -b = -\sin\theta$. Using Figure 2, complete and/or verify the following identities.

1. $\cos(-\theta) = a = \cos\theta$.
2. $\sin(-\theta) =$
3. $\cos(90° + \theta) =$
4. $\sin(90° + \theta) = a = \cos\theta$.
5. $\cos(180° + \theta) =$
6. $\sin(180° + \theta) =$
7. $\cos(90° - \theta) =$
8. $\sin(90° - \theta) =$

9. $\cos(180^\circ - \theta) = -a = -\cos\theta$.

10. $\sin(180^\circ - \theta) =$

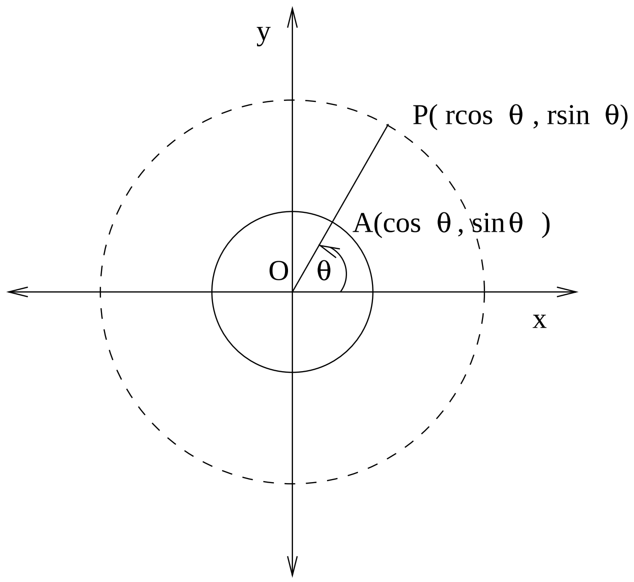
6.2.5 Polar Coordinates of Points in $\mathbf{E}^2$

We assign coordinates to points in $\mathbf{E}^2$ so that we can locate them. Let P be a point in $\mathbf{E}^2$, and $\overline{OP}$ be the line segment between P and the origin. Let r be the distance between P and the origin. (So r is the length of $\overline{OP}$.) Let θ be the angle, measured in the counterclockwise direction, that $\overline{OP}$ makes with the x-axis. (See the figure below.)

The point P lies on the circle of radius r, centered at the origin, given by the equation $x^2 + y^2 = r^2$.

Exercise 6.2.11 Verify that $x = r\cos\theta$ and $y = r\sin\theta$ satisfy this equation.

(r, θ) are called the *polar coordinates* for the point P.



Exercise 6.2.12 Explain exactly when two sets of polar coordinates (r, θ) and (r', θ') describe precisely the same point.

6.2.6 Alternate Definitions of Sine and Cosine

If θ is an acute angle, then we can draw a right triangle, as in Figure 3 (A). The side opposite the angle θ is labeled b, the side adjacent to θ is labeled a and the hypotenuse is labeled c. In Figure 3 (B), we have placed the vertex A at the origin, and the adjacent side on the x-axis. The vertex B is at the point (a, b).

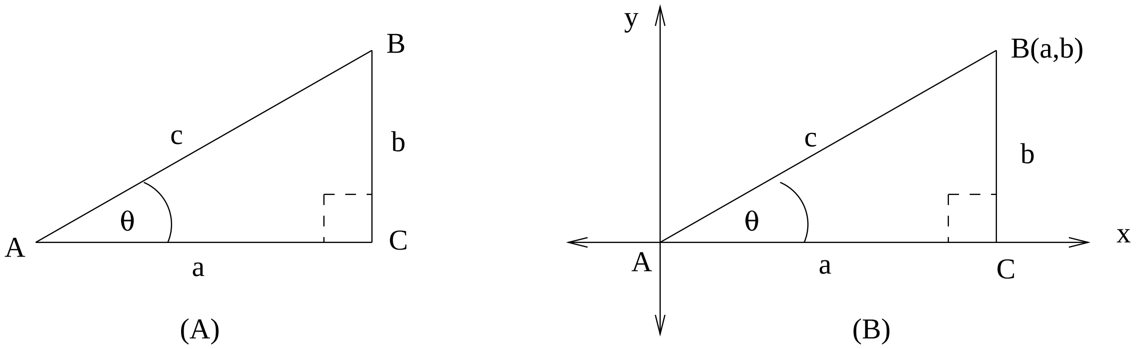


Figure 3: $\theta < 90°$

Alternate definitions for $\sin\theta$ and $\cos\theta$ are:

$$\sin\theta = \frac{\textit{opposite side}}{\textit{hypotenuse}} = \frac{\textit{opp}}{\textit{hyp}} = \frac{\mathbf{O}}{\mathbf{H}} = \frac{b}{c}$$

$$\cos\theta = \frac{\textit{adjacent side}}{\textit{hypotenuse}} = \frac{\textit{adj}}{\textit{hyp}} = \frac{\mathbf{O}}{\mathbf{H}} = \frac{a}{c}$$

Note that when $c = 1$, the point $B(a, b)$ lies on the unit circle, and $\sin\theta = \frac{b}{c} = \frac{b}{1} = b$ is the second coordinate of the point $B(a, b)$, while $\cos\theta = \frac{a}{c} = \frac{a}{1} = a$ is the first coordinate of this point. Thus these alternate definitions are consistent with the old definitions of sine and cosine.

6.2.7 Definitions of Tangent, Secant, and Cosecant

Let θ denote an angle.

Definition: The *tangent* of θ, denoted $\tan\theta$ is

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

Definition: The *secant* of θ, denoted $\sec\theta$ is

$$\sec\theta = \frac{1}{\cos\theta}.$$

Definition: The *cosecant* of θ, denoted $\csc\theta$ is

$$\csc\theta = \frac{1}{\sin\theta}.$$

Exercise 6.2.13 We left out the *cotangent* of θ. Give an appropriate definition as above.

Exercise 6.2.14 Describe the domain and the range of each of the six trig functions.

Exercise 6.2.15 Using the alternate definition of sine and cosine as ratios of sides of right triangles, give alternate definitions of tangent, secant, cosecant, and cotangent.

Exercise 6.2.16 **O**scar **H**ad **A** **H**eap **O**f **A**pples is a mnemonic to remember that $\sin = \frac{O}{H}$, $\cos = \frac{A}{H}$, and $\tan = \frac{O}{A}$. Make up your own mnemonic.

Exercise 6.2.17 Starting with the known trigonometric identity $\sin^2\theta + \cos^2\theta = 1$ we can derive two more identities by dividing both sides of this equation by $\cos^2\theta$ and by $\sin^2\theta$. Try this to find one identity involving $\tan^2\theta$ and another involving $\cot^2\theta$.

6.2.8 The Area of a Triangle Using Sines

It's time to put some algebra and trigonometry to use. In this problem we will use the triangle in Figure 4. In this triangle all angles have measure less than 90°, however, the results hold true for general triangles.

The lengths of $\overline{BC}$, $\overline{AC}$ and $\overline{AB}$ are a, b and c, respectively. Segment $\overline{AD}$ has length c' and $\overline{DB}$ length c''. Segment $\overline{CD}$ is the altitude of the triangle from C, and has length h.

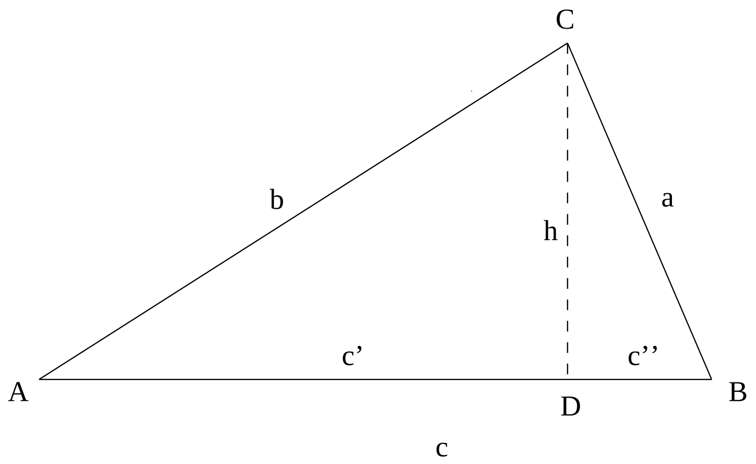


Figure 4: Triangle ABC

The usual formula for the area of a triangle is $\frac{1}{2}$(base)(height).

Exercise 6.2.18 Using the given labeling, Area(ABC) =____________________

Exercise 6.2.19 Since triangle ADC is a right triangle, $\sin A =$________ so $h =$________

Exercise 6.2.20 Thus, Area$(ABC) = \frac{1}{2}ch =$ ____________________

Exercise 6.2.21 What is a formula for Area(ABC) using $\sin B$? Using $\sin C$?? (Note: you will have to use the altitude from A or B).

The conclusion we have made is that the area of a triangle is one-half the product of the lengths of any two sides and the sine of the included angle.

6.2.9 The Law of Sines

Using the triangle in Section 6.2.8, the *Law of Sines* is:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The result holds for arbitrary triangles, but we shall prove it for the triangle ABC in Section 6.2.8.

Exercise 6.2.22 We showed that the area of this triangle was given by three different formulas. What are they?

Exercise 6.2.23 From these three formulas, prove the Law of Sines.

6.2.10 The Law of Cosines

Using the triangle in Section 6.2.8, the *Law of Cosines* is:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Exercise 6.2.24 Show that $c' = b\cos A$.

Exercise 6.2.25 Verify that $c'' = c - c'$.

Exercise 6.2.26 Verify that $h^2 = b^2 - (c')^2$.

Exercise 6.2.27 Apply the Pythagorean Theorem to triangle CDB, then use the facts above to make the appropriate substitutions to prove the Law of Cosines.

Exercise 6.2.28 One last question: What happens when you Apply the Law of Cosines in the case that $\angle A$ is a right angle?

6.2.11 Addition Formulas

There are very useful formulas for the sine and the cosine of the sum of two angles α and β:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Here we will derive these formulas for the case where $A + B < 180°$. In the figures below, $\overline{PA}$ is perpendicular to the x-axis, and $\overline{PB}$ is perpendicular to $\overline{OB}$. Thus triangle OBP is a right triangle in both pictures.

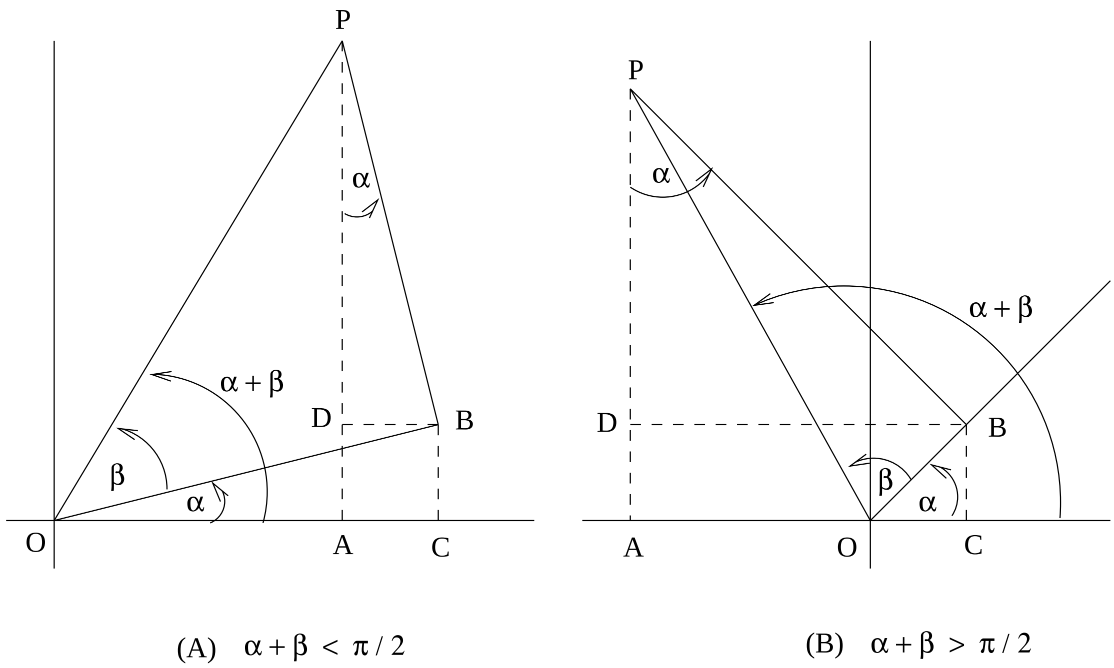


Figure 5: $\alpha + \beta$

First we gather some facts from Figure 5:

1. $AP = AD + DP$.
2. $AD = CB$. Thus $AD + DP = CB + DP$.
3. $OA = OC - AC$.
4. $DB = AC$. Thus $OC - AC = OC - DB$.
5. $\sin\alpha = \frac{DB}{PB}$ (using triangle PDB).
6. $\cos\alpha = \frac{DP}{PB}$ (using triangle PDB).
7. Using triangle OCB, $\sin\alpha =$ ______________________________
8. Using triangle OCB, $\cos\alpha =$ ______________________________
9. $\sin\beta = \frac{PB}{OP}$ and $\cos\beta =$ ________________ (using triangle OBP in (A).
10. Using facts 5,8, and 9,

    $\cos\alpha\cos\beta - \sin\alpha\sin\beta =$ ________________(*)

    Using Figure 5 (A),

    $\cos(\alpha+\beta) = \frac{OA}{OP} =$ ________________using fact 3 above

    $=$ ________________ using fact 4

    $= \cos\alpha\cos\beta - \sin\alpha\sin\beta$ (see (*))

    which proves the first addition formula!
11. Using facts 6,7, and 9,

    $\sin\alpha\cos\beta + \cos\alpha\sin\beta =$ ________________(**)

    Using Figure 5 (A),

    $\sin(\alpha+\beta) = \frac{AP}{OP} =$ ________________ using fact 1 above

    $=$ ________________, using fact 2

    $=\sin\alpha\cos\beta + \cos\alpha\sin\beta$ (see (**))

    which proves the second addition formula!

6.2.12 Heron's Formula

In this section, we will put some of our results together to prove yet another formula for the area of a triangle.

For notation, refer to Figure 4 in Section 6.2.8.

Let $s = \frac{1}{2}(a+b+c)$ (s is half the perimeter). Heron's Formula for the area of triangle ABC is:

$$\text{area}(ABC) = \sqrt{s(s-a)(s-b)(s-c)}$$

First we gather some facts:

1. $\text{area}(ABC) = \frac{1}{2}bc\sin A$ (See Section 6.2.8).
2. $\sin^2 A + \cos^2 A = 1$, thus $\sin A = \sqrt{1-\cos^2 A}$.
3. $a^2 = b^2 + c^2 - 2bc\cos A$, thus

   $$\cos A = \frac{b^2+c^2-a^2}{2bc}.$$

   (This is the Law of Cosines, proven in Section 6.2.10.)
4. $s-a = \frac{1}{2}(-a+b+c)$, $s-b = \frac{1}{2}(a-b+c)$, and $s-c = \frac{1}{2}(a+b-c)$.

Thus, $\text{area}(ABC) = \frac{1}{2}bc\sin A$ (fact 1)

$= \frac{1}{2}bc\sqrt{1-\cos^2 A}$ (fact 2)

$= \frac{1}{2}bc\sqrt{1-\left(\frac{b^2+c^2-a^2}{2bc}\right)^2}$ (fact 3)

$$= \frac{1}{4}\sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2}$$

$$= \frac{1}{4}\sqrt{4b^2c^2 - (b^4 + c^4 + a^4 + 2b^2c^2 - 2a^2b^2 - 2a^2c^2)}$$

$$= \frac{1}{4}\sqrt{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4} \qquad (*)$$

Now we work on the right hand side of Heron's Formula:

$$\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{\tfrac{1}{2}(a+b+c)\tfrac{1}{2}(-a+b+c)\tfrac{1}{2}(a-b+c)\tfrac{1}{2}(a+b-c)}$$

$$= \frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)} \qquad (**)$$

To finish the proof, we must show that $(*) = (**)$.

Exercise 6.2.29 Show that $(a+b+c)(-a+b+c)(a-b+c)(a+b-c) = 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4$.

6.2.13 A Cosine Formula

In this section we remind you of a nice formula to get the cosine of the angle using coordinates of points.

Assume that you have triangle ABC such that the coordinates of the three (distinct) points A, B, and C are $(0,0)$, (x_1, y_1), and (x_2, y_2), respectively. The Law of Cosines can be used to prove that

$$\cos A = \frac{x_1 x_2 + y_1 y_2}{\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}}.$$

Exercise 6.2.30 Use the Law of Cosines to prove this formula. Recall that the length of a line segment joining points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

6.2.14 Determinantal Area Formula

Referring to the triangle ABC of the previous section, we can prove another area formula.

$$\text{area}(ABC) = \frac{1}{2}|x_1 y_2 - x_2 y_1| = \frac{1}{2}\left\| \begin{array}{cc} x_1 & x_2 \\ y_1 & y_2 \end{array} \right\|$$

(I am using the notation $\| \cdot \|$ to denote the absolute value of the determinant.)

Exercise 6.2.31 Use $\text{area}(ABC) = \frac{1}{2}bc \sin A$, the cosine formula from the previous section, and $\sin^2 A + \cos^2 A = 1$ to prove this formula.

6.3 Analytical Euclidean Space $\mathbf{E}^3$

Points in $\mathbf{E}^3$ are ordered triples (x, y, z) of real numbers (called *Cartesian coordinates*). Lines are sets of the form $\{(x_1, y_1, z_1) + t(u, v, w) : t \in \mathbf{R}\}$, where $(u, v, w) \neq (0, 0, 0)$. Planes are sets of the form $\{(x, y, z) : ax + by + cz + d = 0\}$, where $(a, b, c) \neq (0, 0, 0)$.

Exercise 6.3.1 Cylindrical coordinates.

1. Explain how a point (x, y, z) in $\mathbf{E}^3$ can be described by cylindrical coordinates (r, θ, z).
2. Explain exactly when two sets of cylindrical coordinates (r, θ, z) and (r', θ', z') describe precisely the same point.

Exercise 6.3.2 Spherical coordinates.

1. Explain how a point (x, y, z) in $\mathbf{E}^3$ can be described by spherical coordinates $(\rho, \theta\, \phi)$.
2. Explain exactly when two sets of spherical coordinates (ρ, θ, ϕ) and (ρ', θ', ϕ') describe precisely the same point.

Exercise 6.3.3 Locations on the earth are specified by degrees latitude and longitude, where latitude ranges from $0°$ to $90°$ S and from $0°$ to $90°$ N, and longitude ranges from $0°$ to $180°$ E and from $0°$ to $180°$ W. The great semicircle specified by $0°$ longitude is known as the *Prime Meridian.*

1. Assuming that the North Pole has Cartesian coordinates $(0, 0, 1)$, the South Pole has Cartesian coordinates $(0, 0, -1)$, and the Prime Meridian crosses the equator at the point with Cartesian coordinates $(1, 0, 0)$, explain how to convert from spherical coordinates to latitude and longitude.
2. Explain how to convert from latitude and longitude to spherical coordinates.

Exercise 6.3.4 Determine nice coordinates for the eight vertices (corners) of a cube such that the origin $(0, 0, 0)$ is at the center of the cube.

Exercise 6.3.5 Determine nice coordinates for the six vertices of an *octahedron*: a solid whose faces consist of eight equilateral triangles, with exactly four triangles meeting at each vertex. Assume that the origin is at the center of the octahedron.

Exercise 6.3.6 Explain why the following points form the three corners of an equilateral triangle: $(1, 0, 0)$, $(\cos 120°, \sin 120°, 0)$, $(\cos 240°, \sin 240°, 0)$.

Exercise 6.3.7 Using a construction kit like Polydron, Roger's Connection, Zometool, or Googolplex, try to build as many objects as you can with the following properties:

1. Each face (side) is an equilateral triangle.
2. Each equilateral triangle is joined to an equilateral triangle along each of its three faces (these three adjoining triangle are its *neighbors*).
3. No two neighbors are coplanar.
4. The plane determined by any of the faces does not intersect the rest of the object anywhere else (this forces the object to be *convex*).

How many such objects can you construct? How does your answer change if you drop some of these conditions?

Determine exact coordinates (e.g., perhaps involving sin, cos, $\sqrt{\ }$, etc.—not just numerical approximations) for the corners of each object.

Exercise 6.3.8 Prove algebraically that each incidence axiom I-1 through I-5 holds in $\mathbf{E}^3$. Watch out for various special cases in the event that you divide by an expression that might equal 0.

Exercise 6.3.9 Assume that (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) are three distinct non-collinear points. Prove:

An equation of the plane containing (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) is

$$\begin{vmatrix} x & x_1 & x_2 & x_3 \\ y & y_1 & y_2 & y_3 \\ z & z_1 & z_2 & z_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0.$$

6.4 Other Geometrical Worlds

6.4.1 The Sphere $\mathbf{S}^2$

POINTS in $\mathbf{S}^2$ are ordered triples (x, y, z) such that $x^2+y^2+z^2 = 1$. A LINE is a set of points in $\mathbf{S}^2$ that also satisfy an equation of the form $ax + by + cz = 0$, where $(a, b, c) \neq (0, 0, 0)$; i.e., intersections of $\mathbf{S}^2$ with planes in $\mathbf{R}^3$ through the origin.

POINTS in $\mathbf{S}^2$ can also be described by spherical coordinates $(1, \theta, \phi)$; i.e., points with spherical coordinates in which $\rho = 1$.

6.4.2 The Punctured Sphere $\mathbf{U}^2$

POINTS in $\mathbf{U}^2$ are ordered triples (x, y, z) such that $x^2 + y^2 + z^2 = 1$, except for the triple $N = (0, 0, 1)$ which is excluded. A LINE is a set of points in $\mathbf{U}^2$ that also lie on a plane in $\mathbf{R}^3$ that passes through N and is not tangent to the sphere. What is the general equation of such a plane?

6.4.3 The Open Hemisphere $\mathbf{H}^2$

POINTS in $\mathbf{H}^2$ are ordered triples (x, y, z) such that $x^2 + y^2 + z^2 = 1$ and $z > 1$. A LINE is a set of points in $\mathbf{H}^2$ that also lie on a plane in $\mathbf{R}^3$ that is perpendicular to the xy-plane and intersects $\mathbf{H}^2$. What is the general equation of such a plane?

6.4.4 The Projective Plane $\mathbf{P}^2$

POINTS in $\mathbf{P}^2$ are ordinary lines in $\mathbf{R}^3$ that pass through the origin. So each POINT is a set of the form $\{t(x,y,z) : t \in \mathbf{R}\}$, for some ordered triple $(x,y,z) \neq (0,0,0)$. LINES are ordinary planes in $\mathbf{R}^3$ that pass through the origin. So each LINE is a set of the form $\{(x,y,z) : ax+by+cz=0\}$, where $(a,b,c) \neq (0,0,0)$.

6.4.5 Analytical Euclidean 4-Space: $\mathbf{E}^4$

Points in $\mathbf{E}^4$ are ordered quadruples (x_1, x_2, x_3, x_4) of real numbers. Lines are sets of the form $\{(x_1, x_2, x_3, x_4) + t(u_1, u_2, u_3, u_4) : t \in \mathbf{R}\}$, where $(u_1, u_2, u_3, u_4) \neq (0, 0, 0, 0)$. Planes are sets of the form $\{(x_1, x_2, x_3, x_4) + s(u_1, u_2, u_3, u_4) + t(v_1, v_2, v_3, v_4) : s, t \in \mathbf{R}\}$, where $(u_1, u_2, u_3, u_4) \neq (0, 0, 0, 0)$, $(v_1, v_2, v_3, v_4) \neq (0, 0, 0, 0)$, and (u_1, u_2, u_3, u_4) and (v_1, v_2, v_3, v_4) are not multiples of each other.

6.4.6 Analytical Euclidean n-Space: $\mathbf{E}^n$

Assume n is an integer greater than 2. Points in $\mathbf{E}^n$ are ordered n-tuples $x = (x_1, x_2, x_3, \ldots, x_n)$. Lines are sets of the form $\{x + tu : t \in \mathbf{R}\}$, where x is an n-tuple and u is a nonzero n-tuple. Planes are sets of the form $\{x + su + tv : s, t \in \mathbf{R}\}$, where x is an n-tuple, u and v are nonzero n-tuples, and u and v are not multiples of each other.

7 Distance

Exercise 7.0.1 What is the distance between two points in a field?

Exercise 7.0.2 What is the distance between two locations in town? Does your answer change if there are any one-way streets? Does your answer change if you are walking, riding a bicycle, or driving a car?

Exercise 7.0.3 What is the distance between two cities in the state?

Exercise 7.0.4 What is the distance between two cities on the earth?

Exercise 7.0.5 What is the distance traveled by a thrown rock? What is the distance along a curve in the shape of the St. Louis arch?

Exercise 7.0.6 What is the distance from the earth to the moon?

Exercise 7.0.7 What is the distance between a speaker mounted on a wall of a room and the stereo system on the opposite wall?

Exercise 7.0.8 What is the distance between two computers on the internet?

Exercise 7.0.9 What does distance have to do with error-correcting codes?

Exercise 7.0.10 Think about common (and uncommon!) notions of distance. What properties do we expect something called "distance" to satisfy?

Exercise 7.0.11 Given two points, find the set of all points equidistant from both of them.

Exercise 7.0.12 Given three points, find the set of all points equidistant from all three of them.

Exercise 7.0.13 Given a finite set of points ("schools"), divide the plane up into regions ("school districts") according to which school is closest.

Exercise 7.0.14 Give three points, find a point so that the sum of the distances to the three points is minimized.

Exercise 7.0.15 Given an angle formed by two rays, find the set of all points equidistant from both rays.

Exercise 7.0.16 Given a triangle, find the set of all points equidistant from all three sides.

Exercise 7.0.17 Given a point and a line, find the set of all points equidistant from both of them.

Exercise 7.0.18 Given two points, find the set of all points so that the sum of the distances to the two given points is a given constant c.

Exercise 7.0.19 Given three points, find the shortest way to "connect them up." You may need to insert more points.

Exercise 7.0.20 Given four points, find the shortest way to "connect them up." Try starting first with the four corners of a square.

Exercise 7.0.21 A camper finds herself in the angle formed by the edge of a meadow and the bank of a river. Her tent is also in this angle. Describe how to construct the shortest path from her current location to her tent, given that she wishes to stop by the river on the way. Now describe how to construct the shortest path from her current location to her tent, given that she wishes first to stop by the river, and then after that stop by the meadow, on the way to her tent.

Exercise 7.0.22 How is the notion of distance introduced and developed in the K–16 curriculum?

7.1 The Metric Axioms

Here is an outline of some of the main results of Section 2.4 of Kay.

First, we have the Metric Axioms.

Axiom D-1 — Each pair of points A, B is associated with a unique real number AB, called the *distance* from A to B.

Axiom D-2 — For all points A and B, $AB \geq 0$ unless with equality only if $A = B$.

Axiom D-3 — For all points A and B, $AB = BA$.

Axiom D-4 — Ruler Postulate: The points of each line ℓ may be assigned to real numbers x, $-\infty < x < \infty$, called *coordinates*, in such a manner that

1. Each point on ℓ is assigned to a unique coordinate.
2. Each coordinate is assigned to a unique point on ℓ.
3. Any two points on ℓ may be assigned to zero and a positive coordinate, respectively.
4. If points A and B on ℓ have coordinates a and b, respectively, then $AB = |a - b|$.

Definition: Distance is said to satisfy the *Triangle Inequality* if $AB + BC \geq AC$ holds for all triples of points A, B, C.

We won't have to make this property an axiom since it will eventually be proved.

Exercise 7.1.1 What would be a good definition of one point being *between* two others?

Definition of Betweenness: For any three points A, B, and C in space, we say that B is *between* A and C, and we write A-B-C, if and only if A, B, and C are distinct, collinear points, and $AC = AB + BC$.

Definition: If A, B, C, and D are four distinct collinear points, let the betweenness relations A-B-C-D represent the composite of all four betweenness relations A-B-C, A-B-D, A-C-D, and B-C-D.

Theorem 7.1.1 *If A-B-C, then C-B-A, and neither A-C-B nor B-A-C.* (This is Theorem 1 of Kay.)

Theorem 7.1.2 *If A-B-C, A-C-D, and the inequalities $AB+BD \geq AD$ and $BC + CD \geq BD$ hold, then A-B-C-D.* (This is Theorem 2 of Kay.)

Exercise 7.1.2 In terms of betweenness, what would be good definitions of a *segment*, *ray* or *line* determined by two points. How would you define an *angle* determined by three points?

Definition:

Segment AB**:** If A and B are distinct points, the *segment* AB is $\overline{AB} = \{A, B\} \cup \{C : A\text{-}C\text{-}B\}$. Points A and B are called the *endpoints* of the segment.

Ray AB**:** If A and B are distinct points, the *ray* AB is $\overrightarrow{AB} = \{A, B\} \cup \{C : A\text{-}C\text{-}B\} \cup \{D : A\text{-}B\text{-}D\}$. Point A is called the *endpoint* or *origin* of the ray.

Line AB**:** If A, B, and C are distinct points such that A-B-C, then $\overleftrightarrow{AB} = \overrightarrow{BA} \cup \overrightarrow{BC}$. Actually, this must be proven to be equivalent to the original definition of $\overleftrightarrow{AB}$ as the unique line containing both A and B.

Angle ABC**:** If A, B, and C are noncollinear points, the *angle* ABC is $\angle ABC = \overrightarrow{BA} \cup \overrightarrow{BC}$. Point B is called the *vertex* of the angle. Note that our definition of angle explicitly excludes the possibility that A, B, and C are collinear.

Definition: The *extension* of segment $\overline{AB}$ is either the ray $\overrightarrow{AB}$ (in the direction of B), the ray $\overrightarrow{BA}$ (in the direction of A), or the line $\overleftrightarrow{AB}$ (in both directions). The *extension* of ray $\overrightarrow{AB}$ is just the line $\overleftrightarrow{AB}$.

Theorem 7.1.3 *For any line ℓ and any coordinate system under the Ruler Postulate, if $A[a]$, $B[b]$, and $C[c]$ are three points on line ℓ, with their coordinates, then A-B-C iff either $a < b < c$ or $c < b < a$.* (This is Theorem 3 of Kay.)

Theorem 7.1.4 *If* $C \in \overrightarrow{AB}$ *and* $A \neq C$*, then* $\overrightarrow{AB} = \overrightarrow{AC}$. (This is Theorem 4 of Kay.)

Theorem 7.1.5 (Segment Construction Theorem) *If* $\overline{AB}$ *and* $\overline{CD}$ *are two segments and* $AB < CD$*, then there exists a unique point* E *on ray* $\overrightarrow{CD}$ *such that* $AB = CE$ *and* C-E-D. (This is Theorem 5 of Kay.)

7.2 Distance in $\mathbf{E}^2$

7.2.1 The Distance Formula

Definition: The distance AB between the points $A = (x_1, y_1)$ and (x_2, y_2) in $\mathbf{E}^2$ is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Exercise 7.2.1 Given the points A and B above, consider a third point $C = (x_2, y_1)$ and use triangle ABC to prove the distance formula from the Pythagorean theorem.

Exercise 7.2.2 Verify that Axioms D-1, D-2, and D-3 hold.

Exercise 7.2.3 Given $A = (x_1, y_1)$, $B = (x_1, y_2)$, recall that we proved that

$$\overleftrightarrow{AB} = \{A + t(B - A) : t \in \mathbf{R}\}.$$

Prove the following theorem:

Assume that $C \in \overleftrightarrow{AB}$. Then A-C-B if and only if $C = A + t(B - A)$ where $0 < t < 1$.

Exercise 7.2.4 Verify that Axiom D-4 holds.

Exercise 7.2.5 For two ordered pairs $A = (x_1, y_1)$ and $B = (x_2, y_2)$, define

$$A \cdot B = x_1 x_2 + y_1 y_2$$

For an ordered pair $A = (x_1, y_1)$, define

$$\|A\| = \sqrt{x_1^2 + y_1^2} = \sqrt{A \cdot A}$$

Prove the following theorem directly from the definitions:

$$A \cdot B \leq \|A\| \|B\|$$

Exercise 7.2.6 Prove:

$$(A + B) \cdot (A + B) = \|A\|^2 + 2A \cdot B + \|B\|^2$$

Exercise 7.2.7 Observe the obvious fact that

$$AB = \|B - A\| = \sqrt{(B - A) \cdot (B - A)}$$

Prove the Triangle Inequality holds for any three points A, B, C:

$$AC \leq AB + BC$$

Suggestion: First prove that

$$\|D + E\| \leq \|D\| + \|E\|$$

Then let $D = B - A$ and $E = C - B$.

Exercise 7.2.8 Explore the consequences of defining the distance AB between the points $A = (x_1, y_1)$ and (x_2, y_2) in $\mathbf{E}^2$ to be

$$AB = |(x_2 - x_1)| + |(y_2 - y_1)|.$$

Exercise 7.2.9 Explore the consequences of defining the distance AB between the points $A = (x_1, y_1)$ and (x_2, y_2) in $\mathbf{E}^2$ to be

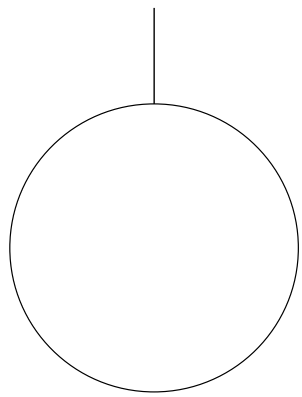
$$AB = \max\{|(x_2 - x_1)|, |(y_2 - y_1)|\}.$$

7.2.2 What is the Distance to the Horizon?

> *It was the first time that Poole had seen a genuine horizon since he had come to Star City, and it was not quite as far away as he had expected.... He used to be good at mental arithmetic—a rare achievement even in his time, and probably much rarer now. The formula to give the horizon distance was a simple one: the square root of twice your height times the radius—the sort of thing you never forgot, even if you wanted to...*
>
> —Arthur C. Clarke, *3001*, Ballantine Books, New York, 1997, page 71

Exercise 7.2.10 In the above passage, Frank Poole uses a formula to determine the distance to the horizon given his height above the ground.

1. Use algebraic notation to express the formula Poole is using.
2. Beginning the diagram below, derive your own formula. You will need to add some more elements to the diagram.



3. Compare your formula to Poole's; you will find that they do not match. How are they different?
4. When I was a boy it was possible to see the Atlantic Ocean from the peak of Mt. Washington in New Hampshire. This mountain is 6288 feet high. How far away is the horizon? Express your answer in miles. Assume that the radius of the Earth is 4000 miles. Use both your formula and Poole's formula and comment on the results. Why does Poole's formula work so well, even though it is not correct?

7.2.3 The Snowflake Curve

Begin with an equilateral triangle. Let's assume that each side of the triangle has length one. Remove the middle third of each line segment and replace it with two sides of an "outward-pointing" equilateral triangle of side length $1/3$. Now you have a six-pointed star formed from 12 line segments of length $1/3$. Replace the middle third of each of these line segments with two sides of outward equilateral triangle of side length $1/9$. Now you have a star-shaped figure with 48 sides. Continue to repeat this process, and the figure will converge to the "Snowflake Curve." Shown below are the first three stages in the construction of the Snowflake Curve.

Exercise 7.2.11

1. In the limit, what is the length of the Snowflake Curve?
2. In the limit, what is the area enclosed by the Snowflake Curve?

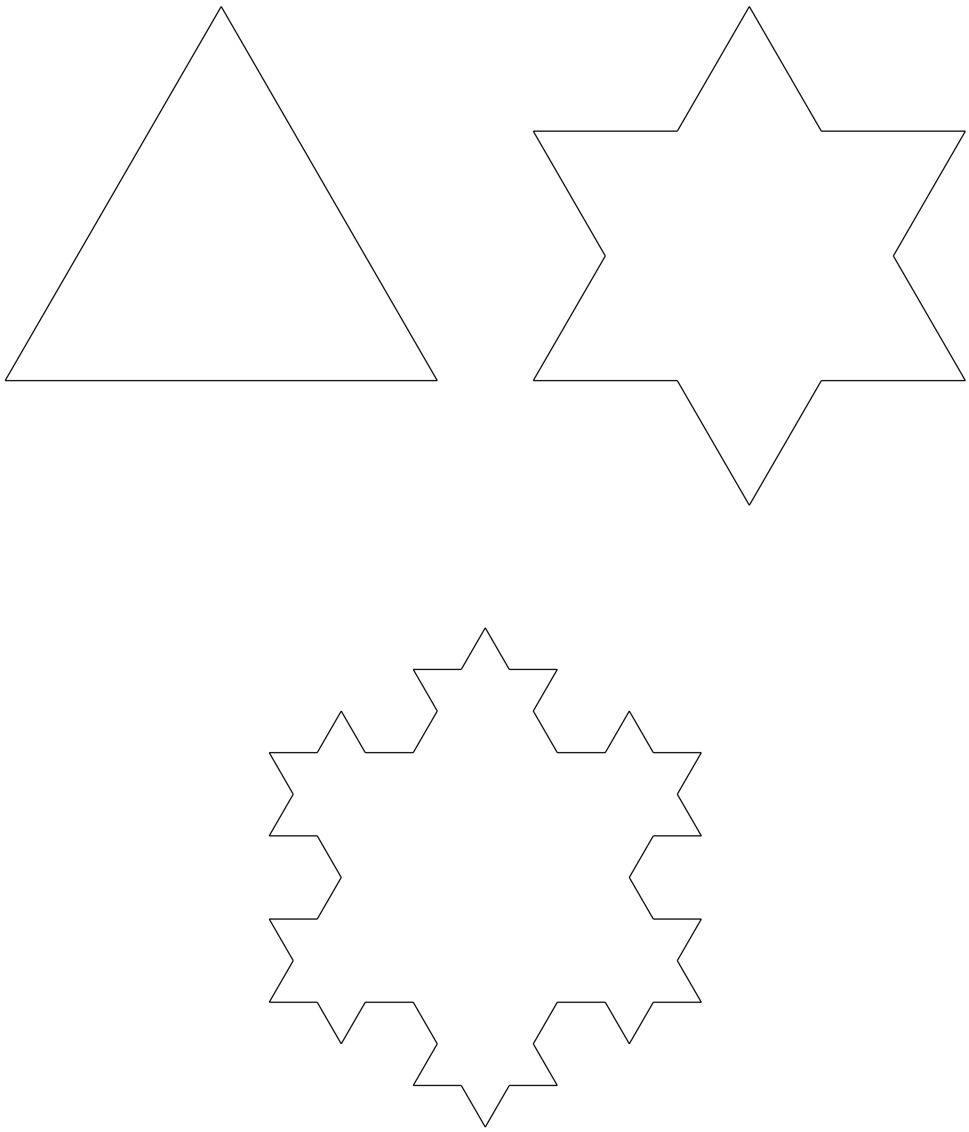


Figure 6: Constructing the Snowflake Curve

7.2.4 The Longimeter

How can we measure the lengths of curves in "real life?" There are devices consisting of wheels with some sort of dial that you can roll over a map to estimate distances, and larger versions that you can roll in front of you on, e.g., paths, to measure distance (what are these things called?). You can also estimate the distance that you walk by wearing a pedometer.

Here is another way to estimate the length of a curve on a map, using a simple device called a *longimeter*. On a transparent sheet of plastic create a square grid, each square having side length of, say 1 mm. Superimpose this grid your curve in three different orientations, differing one from the other by a rotation of 30°. In each of the three cases, count how many squares the curve passes through. Let the sum of these three numbers be S. Then an estimate of the length of the curve is $S/3.82$ mm.

In the example below, I rotated the figure rather than the grid. Each square has side length 0.25 in. The sum S is $16 + 16 + 15 = 47$, so the estimate of the length of the curve is $47/3.82 \approx 12.30$ units of length 0.25 in, or 3.07 in.

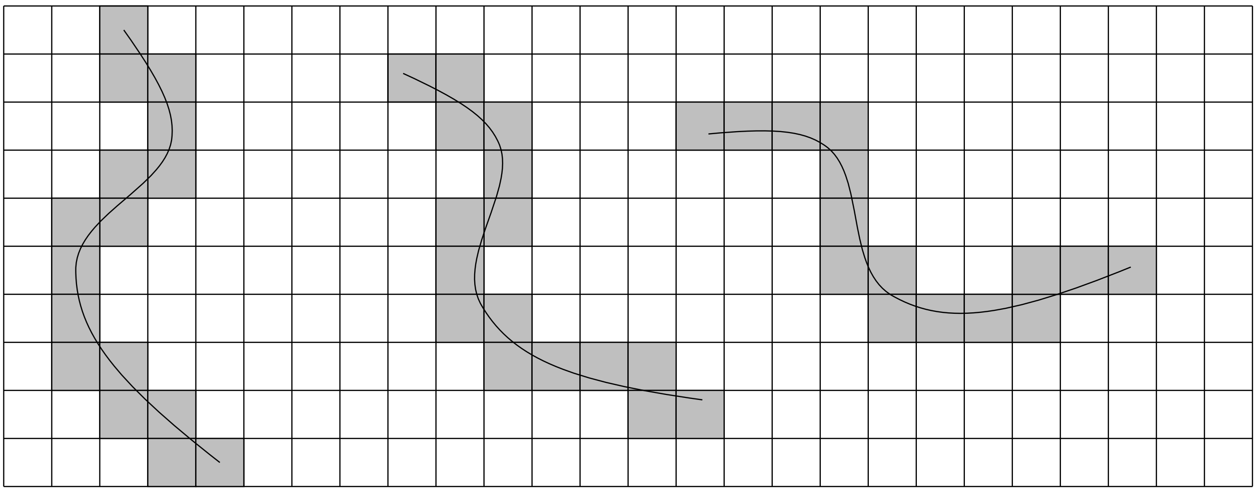


Figure 7: Using a Longimeter

Exercise 7.2.12 Research question: Read the reference below and write up an explanation of why this method works. In particular, where does the number 3.82 come from? (This is not explicitly explained in the book.)

Reference: H. Steinhaus, *Mathematical Snapshots*, Oxford University Press, New York, 1989, pp. 105–107.

7.2.5 Fractals

The notion of "length" of certain naturally occurring objects can, however, be tricky, and can lead one into the notion of fractals. The following quote comes from a book by Mandelbrot:

> *To introduce a first category of fractals, namely curves whose fractal dimension is greater than* 1*, consider a stretch of coastline. It is evident that its length is at least equal to the distance measured along a straight line between its beginning and its end. However, the typical coastline is irregular and winding, and there is no question it is much longer than the straight line between its end points.*
>
> *There are various ways of evaluating its length more accurately... The result is most peculiar: coastline length turns out to be an elusive notion that slips between the fingers of one who wants to grasp it. All measurement methods ultimately lead to the conclusion that the typical coastline's length is very large and so ill determined that it is best considered infinite....*
>
> *Set dividers to a prescribed opening* ϵ*, to be called the yardstick length, and walk these dividers along the coastline, each new step starting where the previous step leaves off. The number of steps multiplied by* ϵ *is an approximate length* $L(\epsilon)$*. As the dividers' opening becomes smaller and smaller, and as we repeat the operation, we have been taught to expect* $L(\epsilon)$ *to settle rapidly to a well-defined value called the* true length. *But in fact what we expect does not happen. In the typical case, the observed* $L(\epsilon)$ *tends to increase without limit.*
>
> *The reason for this behavior is obvious: When a bay or peninsula noticed on a map scaled to* $1/100,000$ *is reexamined on a map at* $1/10,000$*, subbays and subpeninsulas become visible. On a* $1/1,000$ *scale map, sub-subbays and sub-subpeninsulas appear, and so forth. Each adds to the measured length.*
>
> —B.B. Mandelbrot, "How Long is the Coast of Britain," *The Fractal Geometry of Nature*, W.H. Freeman and Company, New York, 1983, Chapter 5, p. 25.

7.3 Distance in $\mathbf{E}^3$

7.3.1 Formulas

You can easily verify that the analogs of the formulas and theorems in $\mathbf{E}^2$ hold for $\mathbf{E}^3$ as well:

Definition: The distance AB between the points $A = (x_1, y_1, z_1)$ and (x_2, y_2, z_2) in $\mathbf{E}^3$ is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Assume that $C \in \overleftrightarrow{AB}$. Then A-C-B if and only if $C = A + t(B - A)$ where $0 < t < 1$.

$$A \cdot B = x_1 x_2 + y_1 y_2 + z_1 z_2$$

$$\|A\| = \sqrt{x_1^2 + y_1^2 + z_1^2} = \sqrt{A \cdot A}$$

$$A \cdot B \leq \|A\|\|B\|$$

$$\|D+E\| \leq \|D\| + \|E\|$$

$$AC \leq AB + BC$$

Assume that you have triangle ABC such that the coordinates of the three (distinct) points A, B, and C are $(0,0,0)$, (x_1, y_1, z_1), and (x_2, y_2, z_2), respectively. Then

$$\cos A = \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2}\sqrt{x_2^2 + y_2^2 + z_2^2}}$$

7.3.2 The Platonic Solids

There are exactly five different three-dimensional polyhedra that satisfy the following conditions:

- Each face is a regular polygon.
- All faces are congruent to each other.
- The same number of faces meets at each vertex (corner).
- No two neighboring faces are coplanar.
- The plane determined by any of the faces does not intersect the rest of the object anywhere else (the object is convex).

These polyhedra are called the *Platonic solids.*

Exercise 7.3.1 By considering the angles of regular polygons, confirm that there cannot be more than five such polyhedra:

- Four triangles, with three meeting at each vertex (*tetrahedron*).
- Eight triangles, with four meeting at each vertex (*octahedron*).
- Twenty triangles, with five meeting at each vertex (*icosahedron*).
- Six squares, with three meeting at each vertex (*cube*).
- Twelve pentagons, with three meeting at each vertex (*dodecahedron*).

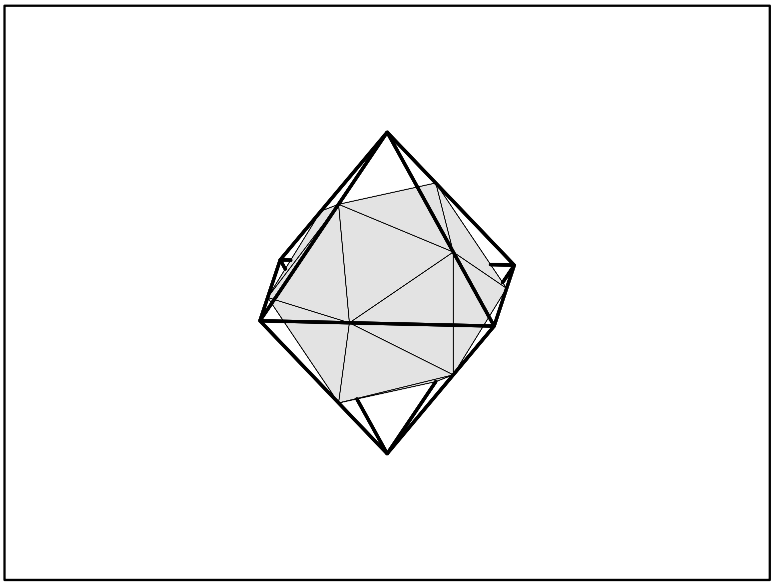
To prove that all five objects can, in fact, be constructed, we can find coordinates for the vertices.

For the cube we have seen that we can use the points $(\pm 1, \pm 1, \pm 1)$.

For the octahedron we have seen that we can use the points $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$.

Exercise 7.3.2 Find coordinates for a regular tetrahedron.

Exercise 7.3.3 Determine coordinates for the icosahedron as follows. Begin with an octahedron with coordinates as above. Divide each of the 12 edges in the same ratio by 12 new points, as shown in the following figure.



Prove that if the ratio is properly chosen, these points will be the vertices of an icosahedron. Suggestion: Given the two end points of an edge of the octahedron, write the parametric formula for the line passing through them. Then the goal is to find the correct value of "t" (necessarily between 0 and 1) to locate the appropriate point on the edge. Also, a physical model of the octahedron can help you visualize the construction.

Exercise 7.3.4 Place a point at the center of each of the 20 triangles of the icosahedron. Verify that these 20 points will be the vertices of a dodecahedron.

7.3.3 The Global Positioning System (GPS)

Orbiting the earth is a collection of 24 satellites broadcasting signals that can be picked up and analyzed by portable receivers. You can buy such a receiver from stores like Radio Shack for a few hundred dollars. Using the signals this receiver is able to determine its distance from several of the satellites. Then, since the position in space of each of these satellites is also known, the receiver is able to compute its location in three-dimensional space (e.g., latitude, longitude, altitude).

Exercise 7.3.5 For any given position determination you do not need to know the distance to all 24 of the satellites. How many distances do you need at any one time? How can you use these distances to calculate your position? Specifically, assume that your (unknown) position is (x_0, y_0, z_0) and that you know the distance d_i to satellite i at known location (x_i, y_i, z_i) for $i = 1, \ldots, n$. What value of n is necessary to determine your position, and what specific calculations must you perform? Suggestion: Try solving this problem first in $\mathbf{E}^1$, then in $\mathbf{E}^2$.

Reference: Thomas A. Herring, "The Global Positioning System," *Scientific American*, February 1996, pp. 44–50. See also various websites.

7.3.4 Earthquake Location

Exercise 7.3.6 Suppose an earthquake occurs somewhere at unknown location (x_0, y_0, z_0) and unknown time t_0. You have several earthquake detectors; detector i is at location (x_i, y_i, z_i) and the tremor arrives at that position at time t_i. Let's make the simplifying assumption that the speed of the tremor v as it travels outward from the earthquake is constant. How can you calculate the location and time of the earthquake from the readings of a set of detectors? Do you need to know the speed v in advance or can you solve for it? What is the minimum number of detectors needed to make the calculations? Do you need to make any special assumptions about the detectors (e.g., do they need to be noncollinear or noncoplanar)? Suggestion: Try solving this problem first in $\mathbf{E}^1$, then in $\mathbf{E}^2$.

7.4 Distance in $\mathbf{S}^2$

A good book about "living" in a spherical world is *Sphereland* by Burger.

7.4.1 Distance Around a Circle

Suppose you have a circle of radius 1. Its circumference is $C = 2\pi r = 2\pi$, which is a bit bigger than 6.2.

Exercise 7.4.1 Explain why the formula for the circumference of a circle provides the *definition* of π.

The measure of a central angle that cuts off a piece of the circumference of length 1 is called a *radian*. Therefore, there are 2π radians around the center of a circle and we can convert back and forth between degrees and radians by

$$\theta(\text{in radians}) = \frac{\pi}{180^\circ}\theta(\text{in degrees})$$

$$\theta(\text{in degrees}) = \frac{180^\circ}{\pi}\theta(\text{in radians})$$

Using radians makes many formulas look "nicer." For example,

Suppose C is a circle of radius r. The length ℓ of an arc intercepted by a central angle θ is given by

$$\ell = r\theta \quad (\text{if } \theta \text{ is measured in radians})$$

$$\ell = \frac{\pi}{180^\circ}r\theta \quad (\text{if } \theta \text{ is measured in degrees})$$

Another motivation for expressing angles is radians are the Taylor series formulas for sine and cosine:

For angle x measured in radians:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Exercise 7.4.2 Derive these Taylor series.

Exercise 7.4.3 Sum the squares of the above series to verify that $\sin^2 x + \cos^2 x = 1$.

This might remind you of the Taylor series for e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots$$

Exercise 7.4.4 Derive this Taylor series.

Exercise 7.4.5 Use the above series to show that $e^a e^b = e^{a+b}$.

From substitution (and some observations about convergence), one gets the beautiful formula for all complex numbers x:

$$e^{ix} = \cos x + i \sin x$$

In particular, setting $x = \pi$ yields an expression containing the perhaps five most important constants in mathematics:

$$e^{i\pi} + 1 = 0$$

These formulas provide a connection between two representations for complex numbers on the one hand, and Cartesian and polar coordinates on the other.

Any complex number $z = a + bi$ can be represented by a point (a, b) in the Cartesian plane. But by the above formula, you can set $r = \sqrt{a^2 + b^2}$ and find θ such that $\cos\theta = a/r$ and $\sin\theta = b/r$. That is, (r, θ) are polar coordinates for the point (a, b). Then $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$.

Exercise 7.4.6 Suppose $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, corresponding to the points $P_1 = (a_1, b_1)$ and $P_2 = (a_2, b_2)$, respectively, in the Cartesian plane. Explain how to find $z = z_1 + z_2$ geometrically.

Exercise 7.4.7 Suppose $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, corresponding to the points P_1, P_2 in the Cartesian plane with polar coordinates (r_1, θ_1), (r_2, θ_2), respectively. Explain how to find $z = z_1 z_2$ geometrically.

Exercise 7.4.8 From what you learned in the previous exercise,

1. Show geometrically that $i^2 = -1$.

2. Find three complex numbers such that $z^3 = 1$.
3. Find three complex numbers such that $z^3 = i$.
4. Explain how to find all solutions to any equation of the form $z^n = z_0$ where n is a positive integer and z_0 is a particular complex number.

Exercise 7.4.9 Suppose $z = z_1 z_2$ where $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_2}$. Use this to prove the angle sum formulas for sine and cosine.

7.4.2 Distance Axioms in S^2

Recall that $\mathbf{S}^2$ is a sphere of radius 1 centered at the origin. Let A and B be any two points on $\mathbf{S}^2$. Find a great circle containing both A and B (remember that great circles are "lines" in $\mathbf{S}^2$). This circle is divided into two arcs by the points A and B. Define the distance between A and B to be the length of the shorter of these arcs. Note that if A and B are not exactly opposite one another (antipodal), then there is a unique great circle containing both of them, so the distance between them is well-defined. If, on the other hand, A and B are antipodal, then there is an infinite number of great circles containing them, but the lengths of all the great-circular arcs joining A and B are the same. So even in this case the distance AB is well-defined.

Exercise 7.4.10 Which of the metric axioms hold?

Exercise 7.4.11 Suppose $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ are two points on $\mathbf{S}^2$. Determine an explicit formula for the distance AB.

Exercise 7.4.12 Look up the latitude and longitude of Lexington, KY and Tokyo, Japan. Look up the diameter, radius, or circumference of the earth. Use this information to determine the distance between these two cities.

Exercise 7.4.13 Does the triangle inequality hold?

Exercise 7.4.14 What is the formula for the circumference of a circle in $\mathbf{S}^2$ in terms of its "radius"?

7.4.3 What is the Size of the Earth?

The kilometer was first defined as $\frac{1}{10{,}000}$ of the distance from the North Pole to the equator of the Earth. (Can you find a reference that verifies this statement?)

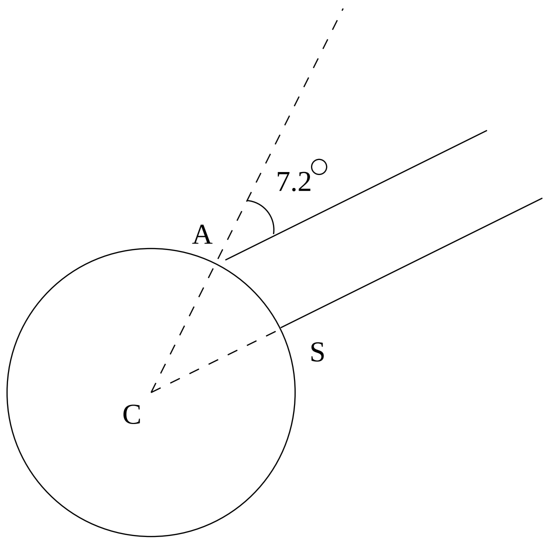
Exercise 7.4.15 What is the circumference of the Earth in kilometers? In miles? Use the conversion 1 mile equals 1.6 kilometers.

Exercise 7.4.16 What is the radius of the Earth in kilometers? In miles?

Exercise 7.4.17 Research question: What is the current official definition of a kilometer and a mile?

7.4.4 Eratosthenes' Estimate of the Size of the Earth

Eratosthenes in about 200 B.C. observed that on a certain day the sun was directly overhead in the city of Syene, but rays from the sun struck the ground at an angle of 7.2° from the vertical in the city of Alexandria, which lay 5000 stadia due north of Syene. Assuming that the Earth was spherical, he used this to estimate its size.



Exercise 7.4.18 Use this information to determine the radius and circumference of the Earth, in stadia.

Exercise 7.4.19 While we do not know the length of a stadium precisely, one estimate is that 1 stadium equals 157.5 meters. Use this to estimate the radius and circumference of the Earth, in kilometers. How do these figures compare with current estimates?

7.5 Distance in U^2

7.5.1 The Distance Axioms

Recall that POINTS in $\mathbf{U}^2$ are points on the unit sphere centered at the origin, except for the point $N = (0, 0, 1)$. For two points $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ in $\mathbf{U}^2$, define the distance AB to be

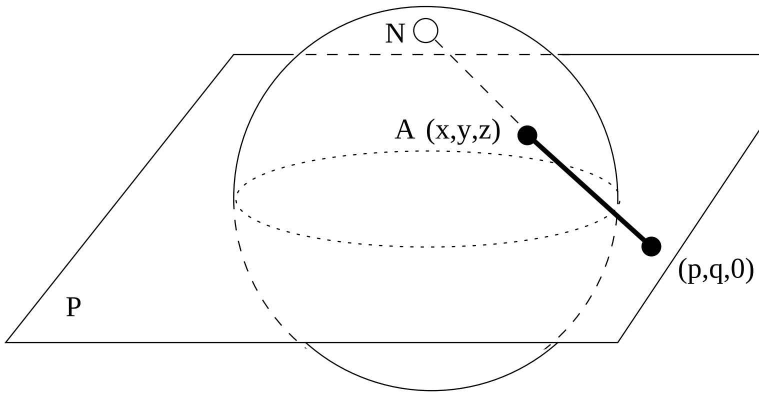
$$AB = \sqrt{\left(\frac{x_2}{1-z_2} - \frac{x_1}{1-z_1}\right)^2 + \left(\frac{y_2}{1-z_2} - \frac{y_1}{1-z_1}\right)^2}$$

Note that, peculiar that this definition may appear to be, it is well-defined because neither z_1 or z_2 equals 1.

Exercise 7.5.1 Verify that the Metric Axioms hold.

7.5.2 Stereographic Projection

The easiest way to explain what is going on in $\mathbf{U}^2$ is to define a mapping from $\mathbf{U}^2$ to the analytic Cartesian plane $\mathbf{E}^2$ in the following way. Let P be the plane given by the equation $z = 0$; i.e., the plane containing the "equator" of the unit sphere. For each point $A = (x, y, z)$ in $\mathbf{U}^2$, consider the ordinary ray $\overrightarrow{NA}$ in $\mathbf{E}^3$. Let $(p, q, 0)$ be the point where this ray intersects P.



Then you can prove that

$$
\begin{aligned}
p &= \frac{x}{1-z} \\
q &= \frac{y}{1-z}
\end{aligned}
$$

Exercise 7.5.2 Prove the above formulas.

We use these formulas to define the map $\phi : (x, y, z) \to (p, q)$ from $\mathbf{U}^2$ to $\mathbf{E}^2$. This map is called *stereographic projection*.

The inverse $\phi^{-1} : (p, q) \to (x, y, z)$ is given by

$$\begin{aligned} x &= \frac{2p}{p^2 + q^2 + 1} \\ y &= \frac{2q}{p^2 + q^2 + 1} \\ z &= \frac{p^2 + q^2 - 1}{p^2 + q^2 + 1} \end{aligned}$$

Exercise 7.5.3 Prove that (x, y, z) given by the above formulas is a point on the unit sphere centered at the origin.

Exercise 7.5.4 Prove that ϕ^{-1} is the inverse of ϕ by showing that $\phi \circ \phi^{-1}$ and $\phi^{-1} \circ \phi$ are each the identify map.

Now we know that ϕ maps $\mathbf{U}^2$ bijectively onto $\mathbf{E}^2$. What do LINES in $\mathbf{U}^2$ map to? A LINE in $\mathbf{U}^2$ is a circle on the unit sphere passing through N (with the point N itself deleted). It is the intersection of the unit sphere with a plane Q through N. Each of the rays $\overrightarrow{NA}$ for points A on the circle lie in this plane, so the intersections of these rays with the plane P equal the intersection $P \cup Q$, which is an ordinary line. So LINES in $\mathbf{U}^2$ map to ordinary lines in $\mathbf{E}^2$.

Conversely, suppose you are given an ordinary line ℓ in $\mathbf{E}^2$. Regard this line as sitting in P. There is a unique plane containing this line and passing through N. This plane intersects the unit sphere in a circle passing through N. Deleting N from this circle gives $\phi^{-1}(\ell)$.

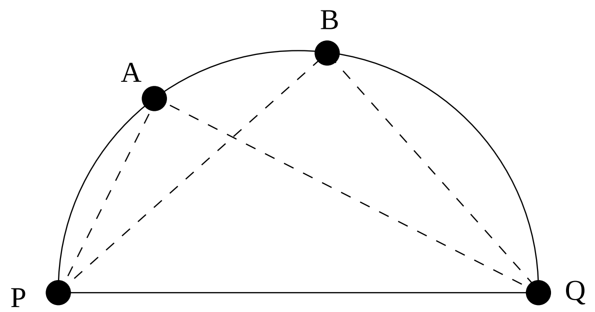
Looking at the formula for ϕ, the distance between two points A, B in $\mathbf{U}^2$ is the same as the ordinary distance between the points $\phi(A), \phi(B)$ in $\mathbf{E}^2$. So all the distance axioms in $\mathbf{U}^2$ hold because they hold in $\mathbf{E}^2$. For the same reason, the triangle inequality also holds.

Exercise 7.5.5 We know that ϕ maps circles on the unit sphere through N (but not including N) to ordinary lines in $\mathbf{E}^2$, and conversely that ϕ^{-1} maps ordinary lines in $\mathbf{E}^2$ to circles on the unit sphere through N (but not including N). Let C be a circle on the unit sphere that does not pass through N. Describe $\phi(C)$.

Exercise 7.5.6 Study the construction of an astrolabe, as described, for example, in the *Cambridge Illustrated History of Astronomy* by Michael Hoskin.

7.6 Distance in $\mathbf{H}^2$

Recall that POINTS in $\mathbf{H}^2$ are points (x, y, z) on the unit sphere centered at the origin such that $z > 1$; i.e, points in the upper hemisphere, excluding the equator. LINES in $\mathbf{H}^2$ are open half-circles perpendicular to the equator. For two points A, B, consider the unique semicircle that contains both of them, and let P and Q be the endpoints of the semicircle on the equator as shown below:



Define the distance AB to be

$$AB = \ln\left(\frac{AQ}{AP}\frac{BP}{BQ}\right)$$

where AP, AQ, BP, and BQ are the ordinary lengths of line segments.

Exercise 7.6.1 Verify that Axioms D-1 – D-3 hold for this model.

Exercise 7.6.2 For two points A, C, consider the unique perpendicular semicircle that contains both of them, and let B be a point on the arc of the semicircle between A and C. Prove that A-B-C.

Exercise 7.6.3 Prove that if A remains fixed and B moves toward Q, then AB tends to infinity.

Exercise 7.6.4 Prove that Axiom D-4 holds for this model.

Exercise 7.6.5 Suppose $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ in the figure above.

1. Determine the equation of the plane containing both of them that is perpendicular to the equator.
2. Determine the coordinates of the points P and Q.
3. Write a formula for AB in terms of the coordinates of A and B.

Exercise 7.6.6 Does the triangle inequality hold for this model?

Exercise 7.6.7 Suppose the model $\mathbf{H}^2$ is constructed on the *lower* hemisphere of the sphere, instead of the *upper*. What does the model look like under the action of stereographic projection?

7.7 Space-Time Distance

Consider a space-time model in which points are given by four coordinates (x, y, z, t), where (x, y, z) is the location of an event and t is the time of the event. For two points (events) $A = (x_1, y_1, z_1, t_1)$ and $B = (x_2, y_2, z_2, t_2)$, define the distance between them to be

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_2 - t_1)^2}$$

where c is the speed of light.

Define the line $\overleftrightarrow{AB}$ to be the set of all points C such that $CA + AB = CB$ or $AC + CB = AB$ or $AB + BC = AC$.

Exercise 7.7.1 Explore which incidence and distance axioms hold.

8 Angles

Exercise 8.0.2 We have already discussed angles in $\mathbf{E}^2$ and $\mathbf{E}^3$ a fair amount. What would be reasonable definitions of measures of angles in $\mathbf{S}^2$, $\mathbf{U}^2$, $\mathbf{P}^2$ and $\mathbf{H}^2$? What would be a reasonable definition of the measure of a "solid angle" in $\mathbf{E}^3$?

8.1 The Angle Axioms

This is a summary of Section 2.5 of Kay.

Axiom A-1: Existence of Angle Measure — Each angle $\angle ABC$ is associated with a unique real number between 0 and 180, called its *measure* and denoted $m\angle ABC$. No angle can have measure 0 or 180.

Axiom A-2: Angle Addition Postulate — If D lies in the interior of $\angle ABC$, then $m\angle ABD + m\angle DBC = m\angle ABC$. Conversely, if $m\angle ABD + m\angle DBC = m\angle ABC$, then ray $\overrightarrow{BD}$ passes through an interior point of $\angle ABC$.

Axiom D-3: Protractor Postulate — The set of rays $\overrightarrow{AX}$ lying on one side of a given line $\overleftrightarrow{AB}$, including ray $\overrightarrow{AB}$, may be assigned to the entire set of real numbers x, $0 \leq x < 180$, called *coordinates*, in such a manner that

1. Each ray is assigned to a unique coordinate.
2. No two rays are assigned to the same coordinate.
3. The coordinate of $\overrightarrow{AB}$ is 0.
4. If rays $\overrightarrow{AC}$ and $\overrightarrow{AD}$ have coordinates c and d, then $m\angle CAD = |c - d|$.

Axiom D-4 — Linear Pair Axiom: A linear pair of angles is a supplementary pair.

To define the terms used in the Axioms:

Definition: A point D is an *interior point* of $\angle ABC$ iff there exists a segment $\overline{EF}$ containing D as an interior point that extends from one side of the angle to the other ($E \in \overrightarrow{BA}$ and $F \in \overrightarrow{BC}$, $E \neq B$, $F \neq B$).

Definition: For any three rays $\overrightarrow{BA}$, $\overrightarrow{BD}$, and $\overrightarrow{BC}$ (having the same endpoint), we say that ray $\overrightarrow{BD}$ lies *between* rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$, and we write $\overrightarrow{BA}$-$\overrightarrow{BD}$-$\overrightarrow{BC}$, iff the rays are distinct and $m\angle ABD + m\angle DBC = m\angle ABC$.

Definition: Two angles are said to form a *linear pair* iff they have one side in common and the other two sides are opposite rays. We call any two angles whose angle measures sum to 180 a *supplementary pair*, or simply, *supplementary*, and two angles whose angle measures sum to 90, *complementary*.

Definition: A *right angle* is any angle having measure 90. Two (distinct) lines ℓ and m are said to be *perpendicular*, and we write $\ell \perp m$, iff they contain the sides of a right angle. (For convenience, segments are perpendicular iff they lie, respectively, on perpendicular lines. Similar terminology applies to a segment and ray, two rays, and so on.) An *acute angle* is any angle whose measure is less than 90. An *obtuse angle* is any angle whose measure is greater than 90.

Definition: Two angles having the sides of one opposite the sides of the other are called *vertical angles*.

Theorem 8.1.1 (Angle Construction Theorem) *For any two angles $\angle ABC$ and $\angle DEF$ such that $m\angle ABC < m\angle DEF$, there is a unique ray $\overrightarrow{EG}$ such that $m\angle ABC = m\angle GEF$ and $\overrightarrow{EF}$-$\overrightarrow{EG}$-$\overrightarrow{ED}$.* (This is Theorem 1 of Kay.)

Theorem 8.1.2 *Two angles that are supplementary, or complementary, to the same angle have equal measures.* (This is Theorem 2 of Kay.)

Lemma 8.1.1 *If two lines are perpendicular, they form four right angles at their point of intersection.*

Theorem 8.1.3 *If line $\overleftrightarrow{BD}$ meets segment $\overline{AC}$ at an interior point B on that segment, then $\overleftrightarrow{BD} \perp \overline{AC}$ iff the adjacent angles at B have equal measures.* (This is Theorem 3 of Kay.)

Theorem 8.1.4 *Given a point A on line ℓ, there exists a unique line m perpendicular to ℓ at A.* (This is Theorem 4 of Kay.)

Theorem 8.1.5 *Vertical angles have equal measures.* (This is Theorem 5 of Kay.)

Exercise 8.1.1 How is the notion of angle introduced and developed in the K–16 curriculum?

9 The Plane Separation Axiom

This is a summary of Section 2.6 of Kay.

Definition: A set K is called *convex* provided it has the property that for all points $A \in K$ and $B \in K$, the segment joining A and B lies in K ($\overline{AB} \subseteq K$).

Axiom H-1 — Plane Separation Postulate: Let ℓ be any line lying in any plane P. The set of all points in P not on ℓ consists of the union of two subsets H_1 and H_2 of P such that

1. H_1 and H_2 are convex sets.
2. H_1 and H_2 are no points in common.
3. If A lies in H_1 and B lies in H_2, the line ℓ intersects the segment $\overline{AB}$.

Definition: The two sets H_1 and H_2 above are called the two *sides* of ℓ, or also, *half-planes* determined by ℓ.

Theorem 9.0.6 *If point A lies on line ℓ and point B lies in one of the half-planes determined by ℓ then, except for A, the entire segment $\overline{AB}$ or ray $\overrightarrow{AB}$ lies in that half-plane.* (This is Theorem 1 of Kay.)

Corollary 9.0.1 *Let B and F lie on opposite sides of a line ℓ and let A and G be any two distinct points on ℓ. Then segments $\overline{GB}$ and ray $\overrightarrow{AF}$ have no points in common.*

Theorem 9.0.7 (Postulate of Pasch) *Suppose A, B, and C are any three distinct noncollinear points in a plane, and ℓ is any line in that plane that passes through an interior point D of one of the sides, $\overline{AB}$, of the triangle determined by A, B, and C. Assume that ℓ does not contain C. Then line ℓ meets either $\overline{AC}$ at some interior point E, or $\overline{BC}$ at some interior point F, the cases being mutually exclusive.* (This is Theorem 2 of Kay.)

Exercise 9.0.2 How is the notion of convexity introduced and developed in the K–16 curriculum?

10 Area and Volume

10.1 Area in $\mathbf{E}^2$

Exercise 10.1.1 How would you determine the area of some region (e.g., a lake, county, etc.) on a map?

Exercise 10.1.2 How would you determine the area of Africa?

Exercise 10.1.3 What is the area of a square? Why? What about squares with nonintegral, rational, or irrational sides?

Exercise 10.1.4 What is the area of a rectangle?

Exercise 10.1.5 What is the area of a parallelogram?

Exercise 10.1.6 What is the area of a triangle?

Exercise 10.1.7 What is the area of a trapezoid?

Exercise 10.1.8 What is the area of a regular polygon?

Exercise 10.1.9 Can you dissect an equilateral triangle to a square? How about a regular hexagon? Other polygonal shapes?

Exercise 10.1.10 Show that any polygonal region is equidissectable to a square.

Exercise 10.1.11 Prove that two polygonal regions are equidissectable iff they have the same area.

Exercise 10.1.12 Analyze some equidissectability paradoxes.

Exercise 10.1.13 Examine some algebraic theorems that can be proved by equidissectability.

Exercise 10.1.14 Find a way to dissect a circle into a finite number of congruent pieces, not all of them touching the center of the circle.

Exercise 10.1.15 What assumptions have we been making about the nature of area?

Exercise 10.1.16 Suppose you have two overlapping regions A and B. The area of $A \cup B$ is not the sum of the areas of A and B. What is it? Extend to more than two regions.

Exercise 10.1.17 If area is preserved under congruence, then the conditions that force a pair of triangles to be congruent ought to determine the area. How?

Exercise 10.1.18 How can we find the area of a polygonal region if we know the coordinates of the vertices?

Exercise 10.1.19 What is the area of the region surrounded by the snowflake curve?

Exercise 10.1.20 What is the area of a polygonal region, all of whose coordinates are integral?

Exercise 10.1.21 What is the area of a circle of radius r? Why?

Exercise 10.1.22 What do we mean by the area of an irregular figure?

Exercise 10.1.23 Are there sets of points for which area is undefined (as opposed to being zero or infinity)?

Exercise 10.1.24 Study and apply Cavalieri's principle to find the area of certain figures. Find the area bounded by the curves $y = (x+2)^2$, $y = x^2 + 2$, $x = 1$, and $x = 4$. Do this by Cavalieri without calculus. Then discuss using calculus for areas under curves. Also double integrals for more general areas.

Exercise 10.1.25 Express area integrals in terms of polar coordinates. Why does this make sense?

Exercise 10.1.26 How can we measure area "practically" from a map?

Exercise 10.1.27 Can unbounded regions have finite area?

Exercise 10.1.28 Does the set of rational points in a unit square have area?

10.2 Area in $\mathbf{S}^2$

Exercise 10.2.1 What is the formula for the area of a circular region on a sphere?

Exercise 10.2.2 What is the area of a lune (two-sided polygon formed by two half-great circles joining two antipodal points) on a sphere?

Exercise 10.2.3 What is the formula for the area of a "triangle" on a sphere?

Exercise 10.2.4 What is the area of a spherical polygonal region?

Exercise 10.2.5 Can the sphere be mapped to the plane in an area-preserving way? In what ways would such a map of the earth be useful or not useful?

10.3 Volume in $\mathbf{E}^3$

Exercise 10.3.1 What are analogs of all of the above results to volume?

Exercise 10.3.2 What is the effect of scaling upon area, volume, and surface area?

Exercise 10.3.3 What is the volume of a prism?

Exercise 10.3.4 Derive the formula for the volume of a polygonal pyramid by decomposing a triangular prism into three triangular pyramids.

Exercise 10.3.5 Are two polyhedra equidissectable iff they have the same volume?

Exercise 10.3.6 Derive the formulas for the volume and the circumference of a sphere.

Exercise 10.3.7 Find the volume formed by the intersection of two cylinders of unit radius and infinite length whose axes cross at right angles.

Exercise 10.3.8 How are the various ideas associated with area and volume introduced and developed in the K–16 curriculum?

11 Polyhedra

Exercise 11.0.9 How are polyhedra and other three-dimensional objects introduced and developed in the K–16 curriculum?

11.1 Initial Questions

Exercise 11.1.1 What is a polyhedron? What is a convex polyhedron? What kinds of regions can be enclosed by a polyhedral surface?

Exercise 11.1.2 What are three-dimensional analogs of circles, triangles, isosceles triangles, equilateral triangles, scalene triangles, quadrilaterals, trapezoids, parallelograms, rectangles, rhombi, squares?

Exercise 11.1.3 Try to find (construct) convex polyhedra such that every face is an equilateral triangle and adjacent triangles do not lie in the same plane.

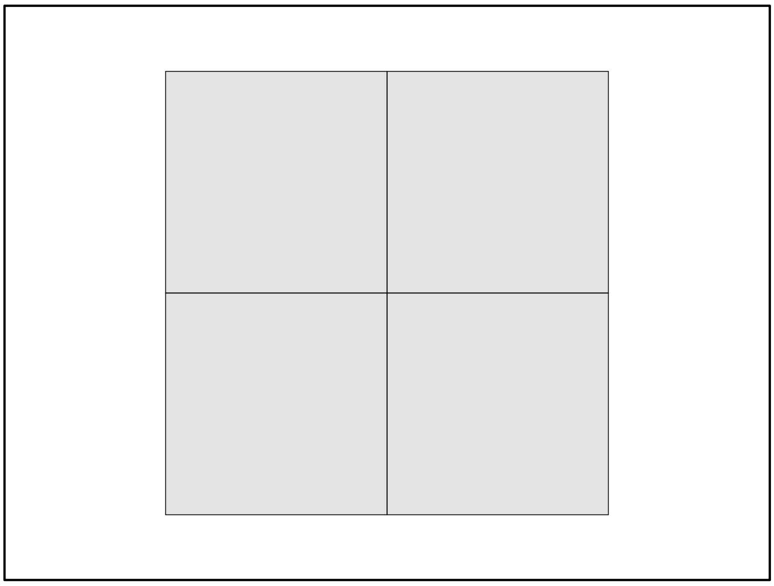
Exercise 11.1.4 What kinds of polygons can you get as cross-sections when you intersect a cube with a plane? Repeat with a tetrahedron and with an octahedron.

Exercise 11.1.5 Construct some tensegrity structures based upon polyhedra.

11.2 Regularity

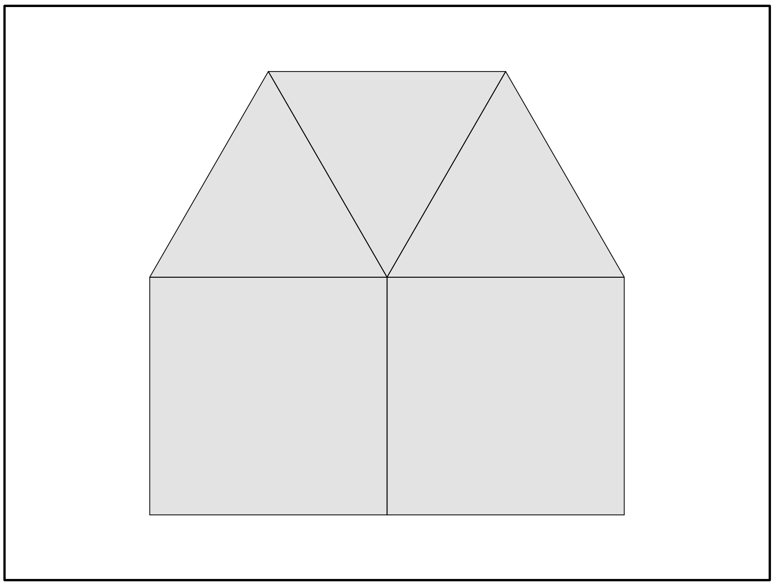
Exercise 11.2.1 What are three-dimensional analogs of regular polygons?

Exercise 11.2.2 Four squares can be fit together perfectly in the plane surrounding a common corner (since each interior angle of a square is 90 degrees). Let's call this a (4,4,4,4) cluster.

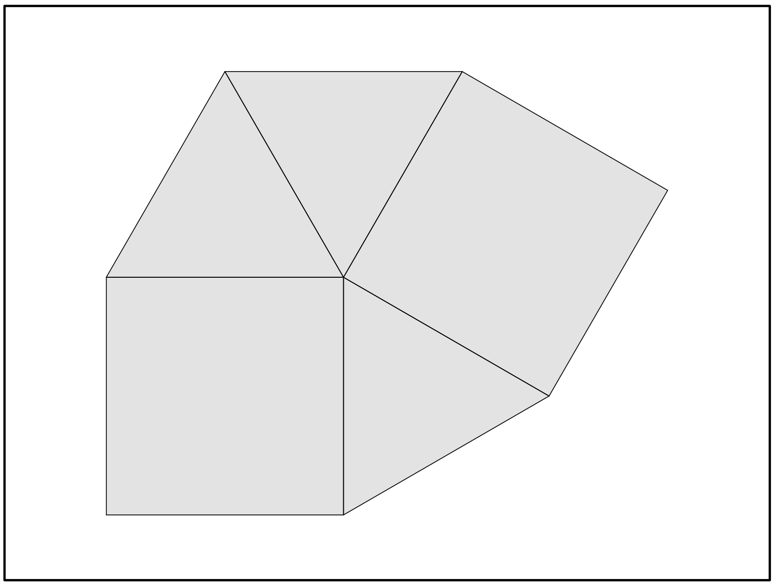


(4,4,4,4) Cluster

Similarly, two squares and three equilateral triangles can fit together perfectly surrounding a common corner. There are essentially two different ways to do this: (4,4,3,3,3) (where the squares are adjacent) and (4,3,4,3,3) (where the squares are not adjacent).



(4,4,3,3,3) Cluster



(4,3,4,3,3) Cluster

Note that we could have called this last cluster $(3, 3, 4, 3, 4)$ as well—it still refers to the same cluster. However, $(4, 4, 3, 3, 3)$ and $(4, 3, 4, 3, 3)$ are *not* the same.

1. You have just seen three clusters. Try to determine all possible clusters that can be formed by placing combinations of regular polygons in the plane surrounding a common corner. Be *systematic* in some fashion, so that you can be certain you have found all of them.

2. Some of the clusters can be extended to cover (tile) the plane so that at every corner point of the tiling, exactly the same cluster appears—the same sequence of polygons, in either clockwise or counterclockwise order. For example, if you tile the plane with squares, you have a $(4, 4, 4, 4)$ cluster at every single corner. Of the clusters you have found, determine which ones can be extended. Make a good drawing of each one you have found.

Exercise 11.2.3 Now we will consider clusters of regular polygons that fit together around a common corner, but with a total angle of less than 360 degrees. Let's call these *space clusters*. For example, the cluster of three squares is $(4, 4, 4)$, and makes a total angle of only 270 degrees. Of course, this cluster can be extended so that the same cluster appears at each corner, eventually closing up to make a cube. Just as in the planar case that we looked at earlier, some space clusters cannot extend to create a polyhedron.

The space cluster $(4, 4, 4)$ consists of only one type of polygon (as opposed to, say, $(3, 4, 3, 4)$ which consists of more than one type of polygon). List all space clusters that consist of only one type of polygon. How do you know that you have them all? Determine which of these appear to extend to enclose a polyhedron.

Exercise 11.2.4 Find coordinates of the vertices and equations of the planes of the faces for the various polyhedra constructed.

Exercise 11.2.5 Describe the symmetries of the various polyhedra constructed.

Exercise 11.2.6 Find tilings of space by combinations of one or more polyhedra.

11.3 Euler's Relation

11.3.1 Initial Investigations

Exercise 11.3.1 Can you construct a polyhedron (not necessarily equilateral) triangles with 9 faces?

Exercise 11.3.2 A tetrahedron has 6 edges. Try to construct a polyhedron with exactly 7 edges.

Exercise 11.3.3 Try to construct a polyhedron for which every face has at least 6 edges.

Exercise 11.3.4 (Euler's Relation) What relationship exists between V (the number of vertices), E (the number of edges), and F (the number of faces) for convex polyhedra?

Exercise 11.3.5 Prove that even if the faces are allowed to be nonregular, there are no more than five polyhedra satisfying the two properties that all faces the same number of sides and the same number of faces meet at each vertex.

11.3.2 Some Basic Inequalities

Exercise 11.3.6 Let F_i denote the number of faces that have i vertices (and hence i edges). Explain why

$$3F_3 + 4F_4 + 5F_5 + 6F_6 + \cdots = 2E. \tag{1}$$

Exercise 11.3.7 Explain why

$$3F_3 + 4F_4 + 5F_5 + 6F_6 + \cdots \geq 3F_3 + 3F_4 + 3F_5 + 3F_6 + \cdots = 3F. \tag{2}$$

Exercise 11.3.8 Conclude

$$2E \geq 3F. \tag{3}$$

Exercise 11.3.9 Let V_i denote the number of vertices at which i faces (and hence i edges) meet. Prove

$$3V_3 + 4V_4 + 5V_5 + 6V_6 + \cdots = 2E. \tag{4}$$

Exercise 11.3.10 Prove

$$2E \geq 3V. \tag{5}$$

Exercise 11.3.11 Use Euler's Relation and (3) to prove

$$F \leq 2V - 4. \tag{6}$$

Exercise 11.3.12 Use Euler's Relation and (5) to prove

$$F \geq \frac{1}{2}V + 2. \tag{7}$$

11.4 Enumerating Possibilities

Exercise 11.4.1 Label the horizontal axis in a coordinate system V and the vertical axis F. Graph the region for which the above two inequalities (6) and (7) hold. Begin listing the whole number solutions in the table below

V	F
4	4
5	5
5	6

Exercise 11.4.2 Can you find a formula for the number of different possible values of F for a given value of V?

Exercise 11.4.3 Prove that no polyhedron has exactly 7 edges.

Exercise 11.4.4 Think of ways to construct polyhedra that match all possible values of V and F in the above table. For example, a tetrahedron has $(V, F) = (4, 4)$ and a pyramid with a pentagonal base has $(V, F) = (5, 5)$. If you chop off one corner of a tetrahedron, the resulting polyhedron has $(V, F) = (6, 5)$. If you build a shallow pyramid over one of the triangles of a tetrahedron, the resulting polyhedron has $(V, F) = (5, 6)$.

11.4.1 Some More Inequalities

Exercise 11.4.5 Use Euler's Relation and (3) to prove that

$$6 \leq 3V - E. \tag{8}$$

Exercise 11.4.6 Use Euler's Relation and (5) to prove that

$$6 \leq 3F - E. \tag{9}$$

Exercise 11.4.7 Use (9) and one of the earlier formulas to prove that

$$12 \leq 3F_3 + 2F_4 + 1F_5 + 0F_6 - 1F_7 - 2F_8 - \cdots. \tag{10}$$

Exercise 11.4.8 Prove that every polyhedron must have at least one face that is a triangle, quadrilateral, or pentagon.

Exercise 11.4.9 Prove that every polyhedron must have at least one vertex at which exactly 3, 4, or 5 edges meet.

Exercise 11.4.10 A truncated icosahedron (soccer ball) is an example of a polyhedron such that (1) each face is a pentagon or a hexagons, and (2) exactly three faces meet at each vertex. Prove that any polyhedron with these two properties must have exactly 12 pentagons. Can you think of a polyhedron that has 12 pentagons but a different number of hexagons than a truncated icosahedron (which has 20 hexagons)?

Exercise 11.4.11 Find formulas for V, E, and F for the semiregular polyhedra in terms of the number of polygons of each type in the vertex cluster.

11.4.2 Angle Deficit

You may remember from plane geometry that for any polygon, the sum of the exterior angles (the amount by which the interior angle falls short of 180 degrees) always equals 360 degrees. Is there an analog for polyhedra? What "deficit" should we measure?

For each vertex we will calculate by how much the sum of the interior angles of the polygons meeting there falls short of 360 degrees. Then we will sum these shortfalls over all the vertices.

Exercise 11.4.12 Remember that the sum of the interior angles of a polygon with n sides is $(n-2)180$ degrees. Prove that

$$S = 180(F_3 + 2F_4 + 3F_5 + 4F_6 + \cdots). \qquad (11)$$

Exercise 11.4.13 Now use Euler's Relation and (1) to show that

$$360V - S = 720 \qquad (12)$$

for all polyhedra.

12 Congruence

12.1 Introduction

Exercise 12.1.1 How are congruence and symmetry introduced and developed in the K–16 curriculum?

Exercise 12.1.2 What do we mean by two figures being *congruent* if they are not triangles?

An *isometry* between two figures is a one-to-one onto distance-preserving mapping f from one figure to the other. *Distance-preserving* means that for every two points A and B in the first figure, the distance from A to B equals the distance from $f(A)$ to $f(B)$.

Exercise 12.1.3 Show that every isometry of the entire plane to itself is uniquely determined by its action on any three noncollinear points.

Exercise 12.1.4 Show that every isometry of two figures in the plane extends to an isometry of the entire plane to itself. When is this extension unique?

Exercise 12.1.5 Show that the composition of every pair of isometries is again an isometry.

Exercise 12.1.6 What is the *identity* isometry under composition?

Exercise 12.1.7 Show that every isometry has an inverse isometry under composition.

Exercise 12.1.8 What do we mean by a figure (set of points) being symmetric?

If there is a nontrivial isometry (not the identity mapping) that maps the figure to itself; i.e., it is self-congruent in a non-trivial way.

A *symmetry* of a figure is an isometry that maps the figure to itself.

Exercise 12.1.9 Show that the composition of every pair of symmetries is again a symmetry.

12.2 Isometries in a Line

Exercise 12.2.1 Classify the isometries of a line (not regarded as sitting in a plane).

Exercise 12.2.2 Find a formula for translation by an amount a. Find a formula for reflecting across a point with coordinate a.

Exercise 12.2.3 If you know the action of an isometry on two distinct points, how can you determine the isometry algebraically? Geometrically?

Exercise 12.2.4 What is the outcome of composing any two of these isometries?

Exercise 12.2.5 Show than an isometry is a translation iff it is the composition of two reflections. Conclude that every isometry is the composition of at most two reflections.

Exercise 12.2.6 When do two isometries commute?

Exercise 12.2.7 What is the identity isometry? What is the inverse of each isometry?

Exercise 12.2.8 A *figure* is a subset of the line. Find a figure with a finite set of symmetries. Make a "multiplication" table for its symmetries.

Exercise 12.2.9 Find a figure with a countable set of symmetries.

Exercise 12.2.10 Find a figure with an uncountable set of symmetries.

Exercise 12.2.11 Describe the group of symmetries of a line segment.

Exercise 12.2.12 A *repeating line pattern* is a set of points on the line such that the set of translational symmetries is generated by a single translational symmetry (and its inverse). There may or may not be reflectional symmetry. Classify line patterns by the types of symmetries they can have.

12.3 Isometries in a Strip

Exercise 12.3.1 Classify the isometries of a horizontal strip (a region between two horizontal parallel lines in the plane).

Exercise 12.3.2 Assume that the strip is defined by the horizontal lines $y = 1$ and $y = -1$. Find formulas for translation by an amount a in the direction of the axis of the strip, the horizontal reflection, the vertical reflection about the line $x = b$, rotation by 180 degrees about the point $(c, 0)$, and the glide reflection about the axis with displacement d.

Exercise 12.3.3 An isometry is uniquely determined by its action on how many points? What assumptions do we need to make about these points? If you know the action of an isometry on these points, how can you determine the isometry algebraically? Geometrically?

Exercise 12.3.4 What is the outcome of composing any two of these isometries?

Exercise 12.3.5 When do two isometries commute?

Exercise 12.3.6 What is the identity isometry? What is the inverse of each isometry?

Exercise 12.3.7 A *repeating strip pattern* or *frieze* pattern is a figure (set of points) on the strip such that the set of translational symmetries is generated by a single translational symmetry (and its inverse). There may or may not be other symmetries. Classify strip patterns by the types of symmetries they can have.

12.4 Isometries in the Plane

Exercise 12.4.1 Classify the isometries of a plane.

Exercise 12.4.2 If you know the action of an isometry on three noncollinear points, how can you determine the isometry geometrically?

Can we geometrically show that every isometry must be one of these four? Not sure about glide reflections yet.

Exercise 12.4.3 Show that every isometry is the composition of at most three reflections.

Exercise 12.4.4 Find formulas for translation by the displacement (vector) (a, b), for rotation by θ about the point (a, b), for reflection across the line $ax + by + c = 0$, and for glide reflection across the line $ax + by + c = 0$ by an amount d.

Exercise 12.4.5 Show that every isometry is the composition of a rotation about the origin or a reflection through a line passing through the origin, followed by a translation.

Exercise 12.4.6 If you know the action of an isometry on three noncollinear points, how can you determine the isometry algebraically?

Exercise 12.4.7 What is the outcome of composing any two of these isometries?

Exercise 12.4.8 What is the composition of two reflections using a pair of non-intersecting lines? Using a pair of intersecting lines?

Exercise 12.4.9 Characterize when two isometries commute.

Exercise 12.4.10 Let R be the reflection through line ℓ and S be a 180 degree rotation about point P. Show that P lies on ℓ iff $(SR)^2$ is the identity.

Exercise 12.4.11 What is the identity isometry? What is the inverse of each isometry?

Exercise 12.4.12 A *figure* is a subset of the plane. Find a figure with a finite set of symmetries. Make a "multiplication" table for its symmetries.

Exercise 12.4.13 Describe the group of symmetries of an equilateral triangle. Of a square. Of a regular pentagon. Of a regular n-gon. Of a three-blade "propeller." Of a circle.

Exercise 12.4.14 A *repeating plane pattern* or *wallpaper pattern* is a set of points in the plane such that the set of translational symmetries is generated by two translations in two noncollinear directions (and their inverses). There may or may not be other symmetries. Classify some wallpaper patterns by their symmetries.

Exercise 12.4.15 Explain why the only angles of rotational symmetry available for wallpaper patterns are 180 degrees, 120 degrees, 90 degrees, and 60 degrees.

Exercise 12.4.16 Show that you can tile the plane with congruent copies of any triangle. Analyze the resulting symmetries.

Exercise 12.4.17 Show that you can tile the plane with congruent copies of any quadrilateral. Analyze the resulting symmetries.

Exercise 12.4.18 Analyze the symmetries of the eleven regular and semiregular tilings.

12.5 Axioms for Reflections

After stating the Incidence Axioms, we could have defined the notion of a one-to-one onto mapping of the plane to itself, made "reflection" a particular type of mapping satisfying certain axioms, and then defined two figures to be congruent if one can be mapped to the other via a finite sequence of reflections. Distance must then be defined so that it is preserved by congruence. For one way to do this; see Ewald, *Geometry: An Introduction*, in which perpendicularity is also an undefined term satisfying certain axioms.

Exercise 12.5.1 What would be a reasonable definition of reflection in the geometric world $\mathbf{S}^2$?

Exercise 12.5.2 Here is how to define reflection in $\mathbf{H}^2$: Let ℓ be a LINE in $\mathbf{H}^2$. If ℓ is an arc of a great circle, define reflection through ℓ as in the previous exercise. If ℓ is not an arc of a great circle, then there exists an ordinary point Q in the ordinary plane determined by the equator of $\mathbf{H}^2$ from which all lines joining points in ℓ to Q are tangent to the hemisphere. Now let m be any line through Q that intersects the hemisphere in two points P and P'. We say that P and P' are reflections of each other with respect to ℓ. (And any point on ℓ is a reflection of itself.) Can you show that distance is preserved under reflection?

We have also seen that using stereographic projection, $\mathbf{H}^2$ may be viewed as the set of all points strictly within the region bounded by a unit circle C, and that LINES are diameters of C, and arcs of circles meeting C at right angles. Suppose you have LINE ℓ and a point P. If ℓ is a diameter of C, define the reflection of P reflection in the ordinary way. If ℓ is not a diameter of C, find the ordinary center Q and the ordinary radius r of the circle determined by ℓ. Define the reflection of P to be that point P' on the ray $\overrightarrow{QP}$ such that $QP \cdot QP' = r^2$. (The point P' is called the *inversion* of P in the circle.)

For a good approach to non-Euclidean geometry along these lines that lends itself very well to using Geometer's Sketchpad, see the book (available in paperback) *Journey into Geometries* by Marta Sved.

Exercise 12.5.3 Returning to the Euclidean plane, define an isometry f to be an *involution* if f^2 is the identity mapping. For every line ℓ define R_a to be the reflection associated with the line. For every point P define H_P to be the rotation by 180 degrees (half-turn) associated with the point P. Prove the following:

1. A point P is incident to a line a iff $H_P R_a$ is an involution.
2. Two lines a and b are perpendicular iff $R_a R_b$ is an involution.
3. A point P' is the reflection of a point P in a line a iff $H_{P'} = R_a H_P R_a$.

In this way we can test certain geometric properties by testing certain algebraic properties.

Exercise 12.5.4 It is possible to take this exercise further and *define* POINTS to be half-turns and LINES to be reflections (see the book by Ewald). Using the matrix forms of the formulas you developed earlier, define POINTS to be appropriate 3×3 matrices, and LINES to be appropriate 3×3 matrices, and define INCIDENCE according to property 1 above. Prove that Axiom I-1 is satisfied.

12.6 Isometries in Space

Extend as many of the planar results as you can to $\mathbf{E}^3$.

13 Euclidean and Non-Euclidean Geometry

13.1 Introduction

Exercise 13.1.1 How are elements of non-Euclidean geometry introduced and developed in the K–16 curriculum?

In this section we explore the consequences of adopting one or the other of the two following contradictory axioms:

Axiom P-1: If ℓ is any line and $P \notin \ell$, there exists a unique line passing through P not meeting ℓ (in the plane of P, ℓ). (See Kay, Section 4.1.)

Axiom P-2: If ℓ is any line and $P \notin \ell$, there exists more than one line passing through P not meeting ℓ (in the plane of P, ℓ). (See Kay, Section 6.3.)

Exercise 13.1.2 Look through Kay again and classify which theorems can be proved without invoking either P-1 or P-2.

Note in particular that the following theorems can be derived from the axioms preceding P-1 and P-2:

Theorem 13.1.1 *The angle sum of a triangle is less than or equal to 180 degrees.* (This is Theorem 2 in Section 3.4 of Kay.)

Theorem 13.1.2 *If ℓ is any line and $P \notin \ell$, there exists at least one line passing through P not meeting ℓ.*

Exercise 13.1.3 Prove this theorem.

> **Theorem 13.1.3** *The following statements are either all true or all false:*
>
> 1. *There exists at least one triangle with angle sum less than 180 degrees.*
> 2. *Every triangle has angle sum less than 180 degrees.*
> 3. *Rectangles (quadrilaterals with four right angles) do not exist.*

The ASA, SSS, and AAS triangle congruence theorems also fall into this category. The latter is perhaps surprising since at this point we cannot prove or disprove that the angle sum of every triangle is 180 degrees.

13.2 Hyperbolic Geometry: Consequences of Assuming P-2

Here are some consequences of choosing Axiom P-2 instead of Axiom P-1. See Chapter 6 of Kay for these and many other fascinating results.

Theorem 13.2.1 *If ℓ is any line and $P \notin \ell$, there exist an infinite number of lines passing through P not meeting ℓ.*

Theorem 13.2.2 *The angle sum of every triangle is less than 180 degrees.*

Theorem 13.2.3 *There exists a constant k such that the area of every triangle is $k(180 - a - b - c)$, where a, b, and c are the angle measures of the triangle.*

Theorem 13.2.4 *If two triangles have congruent respective angles, then the triangles are congruent.*

Exercise 13.2.1 What are the regular and semiregular tilings in the hyperbolic plane? Draw some of them.

The big question is: Are the axioms for hyperbolic geometry consistent? Is there a model? The answer is that if the axioms for Euclidean geometry are consistent, then the axioms for hyperbolic geometry must also be consistent, because we can construct a model for hyperbolic geometry (e.g., $\mathbf{H}^2$) using constructions within the model for Euclidean geometry. It is also possible to go in the other direction, so we know that Euclidean geometry is consistent if and only if hyperbolic geometry is consistent. But isn't Euclidean geometry consistent? Don't we have the analytical models $\mathbf{E}^2$ and $\mathbf{E}^3$? Knowing that these models work depends upon knowing that the axioms for integers are consistent, but Gödel proved that (in a certain precise way) we cannot be sure of this.

See Appendix D of Kay for a unified axiom system for the Euclidean, spherical, and hyperbolic geometry.

14 Dimension

Exercise 14.0.2 How is the notion of dimension introduced and developed in the K–16 curriculum?

14.1 Higher Dimensions

Exercise 14.1.1 What is the analog of a sphere in four or higher dimensions?

Exercise 14.1.2 What is the analog of a cube in four or higher dimensions? How many vertices, edges, faces, and "hyper-faces" does it have? What does it look like if we unfold it in various ways? What do its shadows in three-dimensional space look like?

Exercise 14.1.3 What is the analog of a tetrahedron in four or higher dimensions? How many vertices, edges, faces, and "hyper-faces" does it have? What does it look like if we unfold it in various ways? What do its shadows in three-dimensional space look like?

Exercise 14.1.4 What is the analog of a convex polyhedron in four or higher dimensions? What is the analog of Euler's relation?

Exercise 14.1.5 How can we visualize higher-dimensional objects? Read *Flatland* by Edwin A. Abbott. A new annotation edition has been published in 2001. Read also Banchoff, *Beyond the Third Dimension: Geometry, Computer Graphics, and Higher Dimensions*.

14.2 Fractional Dimension

Exercise 14.2.1 How should we define the notion of dimension of a set?

Exercise 14.2.2 Let S be a line segment. If we obtain the line segment S' by scaling S up by a factor of k (multiplying all distances between points of S by k, we can dissect S' into k copies of S. What happens if we try this again when S is a square? What happens if we try this again when S is a cube? How can we use these results to motivate a definition of the dimension of these objects?

Exercise 14.2.3 Now consider S to be one of the three "sides" of the snowflake curve, generated from one of the three sides of the original equilateral triangle. Show that when S is scaled by a factor of 3, obtaining S', we can dissect it into four copies of S. What does this suggest the dimension of S is?